SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2514. [2000 : 114, 2001 : 143] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

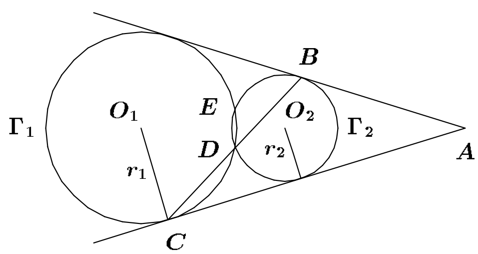
In $\triangle ABC$, the internal bisectors of $\angle ABC$ and $\angle BCA$ meet CA and AB at D and E respectively. Suppose that $AE = BD$ and that $AD = CE$. Characterize $\triangle ABC$.

Correction.

The editor's comment should have read "All of the other solvers, with one exception, also showed that the base angles of the triangle are 72°."

2560*. [2000 : 305] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Lines AB and AC are common tangents to the circles Γ_1 and Γ_2 with distinct radii r_1 and r_2 respectively, as shown.



B is a point of tangency on Γ_2 and C is a point of tangency on Γ_1. The intersection points of the circles, D and E, exist, CDB is a straight line, and $CD = DB$.

Construct such a figure using straightedge and compass.

Combination of solutions by Michel Bataille, Rouen, France and D.J. Smeenk, Zaltbommel, the Netherlands.

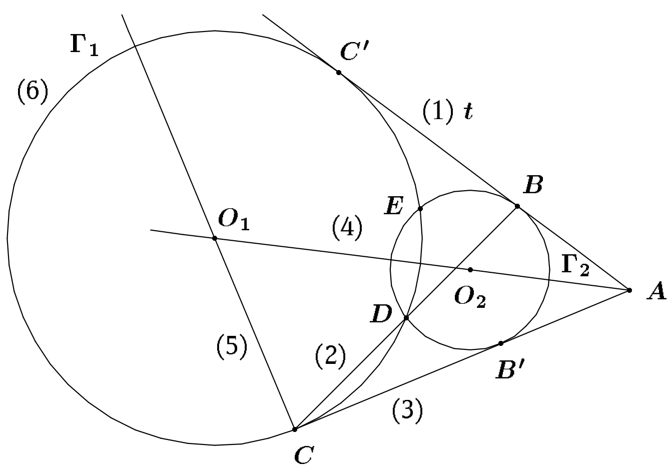
We will construct the configuration given an arbitrary circle Γ_2, its centre O_2, and any two of its points in the role of B and D.

1. Construct the tangent t to Γ_2 at B.
2. Extend the segment BD to a point C beyond D so that D is the midpoint of BC.
3. From C construct either tangent to Γ_2; call the point of tangency B' and let A be the point where it intersects t. (Note that at least one of the tangents must meet t.)

4. Construct the line AO_2.

5. Construct the perpendicular to CA at C, and let O_1 be the point where it meets AO_2. (O_1 exists since CO_1 is parallel to $B'O_2$.)

6. Draw the circle with centre O_1 and radius O_1C.



Claim: The circle in step 6 is the desired circle Γ_1, and the construction is complete.
Proof. This circle is tangent to AC (at C) by construction and to AB (at C', say) by symmetry about the line of centres AO_2O_1. It remains to prove only that Γ_1 intersects Γ_2 in D and in one other point. Assume that Γ_1 intersects the line CB again in D'; we must show that $D' = D$. Since B is exterior to Γ_1, $BD' \cdot BC = BC'^2$; since C is exterior to Γ_2, $CD \cdot CB = CB'^2$. But $CB' = BC'$ (common tangents are equal), and $BD = CD$ (since D is the mid-point of BC); thus the two equations reduce to $BD' = CD = BD$, which implies that $D' = D$ as desired. Finally, D cannot be the unique intersection point of the two circles because that would imply that it lies on the line of centres, and therefore $DB = DB' = DC$; that would force $CB'B$ to be a right angle, which contradicts the fact that $\triangle AB'B$ is isosceles.

Editor's comment. Note that the featured solution proves more than was claimed:

Theorem. Given a pair of circles that have common tangent lines AB and AC with B on one circle and C on the other, suppose BC meets one of the circles again at D; then D lies on the other circle if and only if it is the mid-point of BC.

As a consequence, in the statement of problem 2560 the fact that BCD is a straight line forces $CD = DB$.

