MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5**, or to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario. N2L 3G1**. The electronic address is

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Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1**, or emailed to

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Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 February 2002*. Look for prizes for solutions in the new year.

M8. *Proposed by the Mayhem staff.*

Find all right-angled triangles with integer sides if one of the sides is 2001 units long.

M9. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Find integers a, b, and c (not all equal) with $a + b + c = 2001$, such that a, b, and c form an arithmetic sequence (in that order) and $a + b$, $b + c$, and $c + a$ form a geometric sequence (in that order).

M10. *Proposed by Nicolae Gustia, North York, Ontario.*

(a) Factor fully $2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$.

(b) Find the geometric interpretation of the above expression if a, b, and c are sides of a non-degenerate triangle.

M11. *Proposed by the Mayhem staff.*

Two sequences a_1, a_2, ..., a_{2001} and b_1, b_2, ..., b_{2001} are formed by the following rules:

- $a_1 = 5$ and $a_2 = 3$,
- $b_1 = 9$ and $b_2 = 7$,
- $\dfrac{a_n}{b_n} = \dfrac{a_{n-1} + a_{n-2}}{b_{n-1} + b_{n-2}}$ for $n > 2$ such that each $\dfrac{a_n}{b_n}$ is in lowest terms.

What is the smallest fraction of the form $\dfrac{a_n}{b_n}$?

M12. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

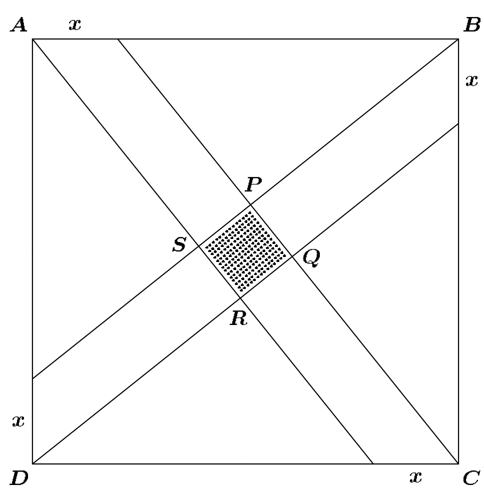
Determine all ordered pairs (x, y) with $\gcd(x, y) = 1$, and $x < y$ such that $2000\left(\frac{x}{y} + \frac{y}{x}\right)$ is an odd integer.

M13. *Proposed by the Mayhem staff.*

Given $n = 9 + 99 + 999 + \cdots + \overbrace{999\cdots99}^{2001\ 9\text{'s}}$, how many of n's digits are 1's?

M14. *Proposed by the Mayhem staff.*

Starting with a square $ABCD$ with unit area we construct four points a distance x from each vertex (located on the clockwise side). Each of these points is connected to the vertex opposite to it so that a small quadrilateral $PQRS$ is formed in the middle of the square. Find the value of x that makes the area of $PQRS$ equal to $\frac{1}{2001}$.



Polya's Paragon

Shawn Godin

Number theory is a beautiful area of mathematics that, for the most part, deals with properties of the positive integers 1, 2, 3, $\ldots$. As a result, many of the toughest problems in the field can be understood by high school students.

If you study number theory, you will find that prime numbers play a central role. Sometimes you will find that breaking things down into primes can shed some light on the original problem. This is helpful because of the following theorem

The Fundamental Theorem of Arithmetic

Every integer $n > 1$ can be expressed as a product of primes in only one way (apart from rearranging the factors).

Thus, $6 = 2{\cdot}3$, $17 = 17$ and $1440 = 2{\cdot}2{\cdot}2{\cdot}3{\cdot}3{\cdot}5{\cdot}5{\cdot}5{\cdot}5 = 2^3{\cdot}3^2{\cdot}5^4$. And there will never be a case where there is more than one such decomposition.

Now let us take a look at this unique factorization in action in a couple of problems.

