PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (*) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard 8½" × 11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than 1 September 2002. They may also be sent by email to crux-editors@cms.math.ca. (It would be appreciated if email proposals and solutions were written in TEX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.

Starting with this issue, we will be giving each problem twice, once in each of the official languages of Canada, English and French. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

2701★. Proposed by Walther Janous, Ursulinegymnasium, Innsbruck, Austria.

Do there exist infinitely many triplets \((n, n+1, n+2)\) of adjacent natural numbers such that all of them are sums of two positive perfect squares?

(Examples are \((232, 233, 234)\), \((520, 521, 522)\) and \((808, 809, 810)\).)

Compare the 2000 Putnam problem A2 [2001 : 3]

Existe-t-il une infinité de triplets \((n, n+1, n+2)\) de nombres naturels consécutifs qui soient tous la somme de deux carrés parfaits non nuls?

(Exemples : \((232, 233, 234)\), \((520, 521, 522)\) et \((808, 809, 810)\).)

Voir le Putnam 2000, problème A2 [2001 : 3]
2702. Proposed by Walther Janous, Ursulengymnasium, Innsbruck, Austria.

Let $\lambda$ be an arbitrary real number. Show that

$$
\left( \frac{s}{r} \right)^{2\lambda} s^2 \geq 3^{3\lambda+1} (s^2 - 8Rr - 2r^2)
$$

where $R$, $r$ and $s$ are the circumradius, the inradius and the semi-perimeter of a triangle, respectively.

Determine the cases of equality.

2703. Proposed by Mihály Bencze, Brasov, Romania.

Suppose that $a$, $b$, $c$, $d$, $u$, $v \in \mathbb{R}$ and $a + c \neq 0$. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(ax + b) + f(cx + d) = ux + v$.

2704. Proposed by Mihály Bencze, Brasov, Romania.

Prove that

$$
R - 2r \geq \frac{1}{12} \left( \sum_{\text{cyclic}} \sqrt{2(b^2 + c^2) - a^2} - s^2 + r^2 + bR - \frac{bRr}{R} \right) \geq 0,
$$

where $a$, $b$ and $c$ are the sides of a triangle, and $R$, $r$ and $s$ are the circumradius, the inradius and the semi-perimeter of a triangle, respectively.

2705. Proposed by Angel Dorito, Geld, Ontario.

The interior of a rectangular container is 1 metre wide and 2 metres long, and is filled with water to a depth of $1/2$ metre. A cube of gold is placed flat in the tub, and the water rises to exactly the top of the cube without overflowing.

Find the length of the side of the cube.
L'intérieur d'un bassin rectangulaire mesure 1m de largeur et 2m de longueur; il est rempli d'eau jusqu'à une hauteur d'un demi-mètre. Un cube en or est posé au fond du bassin et le niveau d'eau monte jusqu'à coïncider exactement avec la hauteur du cube.

Trouver la longueur de l'arête du cube.

2706. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Suppose that $\Gamma_1$ and $\Gamma_2$ are two circles having at least one point $S$ in common. Take an arbitrary line $\ell$ through $S$. This line intersects $\Gamma_k$ again at $P_k$ (if $\ell$ is tangent to $\Gamma_k$, then $P_k = S$).

Let $\lambda$ be a (fixed) real number, and let $R_\lambda = \lambda P_1 + (1 - \lambda) P_2$.

Determine the locus of $R_\lambda$ as $\ell$ varies over all possible lines through $S$.

2707. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $ABC$ be a triangle and $P$ a point in its plane. The feet of the perpendiculars from $P$ to the lines $BC, CA$ and $AB$ are $D, E$ and $F$ respectively.

Prove that

$$\frac{AB^2 + BC^2 + CA^2}{4} \leq AF^2 + BD^2 + CE^2,$$

and determine the cases of equality.

2708. Proposed by Toshio Seimiya, Kawasaki, Japan

Suppose that

1. $O$ is the intersection of diagonals $AC$ and $BD$ of quadrilateral $ABCD$,
2. \( OA < OC \) and \( OD < OB \).
3. \( M \) and \( N \) are the mid-points of \( AC \) and \( BD \), respectively,
4. \( MN \) meets \( AB \) and \( CD \) at \( E \) and \( F \), respectively, and
5. \( P \) is the intersection of \( BF \) and \( CE \).

Prove that \( OP \) bisects the line segment \( EF \).

