**SKOLIAD No. 61**

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to mayhem-editors@cms.math.ca

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by 1 August 2002. A copy of *MATHEMATICAL MAYHEM Vol. 3* will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

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Our item this issue is the 2001 W.J. Blundon Mathematics Contest. My thanks go out to Don Rideout of Memorial University for forwarding the material to me.

**THE EIGHTEENTH W.J. BLUNDON MATHEMATICS CONTEST**

Sponsored by The Canadian Mathematical Society, in cooperation with The Department of Mathematics and Statistics, Memorial University of Newfoundland

February 21, 2001

1. (a) At a meeting of 100 people, every person shakes hands with every other person exactly once. How many handshakes are there in total?

(b) How many four-digit numbers are divisible by 8?

2. Show that \( n^2 + 2 \) is divisible by 4 for no integer \( n \).

3. Prove that the difference of squares of two odd integers is always divisible by 8.

4. The inscribed circle of a right triangle \( ABC \) is tangent to the hypotenuse \( AB \) at \( D \). If \( AD = x \) and \( DB = y \), find the area of the triangle in terms of \( x \) and \( y \).

5. Find all integers \( x \) and \( y \) such that

\[
2^x + 3^y = 3^{y+2} - 2^{x+1}.
\]

6. Find the number of points \((x, y)\), with \( x \) and \( y \) integers, that satisfy the inequality \(|x| + |y| < 100\).
7. A flag consists of a white cross on a red field.

The white stripes are of the same width, both vertical and horizontal. The flag measures 48 cm × 24 cm. If the area of the white cross equals the area of the red field, what is the width of the cross?

8. Solve \( \frac{x + 1}{2 + \sqrt{x}} - \frac{1}{2 - \sqrt{x}} = 3 \).

9. Let \( P(x) \) and \( Q(x) \) be polynomials with "reversed" coefficients

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 ,
\]
\[
Q(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-2} x^2 + a_{n-1} x + a_n ,
\]

where \( a_n \neq 0, a_0 \neq 0 \). Show that the roots of \( Q(x) \) are the reciprocals of the roots of \( P(x) \).

10. If \( 1997^{1998} \) is multiplied out, what is the units digit of the final product?

Next we turn to solutions to the contests presented in the November 2001 issue. Following are the official solutions to the 2001 British Columbia Colleges mathematics competitions. [2001 : 440–445]

**BRITISH COLUMBIA COLLEGES**

**Junior High School Mathematics Contest, 2001**

**Final Round – Part A**

Friday May 4, 2001

1. The integer 9 is a perfect square that is both two greater than a prime number, 7, and two less than a prime number, 11. Another such perfect square is:

(a) 25   (b) 49   (c) 81   (d) 121   (e) 169

Soln. If \( n = 3k \pm 1 \), then \( n^2 + 2 = 9k^2 \pm 6k + 2 \), which is not prime. Thus we must have \( n \) a multiple of 3 in order to have \( n^2 + 2 \) prime. This eliminates all but 81. If we check 81, we see that 79 and 83 are both prime. [C]
2. Three circles, \(a\), \(b\), and \(c\), are tangent to each other at point \(P\), as shown.

![Diagram of three circles tangent at point P](image)

The centre of \(b\) is on \(c\) and the centre of \(a\) is on \(b\). The ratio of the area of the shaded region to the total area of the unshaded regions enclosed by the circles is:

(a) \(3 : 13\)  (b) \(1 : 3\)  (c) \(1 : 4\)  (d) \(2 : 9\)  (e) \(1 : 25\)

**Soln.** Let the radius of \(c\) be \(r\). Then the radii of \(b\) and \(a\) are \(2r\) and \(4r\), respectively. Therefore, the areas of \(a\), \(b\), and \(c\) are \(16\pi r^2\), \(4\pi r^2\), and \(\pi r^2\), respectively. Then the area of the shaded region is \(4\pi r^2 - \pi r^2 = 3\pi r^2\), and the area of the unshaded region is \(16\pi r^2 - 3\pi r^2 = 13\pi r^2\). The ratio of shaded to unshaded areas is then \(3 : 13\).

3. Here is a diagram of part of the downtown in a medium sized town in the interior of British Columbia. The arrows indicate one-way streets. The numbers or letters by the arrows represent the number of cars that travel along that portion of the street during a typical week day.

