Challenge Board Solutions

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In this issue we present some of the solutions to the Konhauser Problemfest presented in the April 2001 issue [2001 : 204].

1. Last season, the Minnesota Timberwolves won 5 times as many games as they lost, in games in which they scored 100 or more points. On the other hand, in games in which their opponents scored 100 or more points, the Timberwolves lost 50% more games than they won. Given that there were exactly 34 games in which either the Timberwolves or their opponents scored 100 or more points, what was the Timberwolves' win-loss record in games in which *both* they and their opponents scored 100 or more points?

Solution: Let the desired win-loss record be $a-b$, so that of the games when both the Timberwolves and their opponents scored at least 100 points, the Timberwolves won a and their opponents won b. Also, let c be the number of games in which only the Timberwolves scored at least 100 points and d be the number of games in which only their opponents scored at least 100 points. Thus, we have

$$a+c = 5b\,, \tag{1}$$
$$b+d = 1.5a\,, \tag{2}$$
$$a+b+c+d = 34\,. \tag{3}$$

Thus, we get $c=5b-a$, and $d=1.5a-b$ from (1) and (2), so that (3) yields $3a+10b=68$.

Now a and b are non-negative integers, and in particular $b \le 6$ and $68-10b$ is divisible by 3, which is true only for $b=2$ and $b=5$. If $b=2$, then $a=16$, $c=5b-1=-6<0$, which is a contradiction. Thus, we must have $b=5$ and $a=6$. The Timberwolves were $6-5$ in games in which both they and their opponents scored at least 100 points.

2. Three circles are drawn in chalk on the ground. To begin with, there is a heap of n pebbles inside one of the circles, and there are "empty heaps" (containing no pebbles) in the other two circles. Your goal is to move the entire heap of n pebbles to a different circle, using a series of moves of the following type. For any non-negative integer k, you may move exactly 2^k pebbles from one heap (call it heap **A**) to another (heap **B**), provided that heap **B** begins with fewer than 2^k pebbles, and that after the move, heap **A** ends up with fewer than 2^k pebbles. Naturally, you want to reach your goal in as few moves as possible. For what values of $n \le 100$ would you need the largest number of moves?

Solution: This is a close relative of the classic "Towers of Hanoi" problem. In that problem, a tapering stack of d discs is to be moved from one peg to one of the other two pegs, which are originally empty. No disc is ever allowed to rest on a larger disc, and the discs must be moved one at a time. If b_d is the number of moves required for d discs, one shows that $b_{d+1} = 2b_d + 1$ (that is, to move the bottom disc, first move the other discs onto a single peg, then after the bottom disc is moved, the others are moved back onto it). However, all we need for our present purpose is that $b_{d+1} > b_d$.

The connection between the given problem and the "Towers of Hanoi" problem can be seen by writing n as a sum of distinct powers of 2. Then, n can be written as such a sum in exactly one way.

The distinct powers of 2 correspond to the discs in the "Towers of Hanoi" problem, but *larger powers of* 2 *correspond to smaller discs*. The condition for moving 2^k pebbles corresponds to the fact that the top disc must be moved before any of the discs underneath.

Note that d corresponds to the number of distinct powers of 2 that add up to n (or the number of 1's in the binary expansion of n). So our problem boils down to: what numbers $n \le 100$ have the most digits 1 in their binary expansion?

The powers of 2 that can be used in the binary expansion of numbers $n \le 100$ are 2^6, 2^5, 2^4, 2^3, 2^2, 2^1, 2^0. However, they cannot *all* be used since their sum is $127 > 100$. It is possible to use all but one of them in just two ways:

$$\begin{aligned} 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 &= 95\,, \\ 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 &= 63\,. \end{aligned}$$

Hence, 63 *and* 95 are the desired values of n.