Example 1: How many five-digit positive integers have the property that the product of their digits is 3000?

Solution: If we break it up into primes we get $3000 = 2^3 \cdot 3 \cdot 5^3$. Now any number that divides evenly into 3000 must be of the form $2^x \cdot 3^y \cdot 5^z$, where $0 \le x \le 3$, $0 \le y \le 1$ and $0 \le z \le 3$. Since we are looking for a five-digit number, we are looking for numbers d_i with $0 < d_i < 10$ for $i = 1$, 2, $\ldots$, 5 such that $d_1 \times d_2 \times \cdots \times d_5 = 3000$.

If we look at the condition that $d_i < 10$ it tells us that if we use a factor of 5 in one of the d_i's, there can be no other factors. Thus three of the digits must be 5's.

The other two digits must be made up of 2's and 3's in such a way as their product is $2^3 \cdot 3 = 24$. When it is in this form we see that the term that has a factor of 3 can only be 3 or $3 \cdot 2$. Thus the other two digits can be either 3 and 8 or 6 and 4.

Therefore, to answer the original question, the digits can be 3, 5, 5, 5, 8 (we can create $\frac{5!}{3!} = 20$ five-digit numbers from these) or 4, 5, 5, 5, 6 (similarly with 20 numbers); thus, there are 40 five-digit numbers with the product of their digits equal to 3000.

Example 2: Find the sum of all the divisors of 540000.

Solution: This is based on a classic result from number theory. Notice that $540000 = 2^5 \cdot 3^3 \cdot 5^4$. As in example 1, we note that any divisor of 540000 must be of the form $2^x \cdot 3^y \cdot 5^z$ with $0 \leq x \leq 5$, $0 \leq y \leq 3$ and $0 \leq z \leq 4$. The result we are after is the sum of all of such numbers. If we take a second to examine the product

$$(1+2+2^2+2^3+2^4+2^5)(1+3+3^2+3^3)(1+5+5^2+5^3+5^4)\,,$$

we notice that every term in that expansion is in the required form, and that every number that we are after is there. Therefore, the result we want is then just

$$63 \cdot 40 \cdot 781 \;=\; 1968120\,.$$

The result above is just a tad easier than finding and adding up all 120 (how did I get that?) divisors of 540000, is it not?

We can summarize this result by defining $\sigma(n)$ to be the sum of all the divisors of n. Then, if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of n, we must have

$$\begin{aligned}\sigma(n) \;&=\; (1+p_1+p_1^2+\cdots+p_1^{\alpha_1}) \times (1+p_2+p_2^2+\cdots+p_2^{\alpha_2}) \\ &\qquad \times \cdots \times (1+p_k+p_k^2+\cdots+p_k^{\alpha_k})\,.\end{aligned}$$

A closer examination of each of the brackets in the product reveals that they are all geometric series. Therefore, we can write

$$\sigma(n) \;=\; \left(\frac{1-p_1^{\alpha_1+1}}{1-p_1}\right)\left(\frac{1-p_2^{\alpha_2+1}}{1-p_2}\right)\cdots\left(\frac{1-p_k^{\alpha_k+1}}{1-p_k}\right)\,.$$

Whew! I think we had better stop there for this issue. Besides at least one problem from elsewhere in this issue, here are a couple of others to try.

1. If $n \geq 1$, the notation $n!$ (read n *factorial*) represents the product of all integers from 1 to n. So $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$. (For reasons that we will not go into here, it is useful to define $0! = 1$. Maybe some other time). Find the value of n if $n! = 2^{31} \cdot 3^{15} \cdot 5^7 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$.