![Diagram of downtown streets](image)

Assuming that no car stops or parks and that no cars were there at the beginning of the day, the value of the variable \(W\) is:

(a) \(30\)  (b) \(200\)  (c) \(250\)  (d) \(350\)  (e) \(600\)

**Soln.** Clearly the number of cars entering the diagram must equal the number of cars exiting the diagram; that is,
4. The corners of a square of side $x$ are cut off so that a regular octagon remains. The length of each side of the resulting octagon is:

(a) $\frac{\sqrt{2}}{2}x$  
(b) $2x \left(2 + \sqrt{2}\right)$  
(c) $\frac{x}{\sqrt{2} - 1}$  
(d) $x \left(\sqrt{2} - 1\right)$  
(e) $x \left(\sqrt{2} + 1\right)$

**Soln.** Let the length of the removed corner piece be $a$ (see diagram below). Then a side of the resulting octagon is equal to $x - 2a$.

Using the Theorem of Pythagoras on the right-angled triangle in any corner gives us:

\[
(x - 2a)^2 = a^2 + a^2 = 2a^2
\]
\[
x - 2a = a\sqrt{2}
\]
\[
x = a(2 + \sqrt{2})
\]
\[
a = \frac{x}{2 + \sqrt{2}}.
\]

We are interested in the length of the side of the octagon:

\[
x - 2a = x - \frac{2x}{2 + \sqrt{2}} = \frac{2x + x\sqrt{2} - 2x}{2 + \sqrt{2}} = \frac{x\sqrt{2}}{2 + \sqrt{2}}
\]
\[
= \frac{x\sqrt{2}(2 - \sqrt{2})}{4 - 2} = \frac{x(2\sqrt{2} - 2)}{2} = x(\sqrt{2} - 1).
\]

**Alternate approach:** Let $b$ be the side length of the regular octagon. Since the removed corners are $45^\circ - 45^\circ - 90^\circ$ triangles, the legs have length $b/\sqrt{2}$. Thus

\[
x = \frac{2b}{\sqrt{2}} + b = \left(\frac{2 + \sqrt{2}}{\sqrt{2}}\right)b
\]

or

\[
b = \left(\frac{\sqrt{2}}{2 + \sqrt{2}}\right)x.
\]
Rationalizing the denominator we get:

\[ b = \frac{\sqrt{2}}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = \left( \frac{2\sqrt{2} - 2}{4 - 2} \right) x = \left( \sqrt{2} - 1 \right) x. \]

5. The value of \((0.0\overline{1})^{-1} + 1\) is: (The line over the digit 1 means that it is repeated indefinitely.)

(a) \(\frac{1}{91}\)  (b) \(\frac{90}{91}\)  (c) \(\frac{91}{90}\)  (d) 10  (e) 91

Soln. Let \(x = 0.0\overline{1}\). Then we note that \(10x = 0.\overline{1}\), which can also be written as \(10x = 0.1\overline{1}\). Comparing this form of \(10x\) with \(x\) we see that the decimal fraction expansions agree except for the first digit following the decimal point. Thus we may subtract to obtain \(9x = 0.1\), which means that \(x = 1/90\). Then

\[ (0.0\overline{1})^{-1} + 1 = \left( \frac{1}{90} \right)^{-1} + 1 = 90 + 1 = 91. \]

Alternate method: We first note that \(0.01 < x < 0.02\) or \(\frac{1}{100} < x < \frac{1}{50}\), which means that \(100 > \frac{1}{x} > 50\), and there is only one possible answer in this range. [c]

6. The people living on Sesame Street all decide to buy new house numbers from the same store, and they purchase the digits for their house numbers in the order of their addresses: 1, 2, 3, ..., If the store has 100 of each digit, then the first address which cannot be displayed occurs at house number:

(a) 100  (b) 101  (c) 162  (d) 163  (e) 199

Soln. In order to cover the addresses from 1 to 99, we need 20 of each non-zero digit and 9 zeros. From 100 to 199, we will have the greatest call on the digit 1 since every such address will have at least one digit 1 in it. Therefore, let us examine only the digit 1 first. From 100 to 109 we use 11 ones; from 110 to 119 we use 21 ones; for each subsequent group of ten (up to the address 199) we use a further 11 ones. Thus we want \(20 + 11 + 21 + k(11) \leq 100\), implying that \(k \leq 4\). That is, up to address 159 we have \(20 + 11 + 21 + 4(11) = 96\) ones. Addresses 160, 161, and 162 use up a further 4 ones and we have exhausted the 100 ones we started with. Thus the first address which cannot be displayed is 163. [d]

7. Given \(p\) dots on the top row and \(q\) dots on the bottom row, draw line segments connecting each top dot to each bottom dot. (In the diagram below, the dots referred to are the small open circles.) The dots must be arranged such that no three line segments intersect at a common point
(except at the ends). The line segments connecting the dots intersect at several points. (In the diagram below, the points of intersection of the line segments are the small filled circles.) For example, when \( p = 2 \) and \( q = 3 \) there are three intersection points, as shown below.