3. (a) Begin with a string of 10 A's, B's, and C's, for example

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A B C C B A B C B A
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and underneath, form a new row, of length 1 shorter, as follows: between two consecutive letters that are different, you write the third letter, and between two letters that are the same, you write that same letter again. Repeat this process until you have only one letter in the new row. For example, for the string above, you will now have:

```
A B C C B A B C B A
 C A C A C C A A C
  B B B B C B A B
   B B B A A C C
    B B C A B C
     B A B C A
      C C A B
       C B C
        A A
         A
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Prove that the letters at the corners of the resulting triangle are always either all the same or all different.

Solution: Think of the letters A, B, C as representing the numbers 0, 1, 2, respectively, $\pmod 3$. Then, if in some row we have $\cdots \quad x \quad y \quad \cdots$ where x and y are integers $\pmod 3$, we get $\cdots - x - y \cdots$ in the next row, where $-x-y$ is computed $\pmod 3$. If the original row corresponds to the integers $x_1 \quad x_2 \quad \cdots \quad x_{10} \quad \pmod 3$, then the next rows are:

$$\begin{array}{ccccccc} -x_1 - x_2 & & -x_2 - x_3 & & -x_3 - x_4 & & \cdots \\ & x_1 + 2x_2 + x_3 & & x_2 + 2x_3 + x_4 & & \cdots & \\ & & -x_1 - 3x_2 - 3x_3 - x_4 & & \cdots & & \end{array}$$

and we see the pattern (which can be proved by induction): apart from the $\pm$ signs, the coefficients are those of Pascal's triangle, the binomial coefficients. In particular, the tenth (bottom) row will consist of the single entry

$$-x_1 - \binom{9}{1}x_2 - \binom{9}{2}x_3 - \cdots - \binom{9}{8}x_9 - x_{10} \pmod 3 .$$

But since 9 is a power of the prime 3, all binomial coefficients $\binom{9}{1}$, $\binom{9}{2}$, $\ldots$, $\binom{9}{8}$ are divisible by 3, so that the entry in the bottom row is equal to $-x_1 - x_{10} \pmod 3$. Thus, the three corner numbers $(x_1, x_{10}, -x_1 - x_{10})$ $\pmod 3$ that add to $0 \pmod 3$ and it follows that they are either all the same or all different.

(b) For which positive integers n (besides 10) is the result from part (a) true for all strings of n A's, B's, and C's?

Solution: For $n = 1$, and for $n = 3^k + 1$, $k \geq 0$. For such an n, the n^{th} row will have the form:

$$-x_1 - \binom{3^k}{1}x_2 - \binom{3^k}{2}x_3 - \cdots - \binom{3^k}{3^k - 1}x_{n-1} - x_n \equiv -x_1 - x_n \pmod 3 ,$$

because the binomial coefficients are divisible by 3.

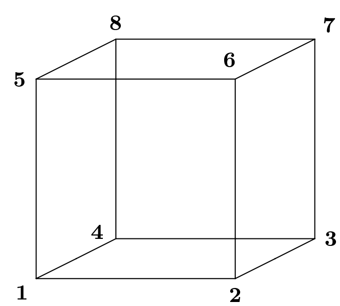
4. When Mark climbs a staircase, he ascends either 1, 2, or 3 stairsteps with each stride, but in no particular pattern from one foot to the next. In how many ways can Mark climb a staircase of 10 steps? (Note that he must finish on the top step. Two ways are considered the same if the number of steps for each stride are the same; that is, it does not matter whether he puts his best or his worst foot forward first.) Suppose that a spill has occurred on the 6^{th} step and Mark wants to avoid it. In how many ways can he climb the staircase without stepping on the 6^{th} step?

Solution: Let a_n be the number of ways for Mark to climb a staircase of n steps. Then $a_1 = 1$, $a_2 = 2$ and $a_3 = 4$. For $n > 3$, consider Mark's last stride. If his last stride was 1–step, then before that he climbed an $n-1$–step staircase, and there are a_{n-1} ways in which that can be done. Using a similar argument for 2– and 3–step last steps, we get $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n > 3$. Using this we can calculate $a_{10} = 274$. Thus there are 274 ways that Mark can climb the stairs before the spill has occurred.