2. Find the largest power of 2 that evenly divides $2001!$

3. Find the positive integer $n < 10^{10}$ that has the most divisors.

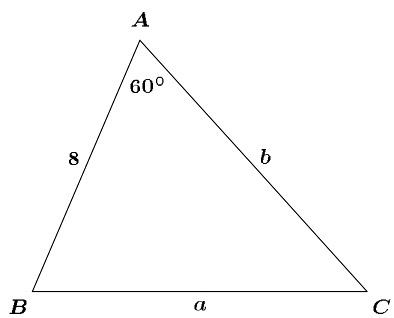
Problem of the Month

Jimmy Chui, student, University of Toronto

Problem.

Problem. In triangle ABC, $AB = 8$, and $\angle CAB = 60^\circ$. Sides BC and AC have integer lengths a and b, respectively. Find all possible values of a and b.

(1997 Euclid, Problem 9)



Comments. Because we are dealing with integer solutions to a problem, there is an extremely good chance that we must somehow form an algebraic equation from this geometric question. We want to find the relation between the three sides of a triangle, given an angle. The Cosine Law gives us this relationship!

Because we are reducing to an algebraic equation, we must be certain not to get answers that make no sense; that is, we must make sure that none of the implicit assumptions are lost in the translation from geometry to algebra. In this case, we must make sure that a and b are positive, and that a, b, and 8 form a legitimate triangle. They will form a triangle if and only if they satisfy the triangle inequality: the sum of any two sides must exceed the third.

Solution. The Cosine Law applied to $\angle CAB$ gives us the equation

$$\begin{aligned} a^2 &= b^2 + 8^2 - 2 \cdot b \cdot 8 \cdot \cos 60^\circ \\ &= b^2 - 8b + 64\,. \end{aligned}$$

Let us factor this equation with all the variables on one side. This is a standard technique for solving equations for integer solutions.

$$\begin{aligned} a^2-(b^2-8b) &= 64 \\ a^2-(b^2-8b+16) &= 64-16 \\ a^2-(b-4)^2 &= 48 \\ (a+b-4)(a-b+4) &= 48\,. \end{aligned}$$

Now, since a and b are both integers, both factors on the left side must be integers as well. Hence, we can equate them to pairs of factors of 48. For example, $48 = 6 \times 8$ and so we solve for:

$$\begin{cases} a+b-4 = 6 \\ a-b+4 = 8 \end{cases}$$

and

$$\begin{cases} a+b-4 = 8 \\ a-b+4 = 6 \end{cases}$$

We would have to do this for every pair of integers, positive and negative, that multiply to 48, and disregard any answers that do not conform to the triangle inequality. But instead of trying all the pairs, why not make some more calculations to get rid of possible pairs?

First, let us assume that we let the two factors $(a+b-4)$ and $(a-b+4)$ be both negative. Then $(a+b-4)+(a-b+4) = 2a$ which would be negative. But a is a measurement, and must be positive! $2a$ must be positive as well. $2a$ cannot be both negative and positive; hence, we can throw out the pairs of negative numbers that multiply to 48.

Second, notice that $(a+b-4)+(a-b+4) = 2a$. The sum of the two factors is even. What does this tell us? The factors must be either both odd or both even! (We say that the factors have the same parity.) Therefore, the couple 1 and 48 can be thrown out, as well as 3 and 16.

This leaves us with the couples 2 and 48, 4 and 12, 6 and 8. Now that we have a lesser number of cases to test, why not just solve for a and b for each pair?

We will get 6 pairs of (a, b) answers from these couples.

$$\begin{cases} a+b-4 = 2 \\ a-b+4 = 24 \end{cases}$$

gives us $a = 13$ and $b = -7$.

$$\begin{cases} a+b-4 = 24 \\ a-b+4 = 2 \end{cases}$$

gives us $a = 13$ and $b = 15$.

$$\begin{cases} a+b-4 = 4 \\ a-b+4 = 12 \end{cases}$$

gives us $a = 8$ and $b = 0$.

$$\begin{cases} a+b-4 &= 12 \\ a-b+4 &= 4 \end{cases}$$

gives us $a = 8$ and $b = 8$.

$$\begin{cases} a+b-4 &= 6 \\ a-b+4 &= 8 \end{cases}$$

gives us $a = 7$ and $b = 3$.

$$\begin{cases} a+b-4 &= 8 \\ a-b+4 &= 6 \end{cases}$$

gives us $a = 7$ and $b = 5$.