Also solved by TOSHIO SEIMIYA, Kawasaki, Japan; Mª JESÚS VILLAR RUBIO, Santander, Spain; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Villar Rubio and Woo interpreted the problem in a different way. They showed how to construct the configuration when given the lengths of the radii r_1 and r_2. In particular, they showed that the distance between the centres satisfies $O_1O_2 = 2\sqrt{r_1r_2}$; furthermore, the construction is possible if and only if $3-\sqrt{8} < \frac{r_1}{r_2} < 3+\sqrt{8}$.

2561. [2000 : 305] *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let M disks from N different colours be placed in a row such that k_i disks are from the i^{th} colour ($i = 1, 2, \ldots, N$) and $k_1 + k_2 + \cdots + k_N = M$.

A move is an exchange of two adjacent disks.

Determine the smallest number of moves needed to rearrange the row such that all disks of the same colour are adjacent to one another.

Editor's remark: There were no solutions submitted to this problem. Indeed, it should have been starred, since the proposer did not submit a solution either. Consequently, the problem remains open.

2562. [2000 : 305] *Proposed by Bernardo Recamán Santos, Colegio Hacienda Los Alcaparros, Bogotá, Colombia.*

(a) Show that for all sufficiently large n, it is possible to find a set of n (not necessarily distinct) positive integers whose sum is the square root of their product.

(b)* Are there infinitely many n for which there is a unique set of n numbers with property (a)?

I. *Solution by Oleg Ivrii, (grade 8) student, Cummer Valley Middle School, North York, Ontario.*

For each $n \geq 3$, let $S = \{n+6, n+6, 9, 1, 1, \ldots, 1\}$ where there are $(n-3)$ 1's, and for each $n \geq 4$, let $T = \{n+2, n+2, 3, 3, 1, 1, \ldots, 1\}$ where there are $(n-4)$ 1's. Then, it is readily verified that both S and T satisfy the given condition. [Ed: Hence the answer to $(b)^\star$ is "no".]

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (slightly adapted by the editor).*

For $n = 1$, clearly the only such set is $\{1\}$. For $n = 2$, no such sets exist since $a + b = \sqrt{ab}$ is impossible for positive numbers by the AM–GM Inequality. For $n = 3$, two such sets are $\{9, 9, 9\}$ and $\{9, 36, 225\}$. For any $n \geq 4$, one such set is $\{n+2, n+2, 3, 3, 1, 1, \ldots, 1\}$ where there are $(n-4)$ 1's.

[Ed: both this set and the first set given in the $n = 3$ case are the same as the ones obtained by Ivrii.]

Furthermore, if n is a composite, $n = ab$, say, where $a \geq 2$ and $b \geq 2$, then another set satisfying the given condition would be

$$\{a+2,\, a+2,\, b+2,\, b+2,\, 1,\, 1,\, \ldots,\, 1\}\,.$$

Finally, if $n \geq 5$ is a prime, then

$$\{\tfrac{n+3}{2},\, \tfrac{n+3}{2},\, 4,\, 2,\, 2,\, 1,\, 1,\, \ldots,\, 1\}\,,$$

where there are $(n-5)$ 1's, would be another such set.

Part (a) *only was also solved by the proposer whose example is*

$$\{2n+4,\, n+2,\, 4,\, 2,\, 1,\, 1,\, \ldots,\, 1\}$$

for each $n \geq 4$.

The corresponding problem regarding a set of n positive integers with the property that their sum equals their product, and the upper bound for this common value was proposed by E.T.H. Wang in 1973 and appeared as E 2447^* (*Bounds for* k*-satisfactory sequences*) *in the* American Mathematical Monthly [80(1973), 953; 82(1975), 78–80]. *Though it is easy to find such sets, the determination of all* n *values for which such a set is unique seems to be a much harder question and, to the best knowledge of this editor, is still an open question. Back then, the editor of that problem announced that "with the aid of computer, it was discovered that up to* 10000*, the only values of* n*'s for which such a set is unique are* $n = 2, 3, 4, 6, 24, 114, 174$ *and* 444*".*

2563. [2000 : 372] *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

You are given that angle x satisfies the equation $a\sin x + b\cos x = c$.

(a) If a, b and c are real numbers, calculate angle x.

(b) Considering a, b and c as line segments, find a straightedge and compass construction for angle x.