Once the spill has occurred, we can work through the same way. If b_n is the number of ways to get to step n and not step on step 6, then we have: $b_n = a_n$ for $n \leq 5$, $b_6 = 0$. Thus, using the recurrence relation for a_n we get $b_7 = 0 + 13 + 7 = 20$, $b_8 = 20 + 0 + 13 = 33$, $b_9 = 0 + 20 + 33 = 53$ and $b_{10} = 20 + 33 + 53 = 106$. Thus, there are 106 ways to go up the stairs and avoid the spill.

5. Number the vertices of a cube from 1 to 8. Let A be the 8×8 matrix whose (i, j) entry is 1 if the cube has an edge between vertices i and j, and is 0 otherwise. Find the eigenvalues of A, and describe the corresponding eigenspaces.

Solution:Let the vertices be numbered as shown.



Then we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We can find a few eigenvectors by inspection; for example $(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)^T$ is an eigenvector for $\lambda = 3$ and $(1\ \ 1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1)^T$ is an eigenvector for $\lambda = -1$. From these a pattern starts to emerge. Note that the entries 1 occur in positions 1, 2, 3, 4 and the entries -1 occur in positions 5, 6, 7, 8. Now, $\{1,\ 2,\ 3,\ 4\} \cup \{5,\ 6,\ 7,\ 8\} = S_1 \cup S_2$ is a decomposition of the vertex set $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ into two disjoint parts S_1, S_2 with the property that each vertex in S_i is connected to two vertices in S_i and one vertex in S_j for $i,\ j \in \{1, 2\}$, $i \neq j$. We now have the observation, whose proof follows from the definition of A given in the problem:

Let $S = S_1 \cup S_2$ be any decomposition of $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ into disjoint subsets S_1, S_2 such that there exist integers k and l with $k + l = 3$ such that each vertex in S_i is connected to k vertices in S_i and l vertices in S_j for $i,\ j \in \{1,\ 2\}$, $i \neq j$. Then the vector with entries 1 in the positions corresponding to S_1 and -1 in the positions corresponding to S_2 is an eigenvector of A for the eigenvalue $k - l$.

Note that while we could interchange the subsets S_1 and S_2, that would just change the eigenvector to its opposite. Except for this, there are eight different decompositions of S of the desired type, as shown in the table below.

S_1	S_2	k	l	λ	Eigenvector
S	ϕ	3	0	3	$(1,1,1,1,1,1,1,1)^T$
$\{1,\ 2,\ 3,\ 4\}$	$\{5,\ 6,\ 7,\ 8\}$	2	1	1	$(1,1,1,1,-1,-1,-1,-1)^T$
$\{1,\ 2,\ 5,\ 6\}$	$\{3,\ 4,\ 7,\ 8\}$	2	1	1	$(1,1,-1,-1,1,1,-1,-1)^T$
$\{1,\ 4,\ 5,\ 8\}$	$\{2,\ 3,\ 6,\ 7\}$	2	1	1	$(1,-1,-1,1,1,-1,-1,1)^T$
$\{1,\ 2,\ 7,\ 8\}$	$\{3,\ 4,\ 5,\ 6\}$	1	2	-1	$(1,1,-1,-1,-1,-1,1,1)^T$
$\{1,\ 4,\ 6,\ 7\}$	$\{2,\ 3,\ 5,\ 8\}$	1	2	-1	$(1,-1,-1,1,-1,1,1,-1)^T$
$\{1,\ 3,\ 5,\ 7\}$	$\{2,\ 4,\ 6,\ 8\}$	1	2	-1	$(1,-1,1,-1,1,-1,1,-1)^T$
$\{1,\ 3,\ 6,\ 8\}$	$\{2,\ 4,\ 5,\ 7\}$	0	3	-3	$(1,-1,1,-1,-1,1,-1,1)^T$

(Geometrically, the eigenvectors for $\lambda = 1$ in this table can be thought of as corresponding to pairs of opposite faces of the cube; the eigenvectors for $\lambda = -1$ can be thought of as corresponding to pairs of diagonal planes through the cube.) It is easily checked that the eigenvectors for $\lambda = 1$ (similarly for $\lambda = -1$) listed above are linearly independent. Therefore, for matrix A we have eigenvalues $\lambda = 3$, $\lambda = 1$ (with multiplicity 3), $\lambda = -1$ (with multiplicity 3) and $\lambda = -3$, and the eigenspaces are spanned by the vectors in the table.

6. Let $f(x)$ be a twice-differentiable function on the open interval $(0, 1)$ such that

$$\lim_{x\to 0^+} f(x) = -\infty\,, \quad \text{and} \quad \lim_{x\to 1^-} f(x) = +\infty\,.$$

Show that $f''(x)$ takes on both negative and positive values.