On suppose que
1. \( O \) est l'intersection des diagonales \( AC \) et \( BD \) d'un quadrilatère \( ABCD \),
2. \( OA < OC \) et \( OD < OB \),
3. \( M \) et \( N \) sont respectivement les points milieu de \( AC \) et \( BD \),
4. \( MN \) coupe \( AB \) et \( CD \) en \( E \) et \( F \), respectivement, et
5. \( P \) est l'intersection de \( BF \) avec \( CE \).

Montrer que \( OP \) coupe le segment \( EF \) en son milieu.

2709. Proposed by Toshio Seimiya, Kawasaki, Japan

Suppose that
1. \( P \) is an interior point of \( \triangle ABC \),
2. \( AP \), \( BP \) and \( CP \) meet \( BC \), \( CA \) and \( AB \) at \( D \), \( E \) and \( F \), respectively,
3. \( A' \) is a point on \( AD \) produced beyond \( D \) such that \( DA' : AD = \kappa : 1 \),
   where \( \kappa \) is a fixed positive number,
4. \( B' \) is a point on \( BE \) produced beyond \( E \) such that \( EB' : BE = \kappa : 1 \),
   and
5. \( C' \) is a point on \( CF \) produced beyond \( F \) such that \( FC' : CF = \kappa : 1 \).

Prove that \( [A'B'C'] \leq \frac{(3\kappa+1)^2}{4} [ABC] \), where \( [PQR] \) denotes the area of \( \triangle PQR \).

On suppose que
1. \( P \) est un point intérieur du triangle \( ABC \),
2. \( AP \), \( BP \) et \( CP \) coupent \( BC \), \( CA \) et \( AB \) en \( D \), \( E \) et \( F \), respectivement,
3. \( A' \) est un point sur \( AD \) situé au-delà de \( D \) de sorte que
   \( DA' : AD = \kappa : 1 \), où \( \kappa \) est un nombre positif fixe,
4. \( B' \) est un point sur \( BE \) situé au-delà de \( E \) de sorte que
   \( EB' : BE = \kappa : 1 \), et
5. \( C' \) est un point sur \( CF \) situé au-delà de \( F \) de sorte que
   \( FC' : CF = \kappa : 1 \).

Montrer que \( [A'B'C'] \leq \frac{(3\kappa+1)^2}{4} [ABC] \), où \( [PQR] \) désigne l'aire du \( \triangle PQR \).
2710. Proposed by Jaroslav Švrček, Palacký University, Olomouc, Czech Republic.

Determine the point $P$ on the semicircle $\Gamma$, constructed externally over the side $AB$ of the square $ABCD$, such that $AP^2 + CP^2$ is maximal.

On le demi-cercle $\Gamma$ construit sur le côté $AB$, à l'extérieur du carré $ABCD$, trouver le point $P$ tel que $AP^2 + CP^2$ soit maximal.


Two circles, centres $O_1$ and $O_2$, of radii $R_1$ and $R_2$ ($R_1 > R_2$), respectively, are externally tangent at $P$. A common tangent to the two circles, not through $P$, meets $O_1O_2$ produced at $Q$, the circle with centre $O_1$ at $A_1$ and the circle with centre $O_2$ at $A_2$.

Prove or disprove that there exist simultaneously integer triangles $QO_1A_1$ and $QO_2A_2$.

Deux cercles, de centre $O_1$ et $O_2$, de rayon respectif $R_1$ et $R_2$ ($R_1 > R_2$), sont extérieurement tangents en $P$. Une tangente commune aux deux cercles et ne passant pas par $P$ coupe la droite $O_1O_2$ en $Q$ et rencontre le cercle de centre $O_1$ en $A_1$ et celui de centre $O_2$ en $A_2$.

Montrer si oui on non il existe simultanément deux triangles $QO_1A_1$ et $QO_2A_2$ dont les côtés sont des entiers.

2712. Proposed by Anreas P. Hatzopolakis, Athens, Greece; and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

Given $\triangle ABC$, let $Y$ and $Z$ be the feet of the altitudes from $B$ and $C$. Suppose that the bisectors of $\angle BYC$ and $\angle BZC$ meet at $X$. Prove that $\triangle BXC$ is isosceles.

On donne un triangle $ABC$ et soit $Y$ et $Z$ les pieds des perpendiculaires abaissées des sommets $B$ et $C$. Soit $X$ le point d'intersection des bissectrices de $\angle BYC$ et $\angle BZC$. Montrer que le triangle $BXC$ est isocèle.

Professor Jordi Dou

We always like to recognise milestones. We have just discovered that we missed Professor Jordi Dou's ninetieth birthday last year. It will be nice to have some problems dedicated to Jordi this year. Please send proposals post haste to the Editor-in-Chief.