![Diagram](image)

When \( p = 3 \) and \( q = 4 \) the number of intersections is:

(a) 7  (b) 12  (c) 18  (d) 21  (e) 27

Soln. If we first consider \( p = 2 \) and \( q = 4 \), we easily see that there are \( 1 + 2 + 3 = 6 \) points of intersection. If we now consider \( p = 3 \) and \( q = 4 \) we see that by considering any pair of the \( p = 3 \) dots, together with the \( q = 4 \) dots opposite, we get 6 points of intersection. Now there are three such distinct pairs which gives us a total of 18 points of intersection.

8. At one time, the population of Petticoat Junction was a perfect square. Later, with an increase of 100, the population was 1 greater than a perfect square. Now, with an additional increase of 100, the population is again a perfect square. The original population was a multiple of:

(a) 3  (b) 7  (c) 9  (d) 11  (e) 17

Soln. Let the first-mentioned population be \( n \). Then \( n = a^2 \) for some integer \( a \). We then also have \( n + 100 = b^2 + 1 \) and \( n + 200 = c^2 \) for some integers \( b \) and \( c \). That is, \( a^2 + 99 = b^2 \) and \( a^2 + 200 = c^2 \), or \( b^2 - a^2 = 99 \) and \( c^2 - a^2 = 200 \). Subtracting these, we get \( b^2 - c^2 = 101 \). Thus \( (c - b)(c + b) = 101 \). Since 101 is prime we see that \( c - b = 1 \) and \( c + b = 101 \), whence \( c = 51 \) and \( b = 50 \). Thus \( n = a^2 = b^2 - 99 = 2401 = 49^2 = 7^4 \).

9. The cashier at a local movie house took in a total of $100 from 100 people. If the rates were $3 per adult, $2 per teenager and 25 cents per child, then the smallest number of adults possible was:

(a) 0  (b) 2  (c) 5  (d) 13  (e) 20

Soln. Let \( a \) be the number of adults, \( t \) be the number of teenagers, and \( c \) be the number of children attending the movie. Then \( a + t + c = 100 \) is the number of persons attending the movie, and \( 3a + 2t + c/4 = 100 \) is the number of dollars taken in by the movie house. Multiplying the second
equation by 4 to clear the fractions, and subtracting the first equation we get: $11a + 7t = 300$, or $t = (300 - 11a)/7$. Since we are seeking integer solutions and we want the smallest possible value for $a$, we may simply examine successive values of $a$ starting with $a = 0$ until we find an integer solution for $t$. The first (that is, the smallest) value of $a$ is $a = 5$, which gives $t = 35$ (and $c = 60$). [c]

10. The island of Arestia has 27 states, each of which belongs to one of two factions, the white faction and the grey faction, who are sworn enemies. The United Nations wishes to bring peace to Arestia by converting one state at a time to the opposite faction; that is, converting one state from white to grey or from grey to white, so that eventually all states belong to the same faction. In doing this they must guarantee that no single state is completely surrounded by states of the opposite faction. Note that a coastal state can never be completely surrounded, and that it may be necessary to convert a state from one faction to the other at one stage and then convert it back to its original faction later. A map of the state of Arestia is shown.

The five shaded states belong to the grey faction, and all of the unshaded states belong to the white faction. The minimum number of conversions necessary to completely pacify Arestia is:

(a) 5  (b) 7  (c) 9  (d) 10  (e) 15

Soln. We can first make a shaded "chain" to the coast by shading one coastal region at the top left and one of the two interior unshaded regions linking the two shaded regions. This requires 2 conversions. This shaded chain of 7 states can now be unshaded one at a time working from the interior to the coast, requiring another 7 conversions for a total of 9 conversions. To see that there can never be fewer than 9 conversions, we note first that we must convert the shaded states to unshaded in order to minimize the number of conversions, and secondly that it is
necessary to convert at least one unshaded coastal state and one unshaded interior state to shaded in order to avoid a shaded state being ultimately surrounded by unshaded states. This means that we would have a minimum of 7 shaded states to be converted to unshaded (in addition to the minimum of 2 unshaded that need to be converted to shaded). Thus we require at least 9 conversions. Thus, 9 is the minimum number of conversions needed to pacify Arezia.