Solution: Take an arbitrary number x_0 in the interval $(0, 1)$ and let $f'(x_0) = m$. Suppose that $f''(x) \geq 0$ for all x in $(0, 1)$. Then, in particular, for all x with $0 < x < x_0$ we must have $f(x) \geq f(x_0) + m(x - x_0)$.

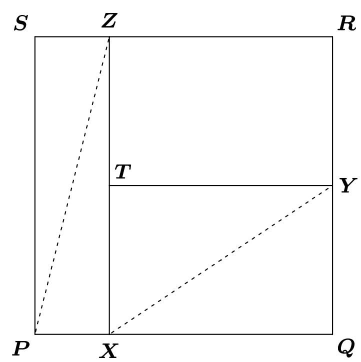
But $f(x) \geq f(x_0) + m(x - x_0)$ shows that, as $x \to 0^+$, we have $f(x) \geq f(x_0) - mx_0$, contradicting the given $\lim_{x\to 0^+} f(x) = -\infty$.

Similarly, if $f''(x) \leq 0$ for all x in $(0, 1)$ then for all x with $x_0 < x < 1$ we must have $f(x) \leq f(x_0) + m(x - x_0)$, and this contradicts $\lim_{x\to 1^-} f(x) = +\infty$.

Thus, we must have $f''(x) > 0$ for some x in $(0, 1)$ as well as $f''(x) < 0$ for some x in $(0, 1)$, and we are done.

7. Three stationary sentries are guarding an important public square which is, in fact, square, with each side measuring 8 rods (recall that one rod equals 5.5 yards). (If any of the sentries see trouble brewing at any location on the square, the sentry closest to the trouble spot will immediately cease to be stationary and dispatch to that location. And like all good sentries, these three are continually looking in all directions for trouble to occur.) Find the maximum value of D, so that no matter how the sentries are placed, there is always some spot in the square that is at least D rods from the closest sentry.

Solution: Divide the square into rectangles as shown, in such a way that $QY = YR = 4$ and $PZ = XY$. We claim once this has been done $D_0 = \frac{1}{2}XY$ is the maximum value of D.



Proof: One way to place the sentries is at the centres of the rectangles $PXZS$, $XQYT$, $TYRZ$; for this placement, every point on the square is at most D_0 from the sentry at the centre of the rectangle it belongs to. Thus, a value $D > D_0$ is not possible. To show that $D = D_0$ actually works, we have to show that there is no placement of the sentries for which every point of the square has distance less than D_0 to the closest sentry. Suppose we did have such a placement. Since there are four corners P, Q, R, S of the square and only three sentries, at least two corners would be guarded by the same sentry. Without loss of generality, let P and S be two corners guarded by the same sentry. Now note that X and Y cannot be guarded by a single sentry, because by our assumption that sentry would have distance less than D_0 to both X and Y, so that $XY < 2D_0 = XY$, a contradiction. Similarly, no other two opposite corners of the three smaller rectangles that we constructed can be guarded by a single sentry. Thus, S, X, Y are all guarded by different sentries, say 1, 2, 3, in that order, and since S and P are guarded by the same sentry P, Z, Y are also guarded by 1, 2, 3 in that order. Thus, Q and R must be guarded by 3. But, we will show that the mid-point O of XQ is too far from Z and R to be guarded by 2 and 3, respectively, which will provide a contradiction.

Let $PX = a$, $XQ = 8 - a$. Then

$$PZ = XY \quad \Longleftrightarrow \quad PX^2 + XZ^2 = XQ^2 + QY^2 ,$$

which gives $a = 1$, so that $PX = 1$, $QX = 7$, $XO = \frac{7}{2}$ and $ZO > ZP = 2D_0$, showing that O is indeed too far from Z. Thus, the answer is $D_0 = \frac{1}{2}\sqrt{65}$.

8. The Union Atlantic Railway is planning a massive project: a railroad track joining Cambridge, Massachusetts and Northfield, Minnesota. However, the funding for the project comes from the will of Orson Randolph Kane, the eccentric founder of the U.A.R., who has specified some strange conditions on the railway; thus the skeptical builders are unsure whether or not it is possible to build a railway subject to his unusual requirements.

