SKOLIAD No. 60

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Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 July 2002*. A copy of ***MATHEMATICAL MAYHEM* Vol. 2** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

This issue's item comes to us from Manitoba. My thanks go to Diane Dowling at St.Paul's College in Winnipeg for forwarding the material to me.

MANITOBA MATHEMATICAL CONTEST, 2001

For Students in Senior 4

9:00 a.m. – 11:00 a.m. Wednesday, February 21, 2001

Sponsored by

The Actuaries' Club of Winnipeg, The Manitoba Association of Mathematics Teachers, The Canadian Mathematical Society, and The University of Manitoba

Answer as much as possible. You are not expected to complete the paper. See both sides of this sheet. Hand calculators are not permitted. Numerical answers only, without explanation, will not be given full credit.

1. (a) Solve the equation $\frac{6}{x} - \frac{1}{x^2} = 9$.

(b) Find real numbers a and b such that $(2^a)(4^b) = 8$ and $a+7b = 4$.

2. (a) Find all real numbers x such that $25|x| = x^2 + 144$.

(b) If x is a real number and $\frac{5}{\sqrt{7+3\sqrt{x}}} = \sqrt{7-3\sqrt{x}}$ what is the value of x?

3. (a) If two of the roots of $x^3+px^2+qx+r = 0$ are equal in absolute value but opposite in sign, prove that $pq = r$. (p, q and r are real numbers.)

(b) If a, b and c are real numbers such that $a + b + c = 14$, $c^2 = a^2 + b^2$ and $ab = 14$, find the numerical value of c.

4. (a) If $0^\circ < \theta < 180^\circ$ and $2\sin^2\theta + 3\sin\theta \geq 2$ what is the largest possible value of θ?

(b) Let O be the origin, P the point whose coordinates are $(2, 3)$ and F a point on the line whose equation is $y = \frac{x}{2}$. If PF is perpendicular to OF find the coordinates of F.

5. How many consecutive zeros are there at the end of the product of all the integers from 16 to 100 inclusive?

6. In triangle ABC, $\angle ACB = 135^\circ$, $CA = 6$ and $BC = \sqrt{2}$. If M is the mid-point of the side AB, find the length of CM.

7. A circle of radius 2 has its centre in the first quadrant and has both coordinate axes as tangents. Another smaller circle also has both coordinate axes as tangents and has exactly one point in common with the larger circle. Find the radius of the smaller circle.

8. A parallelogram has an area of 36 and diagonals whose lengths are 10 and 12. Find the lengths of its sides.

9. a, b, c, d are distinct integers such that $(x-a)(x-b)(x-c)(x-d)=4$ has an integral root r. Prove that $a + b + c + d = 4r$.

10. If x, y and z are positive real numbers, prove that

$$(x + y - z)(x - y)^2 + z(x - z)(y - z) \ \geq \ 0\,.$$

Next we present the official solutions to the Mandelbrot competitions from the October issue [2001 : 385].

The Mandelbrot Competition
Division B Round Two Individual Test
December 1997

1. If a group of positive integers has a sum of 8, what is the greatest product the group can have? **(1 point)**

Solution.

Clearly we do not want any 1's in our group, since they contribute to the sum but not the product. Any 5 in the group can be replaced by a 2 and a 3, which have the same sum but a greater product. Similarly, any 6 can be replaced by two 3's, and so on, since every number contributes more to the product if broken down into 2's and 3's. (A 4 can be replaced with two 2's, with no effect on the sum or product.) Hence, our optimal group should consist of 2's and 3's only. The only such sets of numbers summing to 8 are $2+2+2+2$ or $2+3+3$. The products in these cases are 16 and 18. Hence, $\boxed{18}$ is optimal.

2. There is one two-digit number such that if we add 1 to the number and reverse the digits of the result, we obtain a divisor of the number. What is the number? **(1 point)**

Solution.

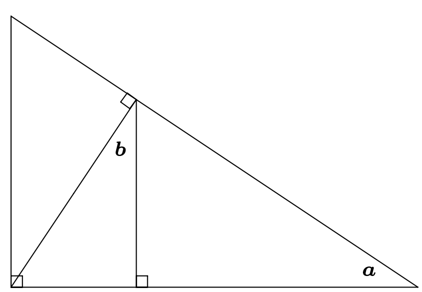
Let the two-digit number n be $\boxed{\boxed{t|u}}$. If u is not 9, then the number obtained by adding one and reversing the digits is $\boxed{\boxed{u+1|t}}$. Since this potential divisor cannot equal n, it must be half of n or less, so that $2(u+1) \leq t$ and $2\left((u+1)\times 10 + t\right) \leq t \times 10 + u$. This restricts us to the numbers $n = 30$, 40, 50, 51, 60, 61, 70, 71, 80, 81, 82, 90, 91, or 92. A quick check shows that none of these numbers works. Hence, u must be 9. If $n = \boxed{\boxed{t|9}}$, then $n+1 = \boxed{\boxed{t+1|0}}$ (unless $t = 9$), so that the divisor would be $t+1$. But n equals $10(t+1) - 1$, which cannot be a multiple of $t+1$. Thus t must also be 9, yielding $n = \boxed{99}$, making the divisor equal to 001, or 1.