We can throw out $(a, b) = (13, -7)$ since b must be positive. We can also throw out $(a, b) = (8, 0)$ since b must be positive. (We should mention that this latter solution is a degenerate solution.)

Now we are left with the other four pairs of solutions. Consider the pair $(a, b) = (13, 15)$. Can we have a triangle with sides 13, 15, and 8? From the triangle inequality, $13 + 15 - 8 = 20 > 0$, $13 + 8 - 15 = 6 > 0$, $15 + 8 - 13 = 10 > 0$. Therefore, the sum of any two sides exceeds the third.

We could do this to the other three pairs of solutions, but we will find that they also satisfy the triangle inequality.

The final check we might want to do would be to verify that $\angle CAB = 60°$. However, we arrived at all these solutions from the Cosine Law, and so it is obvious that these solutions will satisfy the Cosine Law. Hence it is not necessary to do this check.

We conclude that the only values for a and b are $(a, b) = (7, 3)$, $(7, 5)$, $(8, 8)$, and $(13, 15)$.

High School Solutions

H273. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, and c be complex numbers such that $a + b + c = 0$. Prove that

$$\begin{vmatrix} 2ab - c^2 & b^2 & a^2 \\ b^2 & 2bc - a^2 & c^2 \\ a^2 & c^2 & 2ac - b^2 \end{vmatrix} = 0\,.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let D denote the given determinant. Adding the second and third column to the first column, we have:

$$D = \begin{vmatrix} (a+b)^2 - c^2 & b^2 & a^2 \\ (b+c)^2 - a^2 & 2bc - a^2 & c^2 \\ (c+a)^2 - b^2 & c^2 & 2ca - b^2 \end{vmatrix}$$

Since $(a+b)^2 - c^2 = (a+b+c)(a+b-c) = 0$, $(b+c)^2 - a^2 = (b+c+a)(b+c-a) = 0$ and $(c+a)^2 - b^2 = (c+a+b)(c+a-b) = 0$, it follows that $D = 0$.

H274. Find a simplified expression for

$$\sum_{i=1}^{\infty} \frac{i}{k^i}$$

in terms of a real number $k > 1$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let S_k denote the sum of the given series, which is clearly convergent by the Ratio Test. Then,

$$\begin{aligned} kS_k &= \sum_{i=1}^{\infty} \frac{i}{k^{i-1}} = \sum_{i=0}^{\infty} \frac{i+1}{k^i} \\ &= 1 + S_k + \sum_{i=1}^{\infty} \frac{1}{k^i} = 1 + S_k + \frac{\frac{1}{k}}{1 - \frac{1}{k}} = 1 + S_k + \frac{1}{k-1}, \end{aligned}$$

from which we get $(k-1)S_k = \frac{k}{k-1}$ and thus, $S_k = \frac{k}{(k-1)^2}$.

H275. How many non-negative integers less than 10^n are there whose digits are in non-increasing order?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The answer is $\binom{n+10}{10} - n$.

Proof. Let $S = \{t \in \mathbb{Z} \mid 0 \le t < 10^n$ such that the digits of t are in non-increasing order $\}$. Note first that any $t \in S$ has k digits for some $k = 1, 2, \ldots, n$. These k digits are all from $\{0, 1, 2, \ldots, 9\}$ with possible repetitions. Conversely, any k digits chosen from $\{0, 1, 2, \ldots, 9\}$ with repetitions permitted, can be arranged in non-increasing order, in only one way, to yield some $t \in S$, except when $k \ge 2$ and all the digits chosen are 0's. From a well known result in combinatorics, the number of such choices, for each fixed k, is $\binom{10+k-1}{k}$ or $\binom{k+9}{9}$. Discarding the cases when all the k digits chosen are 0's and adding 1 to account for the choice of 0 when $k = 1$, we then have

$$\begin{aligned} |S| &= 1+\sum_{k=1}^{n}\left(\binom{k+9}{9}-1\right) = -n+1+\sum_{k=1}^{n}\binom{k+9}{9} \\ &= -n+\sum_{k=0}^{n}\binom{k+9}{9} \end{aligned}$$