BRITISH COLUMBIA COLLEGES

Junior High School Mathematics Contest, 2001

Final Round – Part B

Friday May 4, 2001

1. Find the smallest 3-digit integer which leaves a non-zero remainder when divided by any of 2, 3, 4, 5, or 6 but not when divided by 7.

Soln. Let \( n \) be the 3-digit integer in question. Clearly \( n \) is a multiple of 7. Since it has three digits, we may start with the smallest 3-digit multiple of 7 and examine successive multiples of 7 until the conditions are satisfied. The first 3-digit multiple of 7 is 105, which is also a multiple of 5; the next is 112, which is a multiple of 2; the next is \( 119 = 7 \times 17 \), and this leaves a non-zero remainder when divided by any of 2, 3, 4, 5, or 6. Thus \( n = 119 \).

Alternate Solution: There was at least one student who, in reading the problem, recognized (unlike the problem posers!) that nowhere is there a mention that the smallest 3-digit integer had to be positive. Since any negative number is smaller than any positive one, the student then found the smallest negative 3-digit integer satisfying the conditions. Since 994 is a multiple of 7, so is \( -994 \). Thus this represents the starting point. Since \( -994 \) is a multiple of 2, it is eliminated; the next candidate is \( -987 \), which is a multiple of 3 and is also eliminated; then comes \( -980 \), which is a multiple of 2 again; the next one is \( -973 \), and it satisfies all the conditions.

Strictly speaking, \( -973 \) is the only correct answer! However, since most solvers, as well as the problem posers, read "positive" into the problem, we also allowed 119 as a correct answer.

2. Assume that the land within two kilometres of the South Pole is flat. There are points in this region where you can travel one kilometre south, travel one kilometre east along one circuit of a latitude, and finally travel one kilometre north, and thus arrive at the point where you started. How far is such a point from the South Pole?

Soln. In the vicinity of the South Pole all east-west travel is on a circle centred about the South Pole. Since we wish to have the circumference of such
a circle equal to 1 km, we must have the radius equal to \( \frac{1}{2\pi} \) km. The original point from which the trip starts must be located a further 1 km away from the south pole. Thus we must start \( 1 + \frac{1}{2\pi} \) km from the South Pole.

3. Café de la Pêche offers three fruit bowls:

- Bowl A has two apples and one banana;
- Bowl B has four apples, two bananas, and three pears;
- Bowl C has two apples, one banana, and three pears.

Your doctor tells you to eat exactly 16 apples, 8 bananas and 6 pears each day. How many of each type of bowl should you buy so there is no fruit left over? Find all possible answers. (The numbers of bowls must be non-negative integers.)

**Soln.** Since the number of apples is twice the number of bananas in each bowl as well as in the doctor's dictum, we can ignore the apple constraint, and simply solve the problem for bananas and pears. Since we have in each bowl either 0 or 3 pears, we see that the condition on the pears can be met in exactly one of three ways: two of bowl B and none of bowl C; one of each of bowls B and C; or none of bowl B and two of bowl C. In each case we can then add the number of A bowls to fill out the requirements. Thus, there are three solutions: \((A, B, C) = (4, 2, 0), (5, 1, 1), \) and \((6, 0, 2)\).

4. In the triangle shown, \( \angle BAD = \alpha \), \( AB = AC \) and \( AD = AE \).

Find \( \angle CDE \) in terms of \( \alpha \).

**Soln.** Let \( \angle B = \angle C = x \). Let \( \angle CDE = y \). Since \( \angle AED \) is an exterior angle to \( \triangle EDC \), we have \( \angle AED = x + y \). Since \( \triangle ADE \) is isosceles, we also have \( \angle ADE = x + y \), whence \( \angle ADC = x + 2y \). But \( \angle ADC \) is an exterior angle of \( \triangle ABD \), which means that \( \angle ADC = x + \alpha \). Thus, we have \( x + 2y = x + \alpha \), or \( y = \frac{1}{2} \alpha \). Therefore, \( \angle CDE = \frac{1}{2} \alpha \).