3. Ten slips of paper, numbered 1 through 10, are placed in a hat. Three numbers are drawn out, one after another. What is the probability that the three numbers are drawn in increasing order? **(2 points)**

Solution.

Let the numbers chosen be A, B, and C. There are six orders in which the slips can be chosen: ABC, ACB, BAC, BCA, CAB, CBA. Of these six, only one is in the increasing order we desire. Hence, the probability is $\boxed{\frac{1}{6}}$.

4. The three marked angles are right angles. If $\angle a = 20°$, then what is $\angle b$? **(2 points)**



Solution.

Note that $\angle b$ is complementary to an angle which is complementary to $\angle a$; hence, $\angle b = \angle a$, so that $\angle b = \boxed{20°}$.

5. Vicky asks Charlene to identify all non-congruent triangles $\triangle ABC$ given:

(a) the value of $\angle A$

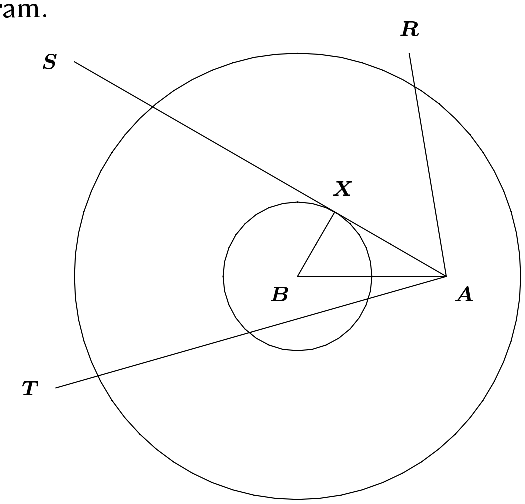
(b) $AB = 10$, and

(c) length BC equals either 5 or 15.

Charlene responds that there are only two triangles meeting the given conditions. What is the value of $\angle A$? **(2 points)**

Solution.

Consider $\overline{AB}$ to be a fixed segment of length 10. Since BC is either 5 or 15, C must lie on one of the two circles with center B and radii 5 and 15, as in the diagram.



There are many possible values for $\angle A$; the possible positions for point C are the points where $\angle A$ intersects the two circles. For some angles, as with $\angle A = \angle BAR$ in the diagram, there is only one possible point C; for others, as with $\angle A = \angle BAT$, there are three. The only value which gives exactly two points C is that which makes the angle tangent to the inner circle, as with $\angle A = \angle BAS$. Call the point of tangency X. Since $AB = 10$, $BX = 5$, and $\angle BXA$ is a right angle, we conclude that triangle ABX is a $30°$-$60°$-$90°$ triangle. Hence, $\angle A = \boxed{30°}$.

6. Five pirates find a cache of five gold coins. They decide that the shortest pirate will become bursar and distribute the coins — if half or more of the pirates (including the bursar) agree to the distribution, it will be accepted; otherwise, the bursar will walk the plank and the next shortest pirate will become bursar. This process will continue until a distribution of coins is agreed upon. If each pirate always acts so as to stay aboard if possible and maximize his wealth, and would rather see another pirate walk the plank than not (all else being equal), then how many coins will the shortest pirate keep for himself? **(3 points)**

Solution.

Call the pirates $p_1, p_2, \ldots, p_5$, with p_1 the shortest and p_5 the tallest. Consider what would happen if only p_4 and p_5 remained. Whatever division strategy p_4 suggested would hold, since p_4's vote alone would constitute half the total vote. Thus, p_4 would simply allot himself all the gold. Next consider the situation where three pirates remained. Whatever distribution p_3 chose, p_5 would have to agree, as long as he got one or more coins. His only alternative would be to go to the two-pirate situation, in which p_5 gets nothing at all. Hence, p_3 would simply take 4 coins for himself and allot 1 coin to p_5, getting a majority vote from himself and p_5. Similarly, with four pirates remaining, p_2, the bursar, would take 4 coins for himself and allot 1 coin to p_4. Again, since p_4 would have otherwise gotten nothing, he would have to support the plan. With five pirates, the bursar, p_1, would allot himself 3 coins and give one coin each to p_3 and p_5. Since each of p_3 and p_5 gets more than they would get by vetoing the plan, they must support it. The shortest pirate gets $\boxed{3}$ coins.