Since it is well known that $\binom{a}{a}+\binom{a+1}{a}+\cdots+\binom{b}{a}=\binom{b+1}{a+1}$ for all a, $b \in \mathbb{N}$ with $a \leq b$ (see, for example, p. 217 of Applied Combinatorics by Alan Tucker, 3rd ed.), we finally obtain $|S| = -n + \binom{n+10}{10}$ as claimed.

H276. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

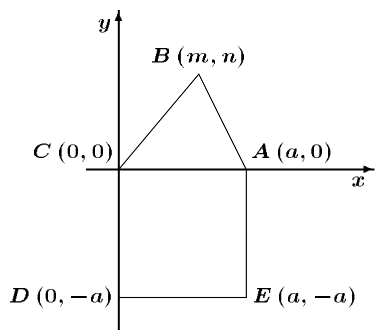
Let $ABCDE$ be a convex pentagon such that $ACDE$ is a square, and

$$\cot\angle BDE + \cot\angle DEB + \cot\angle EBD = 2\,.$$

Show that $\triangle ABC$ is a right triangle.

Solution by the proposer.

Attach our figure to a grid and label the points as in the diagram below.



It is known that

$$\cot(\angle BDE)+\cot(\angle DEB)+\cot(\angle EBD) = \frac{\overline{BD}^2+\overline{DE}^2+\overline{EB}^2}{4[BDE]}\,,$$

where $[BDE]$ is the area of the figure BDE. It follows that

$$\begin{aligned} &\cot(\angle BDE)+\cot(\angle DEB)+\cot(\angle EBD) \\ &= \frac{m^2+(n+a)^2+a^2+(m-a)^2+(n+a)^2}{4\cdot\frac{1}{2}a(a+n)} \\ &= \frac{2m^2-2ma+2n^2+4an+4a^2}{2a(a+n)} = \frac{2m^2-2ma+2n^2}{2a(a+n)}+2\,. \end{aligned}$$

By the hypothesis, we get $0 = \frac{2m^2-2ma+2n^2}{2a(a+n)}$ or $2m^2 - 2ma + 2n^2 = 0$. This implies that $(m^2 + n^2) + \{(m - a)^2 + n^2\} = a^2$, which means that $\overline{BC}^2 + \overline{BA}^2 = \overline{CA}^2$. It follows that $\angle ABC = 90°$.

H277.

(a) Find all right triangles with integer sides with perimeter 60.

(b) Find all right triangles with integer sides with area 600.

Solution by Mihály Bencze, Brasov, Romania.

(a) If we let a, b and c be the sides of the triangle, we have $a, b, c \in \mathbb{Z}^+$ with $a + b + c = 60$ and $a^2 = b^2 + c^2$. Thus we must have

$$\begin{array}{rclcrcl} a & = & x^2 + y^2 & \quad & b & = & 2xy \\ c & = & x^2 - y^2 & & x & > & y \end{array}$$

Combining these gives $x(x + y) = 30$ with solution $x = 5$, $y = 1$, which yields $a = 26$, $b = 10$, $c = 24$.

(b) Using a, b, c, x and y as defined in (a), we get $xy(x^2 - y^2) = 600$.