Kane's will insists that there must be exactly 100 stops (each named after one of his great-grandchildren) between the termini, and he even dictates precisely what the distance along the track between each of these stops must be. (Unfortunately, the tables in the will *do not* list the order in which the stops are to appear along the railway.) Luckily, it is clear that Kane has put some thought into these distances; for any three distinct stops, the largest of the three distances among them is equal to the sum of the smaller two, which is an obvious necessary condition for the railway to be possible. (Also, all the given distances are shorter than the distance along a practical route from Cambridge to Northfield!)

U.A.R.'s engineers have pored over the numbers and noticed that for any four of Kane's stops, it would be possible to build a railway with these four stops and the distances between them as Kane specifies. Prove that, in fact, it is possible to complete the entire project to Kane's specifications.

Solution: Let $d(i, j)$ be the required distance from stop i to stop j along the track. Note that if four stops are arranged successfully along the track, there is a unique greatest distance between two of those four stops: the one that ends up closest to Cambridge, and the one that ends up closest to Northfield.

Now for the proof.

The case $n = 4$ is trivial. Suppose the statement is true for N stops ($N \geq 4$) and we have a table of distances $d(i, j)$ for $N + 1$ stops satisfying the given conditions. Find the largest of these distances. This largest distance can only occur *once*, because if $d(i, j) = d(k, l)$ with $\{i, j\} \neq \{k, l\}$, it would be impossible to arrange stops i, j, k, l (or if two of them were equal, say $j = l$, then stops i, j, k, m with $m \neq i, j, k$) successfully along the track. Without loss of generality, assume that the greatest distance in the table is $d(1, N + 1)$. Let a, b be any two distinct stops with $2 \leq a, b \leq N$. Then, since stops 1, a, b, $N+1$ could be arranged successfully, and $d(1, N+1)$ is the largest distance, we must have either $d(1,a) = d(1,b) + d(a,b)$ or $d(1,b) = d(1,a) + d(a,b)$ (but we cannot have $d(a,b) = d(1,a) + d(1,b)$). Now, ignore stop $N+1$ for the moment and arrange stops 1, 2, $\ldots$, N successfully along the track, which is possible by the induction hypothesis. Since we cannot have $d(a,b) = d(1,a) + d(1,b)$ for any a, $b \in \{2, 3, \ldots, N\}$, we have that 1 cannot be between any two other stops in this successful arrangement; that is, all stops 2, 3, $\ldots$, N must be the same side of stop 1. Since they all have the proper distances to stop 1 and $d(1, N+1)$ is larger than all those distances, we can place stop $N+1$ on the track in such a way that all stops 2, 3, $\ldots$, N are between stop 1 and stop $N+1$, that stops 1 and $N+1$ are the proper distance apart, and that all stops 1, 2, $\ldots$, N are still in the same places.

All we now need to do is to check that stop $N+1$ is the correct distance $d(a, N+1)$ to each of the stops a, $2 \le a \le N$. Then, since $d(1, N+1)$ is less than the distance from Cambridge to Northfield, we can arrange for the stops to follow each other in this order between Cambridge and Northfield, and we will be done.

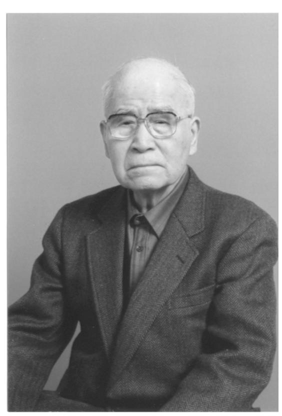
Consider any stop a, $2 \le a \le N$. Since it has the correct distance $d(a,1)$ to stop 1, in the above placement its distance to stop $N+1$ is given by $d(1, N+1) - d(a,1)$. But since $d(1, N+1)$ is the largest of the required distances between stops 1, a, $N+1$, we have

$$d(1, N+1) \ = \ d(a,1) + d(a, N+1) \,,$$

so that

$$d(1, N+1) - d(a,1) \ = \ d(a, N+1) \,,$$

and we are done.



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