7. The twelve positive integers $a_1 \leq a_2 \leq \cdots \leq a_{12}$ have the property that no three of them can be the side lengths of a non-degenerate triangle. Find the smallest possible value of $\frac{a_{12}}{a_1}$. **(3 points)**

Solution.

If a, b and c can be sides of a non-degenerate triangle with $a \leq b \leq c$, we always must have $c < a + b$. Hence, if no three of our integers can form a non-degenerate triangle, we must have $a_i \geq a_j + a_k$ for any three with $i > j$, k. Since the numbers are increasing, it suffices to show that

$$a_1 + a_2 \ \leq \ a_3 \,, \quad a_2 + a_3 \ \leq \ a_4 \,, \quad a_3 + a_4 \ \leq \ a_5$$

and so on. Substituting the first inequality into the second, we have

$$a_1 + 2a_2 \ \leq \ a_4 \,.$$

Substituting this and the second inequality into the third, we get

$$2a_1 + 3a_2 \ \leq \ a_5 \,.$$

Substituting into the fifth inequality:

$$3a_1 + 5a_2 \ \leq \ a_6 \,.$$

We continue in this manner. At each step, we simply increase the coefficients of a_1 and a_2; they must increase as Fibonacci numbers, since at each step we get the new coefficient by adding the previous two. Hence, the final inequality will read

$$55a_1 + 89a_2 \ \leq \ a_{12} \,.$$

To obtain the smallest possible ratio we choose $a_1 = a_2$. This simply yields $144a_1 \ \leq \ a_{12}$, or $a_{12}/a_1 \ \geq \ 144$. Since this limit can be attained, with $a_1 = a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, $\ldots$, $a_{12} = 144$, we find that $\boxed{144}$ is the least value.

The Mandelbrot Competition
Division B Round Two Team Test
December 1997

FACTS: A polynomial $p(x)$ of degree n or less is determined by its value at $n+1$ x–coordinates. For $n=1$ this is a familiar statement; a line (degree one polynomial) is determined by two points. Moreover, the value of $p(x)$ at any other x–value can be computed in a particularly nice way using Lagrange interpolation, as outlined in the essay *An Interpretation of Interpolation*.

We will also need a result from linear algebra which states that a system of n "different" linear equations in n variables has exactly one solution. For example, there is only one choice for x, y, and z which satisfies the equations $x+y+z=1$, $x+2y+3z=4$, and $x+4y+9z=16$.

SETUP: Let $p(x)$ be a degree three polynomial for which we know the values of $p(1)$, $p(2)$, $p(4)$, and $p(8)$. By the facts section there is exactly one such polynomial. According to Lagrange interpolation the number $p(16)$ can be deduced; it equals

$$p(16) \;=\; A_0p(1) + A_1p(2) + A_2p(4) + A_3p(8)\,,$$

for some constants A_0 through A_3. The goal of this team test will be to compute the A_i and use them to find information about $p(16)$ *without ever finding an explicit formula for* $p(x)$.

Problems:

Part i: (4 points) We claim that the A_i can be found by subtracting

$$x^4 - (x-1)(x-2)(x-4)(x-8) \;=\; A_3x^3 + A_2x^2 + A_1x + A_0\,. \tag{1}$$

Implement this claim to compute A_0 through A_3.

Solution.

Apparently we have been handed a magic formula which generates the constants A_i needed for computing $p(16)$. It is referred to frequently in this solution, so we reproduce it here:

$$x^4 - (x-1)(x-2)(x-4)(x-8) = A_3x^3 + A_2x^2 + A_1x + A_0\,. \tag{2}$$

Rather than marvel at this stroke of good fortune so early on, we set about computing the values A_0 through A_3. Carefully multiplying out the left-hand side yields

$$\begin{aligned}
&x^4 - (x-1)(x-2)(x-4)(x-8)\\
&\quad= x^4 - (x^2-3x+2)(x^2-12x+32)\\
&\quad= x^4 - (x^4-15x^3+70x^2-120x+64)\\
&\quad= 15x^3 - 70x^2 + 120x - 64\\
&\quad= A_3x^3 + A_2x^2 + A_1x + A_0\,.
\end{aligned}$$

Therefore we should use $A_3 = 15$, $A_2 = -70$, $A_1 = 120$, and $A_0 = -64$.

Part ii: (4 points) To show that these A_i are in fact the correct numbers we must show that they correctly predict $p(16)$ for four "different" polynomials. We begin with the case $p(x) = x$. Show that the value of $p(16)$ agrees with the prediction $A_0p(1) + A_1p(2) + A_2p(4) + A_3p(8)$. (HINT: try $x = 2$ in (1).)

Solution.