(a) *Solution by Michel Bataille, Rouen, France.*

If $a = b = 0$, then there is no solution when $c \neq 0$, while every real number x is a solution when $c = 0$. Suppose now that $a^2 + b^2 \neq 0$. The given equation may then be written as

$$\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x \;=\; \frac{c}{\sqrt{a^2+b^2}}\,,$$

or

$$\cos(x-\alpha) \;=\; \frac{c}{\sqrt{a^2+b^2}}\,,$$

where α is determined by

$$\sin\alpha \;=\; \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos\alpha \;=\; \frac{b}{\sqrt{a^2+b^2}}\,.$$

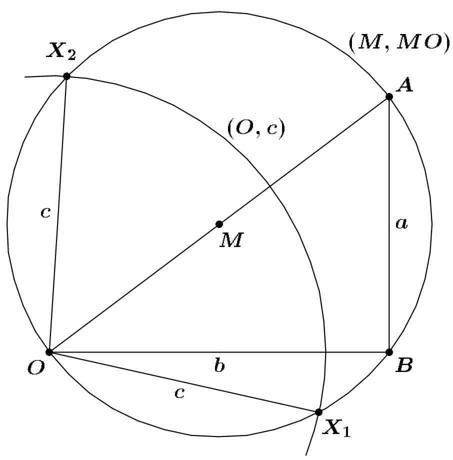
We have the following consequences:

If $c^2 > a^2 + b^2$, the equation has no solution.

If $c^2 \leq a^2 + b^2$ we define β by $\cos\beta = \dfrac{c}{\sqrt{a^2+b^2}}$, and the equation becomes $\cos(x - \alpha) = \cos\beta$, whose solutions are the real numbers $\alpha + \beta + 2k\pi$ and $\alpha - \beta + 2k\pi$ (for all integers k).

(b) *Solution by the Angelo State Problem Group, San Angelo, TX (shortened by the editor).*

We are given the (directed) lengths a, b, c, and will construct the angles α and β [in the notation of solution (a)], under the assumption that $c^2 \leq a^2 + b^2$; then by part (a), $x = \alpha + \beta$ or $x = \alpha - \beta$.



1. Construct right triangle AOB with right angle at B, and legs $OB = b$ and $BA = a$. (Note that $\angle BOA = \alpha$.)

2. Construct circle (O, c) with centre O and radius c.

3. Construct the mid-point M of OA.

4. Construct the circle (M, MO) with centre M and radius MO. Since $c^2 \leq a^2 + b^2$, the two circles will meet at one [when $c^2 = a^2 + b^2$] or two points; call them X_1 and X_2.

Claim: $\angle BOX_1$ and $\angle BOX_2$ are the two solutions for angle x. To see this, note that because the X_i are on the circle with diameter $OA = \sqrt{a^2+b^2}$, the triangles OX_iA are congruent right triangles with leg c and hypotenuse $\sqrt{a^2+b^2}$. Hence $\angle X_1OA = \angle AOX_2 = \beta$ $\left[\text{since their cosine is } \dfrac{c}{\sqrt{a^2+b^2}}\right]$. Combining these angles with α of step 1 we see that $\angle BOX_1 = \alpha - \beta$, and $\angle BOX_2 = \alpha + \beta$, as desired.

Remarks. In the original equation, c can be interpreted as the (scalar) projection of the vector $\langle b, a\rangle$ on the unit vector $\langle \cos x, \sin x\rangle$. Thus the construction reduces to the problem of finding a point X on the circle (O, c) with the property that the line AX is perpendicular to the radius OX; that is, AX is tangent to (O, c) at X. Our construction is the standard way of constructing such a point.

Also solved by MOHAMMED AASSILA, Strasbourg, France (part (a) only); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3 solutions); MENG DAZHE, River Valley High School, Singapore; PAUL DEIERMANN, Cape Girardeau, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria(3 solutions); VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; RENÉ LAUMEN, Deurne, Belgium; HENRY LIU, student, Cambridge, England; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

An alternative approach to part (a) is to replace $\sin x$ *in the given equation by* $\pm\sqrt{1-\cos^2 x}$, *then solve for* $\cos x$, *obtaining* $\cos x = \dfrac{bc \pm a\sqrt{a^2+b^2-c^2}}{a^2+b^2}$; *similarly, via* $\cos x = \pm\sqrt{1-\sin^2 x}$, *one obtains* $\sin x = \dfrac{ac \mp b\sqrt{a^2+b^2-c^2}}{a^2+b^2}$. *Note how the plus and minus signs match up, so that one obtains the same pair of solutions as in the featured approach. For an alternative construction for part (b), let* $u = \sin x$ *and* $v = \cos x$, *then plot the line* $au + bv = c$ *and the unit circle* $u^2 + v^2 = 1$ *in the* uv*-plane. If the points of intersection are denoted by* P_i $(i = 1, 2)$, *then the angles the rays* $\overrightarrow{OP_i}$ *make with the positive* u*-axis are the two solutions to the original equation.*

2564. [2000 : 373] *Proposed by Darko Veljan, University of Zagreb, Zagreb, Croatia.*

(a) Find all integer solutions (a, b, c) of the equation $\binom{a}{2} + \binom{b}{2} = \binom{c}{2}$ such that $2 \le a \le b \le c$.