If $y = 1$ we get $(x - 1)x(x + 1) = 600$, which has no solutions. Similarly,

$$\begin{array}{rcl} y = 2 & \Longrightarrow & (x-2)x(x+2) = 300 \\ y = 3 & \Longrightarrow & (x-3)x(x+3) = 200 \\ y = 4 & \Longrightarrow & (x-4)x(x+4) = 150 \\ y = 5 & \Longrightarrow & (x-5)x(x+5) = 120 \\ y = 6 & \Longrightarrow & (x-6)x(x+6) = 100 \\ y = 8 & \Longrightarrow & (x-8)x(x+8) = 75 \end{array}$$

[Ed. Since $x > y$ we have $xy(x^2 - y^2) = xy(x+y)(x-y) > y \cdot y \cdot y \cdot 1 = y^3$, so that $y^3 < 600$, and $y < 9$.]

Since none of these yield solutions, there are no solutions.

H278. Consider the time as seen on a digital clock in 24–hour mode. (24–hour mode is representing the time relative to 12 midnight. For example, 6:25 am is 06:25, but 6:25 pm is 18:25. Also, 12:45 am counts as 00:45.) Let n be the number we get when we remove the colon from the time T as seen on a digital clock in 24 hour mode. Find all times T such that:

(i) n is a palindrome, [Ed. reads the same backwards as forwards.]

(ii) m, the number of minutes that T is after midnight, is a palindrome, and

(iii) $n = m$.

Solution by the editors.
Since n is a palindrome, T must be of the form:

(a) $AB : BA$ if $0 < A \leq 2, 0 \leq B \leq 5$
(b) $0C : DC$ if $0 < C \leq 9, 0 \leq D \leq 5$
(c) $00 : EE$ if $0 < E \leq 5$
(d) $00 : 0F$ if $0 \leq F \leq 9$

CASE A. $n = 1000A + 100B + 10B + A = 1001A + 110B$ and $m = 60(10A + B) + (10B + A) = 601A + 70B$. If $n = m$, then $400A + 40B = 0$. Since $A > 0$ and $B \geq 0$, there are no solutions for this case.

CASE B. $n = 100C + 10D + C = 101C + 10D$ and $m = 60C + 10D + C = 61C + 10D$. If $n = m$, $40C = 0$, a contradiction, so that there are no solutions for this case.

CASE C. All values of E from 1 to 5 inclusive work.

CASE D. All values of F from 0 to 9 inclusive work.

All times are 00:00, 00:01, 00:02, 00:03, 00:04, 00:05, 00:06, 00:07, 00:08, 00:09, 00:11, 00:22, 00:33, 00:44, 00:55.

H279. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*
Let a and b be integers such that $a \equiv b \pmod 3$. Prove that

$$\tfrac{2}{3}\ (a^2 + ab + b^2)$$

can be expressed as a sum of three non-negative squares.

I. *Solution by the proposer.*
We first note that $\frac{a+2b}{3}$, $\frac{2a+b}{3}$, and $\frac{a-b}{3}$ are integers since $a \equiv b \pmod 3$. By expanding and simplifying, we easily see that

$$\frac{2}{3}(a^2 + ab + b^2) \;=\; \left(\frac{a + 2b}{3}\right)^2 + \left(\frac{2a + b}{3}\right)^2 + \left(\frac{a - b}{3}\right)^2 .$$

II. *Solutions by Mihály Bencze, Brasov, Romania and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Since $a \equiv b \pmod 3$ we have $a = 3k + b$ for some $k \in \mathbb{Z}$. Thus

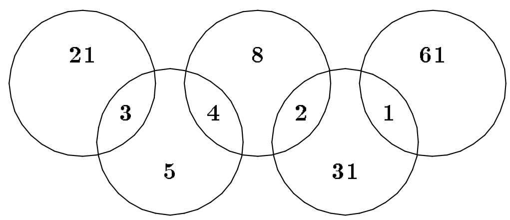
$$\frac{2}{3}(a^2 + ab + b^2) \;=\; 6k^2 + 6kb + 2b^2 \;=\; (2k + b)^2 + k^2 + (k + b)^2 .$$

[Ed. note that the decompositions in I and II are equivalent]

H280. *Proposed by Fotifo Casablanca, Bogotá, Colombia.*
In the spirit of the Olympics: There are 9 regions inside the rings of the Olympics. Put a different positive whole number in each so that the five products of the numbers in each ring form a set of five consecutive integers.