According to Lagrange Interpolation, if $p(x)$ is a polynomial of degree three or less, then we should be able to predict $p(16)$ based on the values of $p(x)$ at $x = 1$, 2, 4 and 8. We will now show that the constants A_i just computed do the job by showing that $p(16)$ always equals the sum $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$. According to the facts section we need only verify that the A_i work in four different cases to know that they will always work. In checking these four cases we employ a somewhat clever method that never actually uses the numbers calculated in part i, just the equation (2) that produced them.

First suppose that $p(x) = x$. Then clearly $p(1) = 1$, $p(2) = 2$, $p(4) = 4$, and $p(8) = 8$. Let us check whether or not $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$, which is the same as $8A_3 + 4A_2 + 2A_1 + A_0$, correctly predicts $p(16)$. Substituting $x = 2$ into equation (2) gives us

$$2^4 - (2-1)(2-2)(2-4)(2-8) \;=\; 8A_3 + 4A_2 + 2A_1 + A_0\,.$$

The left-hand side equals 16 since the $(2-2)$ factor causes the second term to vanish. However, the right hand side is our prediction for $p(16)$. Sure enough, we get $p(16) = 16$, just as we should for the function $p(x) = x$.

Part iii: (5 points) Continuing the previous part, show that the A_i correctly predict $p(16)$ for the three other polynomials $p(x) = 1$, $p(x) = x^2$ and $p(x) = x^3$.

Solution.

In all cases the prediction for $p(16)$ is $A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1)$. Continuing our work from part ii we try $p(x) = 1$, so our prediction becomes $A_3 + A_2 + A_1 + A_0$. Using $x = 1$ in (2) yields

$$1^4 - (1-1)(1-2)(1-4)(1-8) \;=\; A_3 + A_2 + A_1 + A_0\,.$$

The left-hand side reduces to 1, so that the prediction is $p(16) = 1$, which again is correct. When $p(x) = x^2$ the prediction for $p(16)$ becomes $64A_3 + 16A_2 + 4A_1 + A_0$. This can be found quickly by substituting $x = 4$ into equation (2):

$$4^4 - (4-1)(4-2)(4-4)(4-8) \;=\; 64A_3 + 16A_2 + 4A_1 + A_0\,.$$

The now familiar cancellation occurs on the left hand side, leaving us with a prediction of 4^4 for $p(16)$. Since $p(x) = x^2$ we expect to have $p(16) = 16^2$, and indeed $16^2 = (4^2)^2 = 4^4$. The case of $p(x) = x^3$ works in exactly the same manner, substituting $x = 8$ into equation (2), so we encourage the reader to try it as practice. (Naturally teams were expected to show the details for this case as well in their solutions!)

Notice that we were able to do all of our checking without ever using the numerical values of A_0 through A_3. The other more obvious method is to plug in the values for the A_i and do the arithmetic. However, the slick technique can be generalized, while the more routine method cannot.

Part iv: (4 points) Suppose that $p(x)$ is a third degree polynomial with $p(1) = 0$, $p(2) = 1$, and $p(4) = 3$. What value should $p(8)$ have to guarantee that $p(x)$ has a root at $x = 16$?

Solution.

We have now verified that $A_3 = 15$, $A_2 = -70$, $A_1 = 120$, and $A_0 = -64$ are the correct values needed to interpolate $p(16)$. We are also told in this problem that $p(1) = 0$, $p(2) = 1$, and $p(4) = 3$. Furthermore, we want $p(16) = 0$ so that $p(x)$ has a root at $x = 16$. Substituting all of these values into our interpolation formula produces

$$\begin{aligned} p(16) &= A_3p(8) + A_2p(4) + A_1p(2) + A_0p(1) \\ \Longrightarrow \quad 0 &= 15p(8) - 70 \cdot 3 + 120 \cdot 1 - 64 \cdot 0 \\ \Longrightarrow \quad 90 &= 15p(8) \\ \Longrightarrow \quad 6 &= p(8)\,. \end{aligned}$$

Thus we need $p(8) = 6$ to ensure that $p(16) = 0$ so that there is a root at $x = 16$.

Part v: (4 points) Let $p(x)$ be a degree three polynomial with $p(1) = 1$, $p(2) = 3$, $p(4) = 9$, and $p(8) = 27$. Calculate $p(16)$. How close does it come to the natural guess of 81?

Solution.

There are two ways to do this problem—a long way and a short way. The long way involves substituting all of the values for A_0 through A_3 and $p(1)$ through $p(8)$ into the interpolation formula shown above and cranking out the answer. The short way, hinted at by our above work, involves plugging $x = 3$ into equation (2). We opt for the short way, obtaining

$$3^4 - (3-1)(3-2)(3-4)(3-8) = 27A_3 + 9A_2 + 3A_1 + A_0\,.$$

The left-hand side reduces to $81 - (2)(1)(-1)(-5) = 71$, while the expression on the right is exactly the interpolation formula for $p(16)$. Therefore we have $p(16) = 71$, ten less than the intuitive guess of 81.