5. In the multiplication below each of the letters stands for a distinct digit. Find all values of \( JEEP \).

\[
\begin{array}{c}
JEEP \\
\times JEEP \\
\hline
BEEEBEEP
\end{array}
\]
Soln. Since $P^2 = P + 10k$ for some integer $k$, $0 \leq k \leq 8$, we see that $P$ is one of 0, 1, 5, or 6. Now by considering the last two digits of each factor and the product, we have $(10E + P)^2 = 100m + 10E + P$ for some integer $n < 100$.

This means that $20PE + P^2 - 10E - P = 10E(2P - 1) + P(P - 1)$ is a multiple of 100. Let us consider $P = 6$. Then $110E + 30$ is a multiple of 100, implying that $E = 7$. This means that $776^2$ must end in the digits 776, but $776^2$ actually ends in the digits 176. Thus, $P \neq 6$.

Next try $P = 5$. Then $90E + 20$ is a multiple of 100, implying that $E = 2$. This means that $225^2$ must end in the digits 225, but $225^2$ actually ends in the digits 625. Thus $P \neq 5$. Therefore, $P = 0$ or 1. In either case we have $P(P - 1) = 0$, which means that $10E(2P - 1)$ is a multiple of 100. Since $2P - 1 = \pm 1$, we conclude that $E = 0$, implying that $P = 1$, since it must be different from $E$. Thus, we have $P = 1$ and $E = 0$. Then we have $B = J^2$ and $B = 2J$, since $(J001)^2 = (J)^200(2J)001 = B00B001$. Since $J^2 = 2J$ and $J \neq 0$, we conclude that $J = 2$, whence $JEJEP = 2001$.

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**BRITISH COLUMBIA COLLEGES**

Senior High School Mathematics Contest, 2001

Final Round – Part B

Friday May 4, 2001

1. See question #4 above.

Soln. See question #4 on the Junior Final (Part B).

2. A semicircle $BAC$ is mounted on the side $BC$ of the triangle $ABC$. Semicircles are also mounted outwardly on the sides $BA$ and $AC$, as shown in the diagram. The shaded crescents represent the area inside the smaller semicircles and outside the semicircle $BAC$. Show that the total shaded area equals the area of the triangle $ABC$.
Soln. To obtain the area of the shaded region we will compute the sum of the areas of triangle $ABC$, the area of the semicircle on $AB$, and the area of the semicircle on $AC$, and then we will subtract from this sum the area of the semicircle on $BC$. Let $a$, $b$, and $c$ be the lengths of the sides $BC$, $AC$, and $AB$, respectively. Since $\triangle ABC$ is inscribed in a semicircle on $BC$, we see that $\angle BAC = 90^\circ$. Thus by the Theorem of Pythagoras we have $a^2 = b^2 + c^2$. If we now denote the area of $\triangle ABC$ by $[ABC]$, then our desired area is:

$$
A = [ABC] + \frac{1}{2} \pi \left( \frac{a}{2} \right)^2 + \frac{1}{2} \pi \left( \frac{b}{2} \right)^2 - \frac{1}{2} \pi \left( \frac{a}{2} \right)^2 = [ABC] + \frac{1}{8} \pi (b^2 + c^2 - a^2) = [ABC].
$$

3. Five schools competed in the finals of the British Columbia High School Track Meet. They were Cranbrook, Duchess Park, Nanaimo, Okanagan Mission, and Selkirk. The five events in the finals were: the high jump, shot put, 100-metre dash, pole vault and 4-by-100 relay. In each event the school placing first received five points; the one placing second, four points; the one placing third, three points; and so on. Thus, the one placing last received one point. At the end of the competition, the points of each school were totalled, and the totals determined the final ranking.

(a) Cranbrook won with a total of 24 points.

(b) Sally Sedgwick of Selkirk won the high jump hands down (and feet up), while Sven Sorenson, also of Selkirk, came in third in the pole vault.

(c) Nanaimo had the same number of points in at least four of the five events.

Each school had exactly one entry in each event. Assuming there were no ties and the schools ended up being ranked in the same order as the alphabetical order of their names, in what position did Doug Dolan of Duchess Park rank in the high jump?

Soln. Since each school had exactly one entry in each event, we conclude by (a) that Cranbrook had four first place finishes and one second place finish. By (b) it becomes clear that the one second place finish they had was in the high jump. Thus Doug Dolan of Duchess Park could finish no higher than third place in the high jump. Of the total of 75 available points, 24 went to Cranbrook, which leaves 51 points to be shared by the other four schools.