(b) For each integer $n \ge 1$, find at least one integer solution (a, b, c) $(n \le a \le b \le c)$ of the equation $\binom{a}{n} + \binom{b}{n} = \binom{c}{n}$.

(c) For $n = 3$, find at least one further solution for (b).

I. *Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA (adapted by the editor).*

(a) Note first that $\dbinom{a}{2} + \dbinom{b}{2} = \dbinom{c}{2}$ is equivalent to

$$\begin{aligned} a(a-1) &+ b(b-1) = c(c-1) \quad \text{or} \\ a(a-1) &= (c-b)(c+b-1)\,. \end{aligned} \qquad (1)$$

Hence, if we take any integer $a \ge 2$ and factor $a(a-1)$ into $a(a-1) = d_1 d_2$ in all possible ways in which $d_1 < d_2$, $d_1 + d_2$ is odd, and $\frac{d_2-d_1+1}{2} \ge a$, then with $b = \frac{d_2-d_1+1}{2}$ and $c = \frac{d_2+d_1+1}{2}$, one can verify immediately that (1) holds. [Ed: It is easy to see that the factorization described above is possible for all $a \ge 3$. In particular, for any $a \ge 3$, we can always take $d_1 = 1$ and $d_2 = a(a-1)$. This yields the particular solution

$\left(a,\, \frac{a(a-1)}{2},\, \frac{a(a-1)}{2}+1\right)$. This solution was mentioned explicitly by a few solvers. Note also that this is the only solution if $a = 2^n$ such that $2^n - 1$ is a prime; that is, if $a-1$ is a Mersenne prime, since in this case it is easy to see that $d_1 = 1$ and $d_2 = a(a-1)$ is the only factorization of $a(a-1)$ satisfying all the required conditions. For example, the only solution in which $a = 8$ is $(8, 28, 29)$. On the other hand, it is easy to see that there are no solutions when $a = 2$ since (1) becomes $(c-b)(c+b-1) = 2$ which implies that $c - b = 1$ and $c + b - 1 = 2$. Solving, we get $b = 1 < a$, a contradiction.]

Conversely, suppose (1) holds. If we let $d_1 = c - b$ and $d_2 = c + b - 1$, then clearly d_1 and d_2 are divisors of $a(a-1)$, with $d_1 < d_2$, $d_1 + d_2$ being odd, and $\frac{d_2 - d_1 + 1}{2} = b \geq a$. Therefore, the constructions described above would indeed produce all the solutions.

(b) Since $\binom{2n}{n} = \binom{2n-1}{n} + \binom{2n-1}{n-1} = \binom{2n-1}{n} + \binom{2n-1}{n}$, we see that $(a, b, c) = (2n-1,\ 2n-1,\ 2n)$ is a solution.

(c) There are many solutions; for example, $(a, b, c) = (10, 16, 17)$, $(22, 56, 57)$, $(32, 57, 60)$, ... [Ed: Hess gave 11 solutions for this part.]

II. *Solution to* (a) *by the proposer* (*modified slightly by the editor*).

The given equation is equivalent to $a^2 - a + b^2 - b = c^2 - c$.

If we let $c = a + m = b + n$, where $0 \leq n \leq m$, then, from $a^2 - a + b^2 - b = (a+m)^2 - (b+n)$, we get

$$\begin{aligned} 2am &+ a + m^2 - n = b^2 = (a + m - n)^2 \quad \text{or} \\ a^2 &- (2n+1)a + n^2 - 2mn + n = 0 \,. \end{aligned} \tag{2}$$

The discriminant of this quadratic equation is

$$D = (2n+1)^2 - 4(n^2 - 2mn + n) = 8mn + 1 \,.$$

Since D must be a perfect square, we have $8mn + 1 = (2k-1)^2$ for some $k \in \mathbb{N}$, from which we get $mn = \dfrac{k(k-1)}{2} = \dbinom{k}{2}$ for some $k \geq 2$. [Ed: Note that if $k = 1$, then $mn = 0$ implies $n = 0$ or $m = 0$; that is, either $c = b$ or $c = a = b$. In either case we get $\dbinom{a}{2} = 0$, a contradiction.]