Solution by the proposer.

The solution with the smallest possible set of consecutive numbers is as follows



H281 *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Suppose the monic polynomial $A(z) = \sum_{k=0}^{n} a_k z^k$ can be factored into $(z - z_1)(z - z_2) \cdots (z - z_n)$, where $z_1, z_2, \ldots, z_n$ are positive real numbers. Prove that $a_1 a_{n-1} \geq n^2 a_0$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We add the assumption that n is even (or equivalently, $a_0 > 0$) for otherwise it is easy to see that the conclusion is false.

By comparing the coefficients of appropriate terms, we have

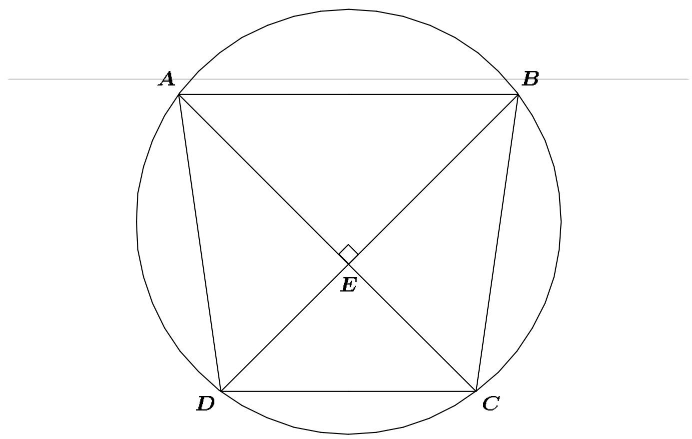
$$\begin{aligned} a_0 &= A(0) = (-1)^n \prod_{k=1}^{n} z_k = \prod_{k=1}^{n} z_k \\ a_1 &= (-1)^{n-1} \sum_{k=1}^{n} \frac{z_1, z_2, \ldots, z_n}{z_k} = -\sum_{k=1}^{n} \frac{a_0}{z_k}, \quad \text{and} \\ a_{n-1} &= -\sum_{k=1}^{n} z^k . \end{aligned}$$

Hence by the Arithmetic-Harmonic-Mean Inequality (or Cauchy–Schwarz Inequality), we have

$$a_1 a_{n-1} = a_0 \left(\sum_{k=1}^{n} \frac{1}{z_k} \right) \left(\sum_{k=1}^{n} z_k \right) \geq n^2 a_0 .$$

H282. Let $ABCD$ be a cyclic quadrilateral such that its diagonals are perpendicular. Let E be the intersection of AC and BD. It is known that $AE + ED = BE + EC$. Show that $ABCD$ is a trapezoid.

Solution by the editors.



We have $AC \perp BD$ and $AE + ED = BE + EC$. Now, since the quadrilateral cyclic, we have

$$\begin{aligned} AE \cdot EC &= BE \cdot ED\,, \\ (BE + EC - ED) \cdot EC &= BE \cdot ED\,, \\ BE \cdot EC + EC^2 - DE \cdot CE - BE \cdot ED &= 0\,, \\ (EC - ED)(EC + BC) &= 0\,. \end{aligned}$$

Thus, $EC = ED$. If $EC = ED$, then $\angle DCE = \angle CDE = 45^\circ$. But $\angle ABE = \angle DCE = 45^\circ = \angle CDE$, so that $AB \parallel CD$.

Similarly, $ED = AE$. Then, $AD \parallel BC$. Thus, $ABCD$ is a trapezoid.