Since they all received different totals, Duchess Park, who came in second must have obtained at least 15 points (since $14 + 13 + 12 + 11 = 50$, which is too small). A similar argument shows that last place Selkirk must have obtained at most 11 points (since $12 + 13 + 14 + 15 = 54$,
which is too large). Since Selkirk obtained 5 points for the high jump and 3 points for the pole vault by (b), and at least 1 point for each of the other three events, they must have a total of at least 11. This, together with our previous remark shows that Selkirk had exactly 11 points. This leaves only 40 points to be shared by Duchess Park, Nanaimo, and Okanagan Mission, and each of them must have at least 12 points.

The only possibility is that Duchess Park had 15 points, Nanaimo had 13 points and Okanagan Mission had 12 points. Since Nanaimo received the same number of points in four of the five events and had a total of 13 points, they must have finished third four times and last once (since four second place finishes would give them too many points, while four fourth place finishes would require them to finish first in the other event to get 13 points, but all the first place finishes went to Cranbrook and Selkirk).

Thus Nanaimo had to finish last in the pole vault, as Selkirk finished third. At this point we have determined that all 1-point, 3-point, and 5-point finishes (except for last place in the high jump) have gone to one of Cranbrook, Nanaimo, or Selkirk. Since the only remaining odd point will generate an odd total, it must go to Duchess Park, which has a total of 15 points.

Thus Doug Dolan of Duchess Park must have finished last in the high jump.

4. A box contains tickets of two different colours: blue and green. There are 3 blue tickets. If two tickets are to be drawn together at random from the box, the probability that there is one ticket of each colour is exactly $\frac{1}{2}$. How many green tickets are in the box? Give all possible solutions.

Soln. Let $g$ be the number of green tickets in the box. Then the total number of tickets in the box is $g + 3$. The number of ways of drawing two tickets from the box (together) is $\binom{g + 3}{2} = \frac{(g + 3)(g + 2)}{2}$. The number of ways of drawing one ticket of each colour is by drawing one of 3 blue tickets and one of $g$ green tickets, which is $3 \cdot g$. Thus the probability of drawing one of each colour when drawing two tickets together is

$$\frac{3g}{(g + 3)(g + 2)/2} = \frac{6g}{(g + 3)(g + 2)}.$$  

We are told that this probability is $\frac{1}{2}$. Therefore, we have

$$\frac{1}{2} = \frac{6g}{g^2 + 5g + 6}$$

$$g^2 + 5g + 6 = 12g$$

$$g^2 - 7g + 6 = 0$$

$$(g - 6)(g - 1) = 0.$$
which means that \( g = 1 \) or \( g = 6 \). Both of these solutions can be verified.

5. In (a), (b), and (c) below the symbols \( m, h, t, \) and \( u \) can represent any integer from 0 to 9 inclusive.

(a) If \( h - t + u \) is divisible by 11, prove that \( 100h + 10t + u \) is divisible by 11.

(b) If \( h + u = m + t \), prove that \( 1000m + 100h + 10t + u \) is divisible by 11.

(c) Is it possible for \( 1000m + 100h + 10t + u \) to be divisible by 11 if \( h + u \neq m + t \)? Explain.

Soln. (a) Note that

\[
100h + 10t + u = 99h + 11t + (h-t+u) = 11(9h+t) + (h-t+u).
\]

Clearly, if \( h - t + u \) is divisible by 11, the entire right hand side is also divisible by 11, which means that \( 100h + 10t + u \) is divisible by 11.

(b) In this case we observe

\[
1000m + 100h + 10t + u
= 1001m + 99h + 11t + (-m + h - t + u)
= 11(91m + 9h + t) + (h + u - m - t)
\]

Again, if \( h + u = m + t \) we see that the right hand side is simply \( 11(91m + 9h + t) \), which is clearly divisible by 11, implying that the right hand is also divisible by 11.

(c) If we examine (1) in part (b) above, we see that to have the right hand side divisible by 11 all we need is that \( h + u - m - t \) is divisible by 11. This can happen without \( h + u = m + t \) as the following example shows: \( h = 9, u = 8, m = 4, \) and \( t = 2; h + u = 17 \) and \( m + t = 6, \) which means that \( h + u \neq m + t, \) but \( h + u - m - t = 11 \) and the expression in (1) above is then divisible by 11.