Then, the solutions of (2) are given by

$$a = \frac{2n + 1 \pm \sqrt{D}}{2} = \frac{2n + 1 + 2k - 1}{2} = n + k \,.$$

[Ed: If we take $-\sqrt{D}$, then $a = n - k + 1$. However, $n^2 \leq mn = \dbinom{k}{2} < k^2$ implies $n < k$ and therefore, $a = n - k + 1 \leq 0$, a contradiction.] It follows that $b = a + m - n = m + k$ and $c = b + n = m + n + k$. Conversely, it is straightforward to verify that these values of a, b, c do yield a solution. Therefore, all solutions are given by $(a, b, c) = (n + k, m + k, m + n + k)$ where $k \geq 2$, $mn = \binom{k}{2}$, $n \leq m$.

Also solved (completely) by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SHOICHI HIROSE, Mita High School, Tokyo, Japan; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, Trinity College, Cambridge, England; and the proposer.

Partial solutions were submitted by HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; PANOS E. TSAOUSSOGLOU, Athens, Greece; and KENNETH M. WILKE, Topeka, KS, USA.

There were also one incomplete and one partially incorrect solution.

Hirose gave a few references, including a paper of his own ([1], [2] below) in which this problem was studied. In particular, he provided a list of all solutions (a, b, c) to part (a) in which $2 \leq a \leq b \leq c \leq 50$. For part (c), he stated that there are five solutions in which $3 \leq a \leq b \leq c \leq 100$, 34 solutions in which $3 \leq a \leq b \leq c \leq 1000$, and 82 solutions in which $3 \leq a \leq b \leq c \leq 10000$. Parts of his claims were substantiated by a computer search by Wilke. Both of them discovered the solution $(a, b, c) = (132, 190, 200)$ to $\binom{a}{4} + \binom{b}{4} = \binom{c}{4}$ besides the "trivial" solution $(7, 7, 8)$ covered by the general solution to part (b).

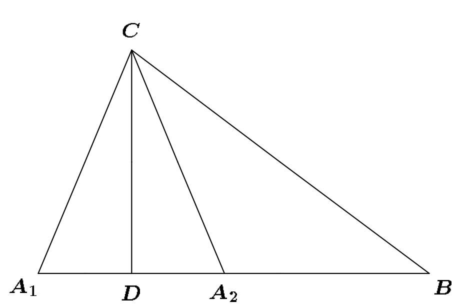
[1]. A.S. Fraenkel. *"Diophantine Equations Involving Generalized Triangular and Tetrahedral Numbers".* Computers in Number Theory. *New York: Academic Press, 1971, pp. 99–114.*

[2]. S. Hirose. *"On Some Polygonal Number which are, at the Same Time the Sums, Differences, and Products of Two Other Polygonal Numbers."* The Fibonacci Quarterly *24 (1986) pp. 99–106.*

2565. [2000 : 373] *Proposed by K.R.S. Sastry, Dodballapur, India.*

A *Heron* triangle has integer sides and integer area. Show that there are exactly three pairs of Heron Triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that $B_1C_1 = B_2C_2$, $A_1C_1 = A_2C_2$, $\angle A_1B_1C_1 = \angle A_2B_2C_2$ and $A_2B_2 - A_1B_1 = 10$.

Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany.



$AD = 5$ implies that $CD = 12$ and $A_nC = 13$ $(n = 1, 2)$. Let $BC = a$ and $BD = d$.

From $CD^2 = BC^2 - BD^2$, we have $144 = a^2 - d^2 = (a+d)(a-d)$. Note that $a+d$ and $a-d$ must have the same parity. Thus,

$$144 = 2 \cdot 72 = 4 \cdot 36 = 6 \cdot 24 = 8 \cdot 18$$

implies that

$$(a, d) = (37, 35) \quad \text{or} \quad (20, 16) \quad \text{or} \quad (15, 9) \quad \text{or} \quad (13, 5)\,.$$

Therefore, we have

First pair	$:(a, b, c) =$	$(37, 13, 40)$	and	$(37, 13, 30)$,
Second pair	:	$(20, 13, 21)$	and	$(20, 13, 11)$,
Third pair	:	$(15, 13, 14)$	and	$(15, 13, 4)$,

and these are all the solutions.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHAI CIPU, IMAR, Bucharest, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Cambridge, England; KENNETH M. WILKE, Topeka, KS, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; CALVIN LIN ZHIWEI, Singapore; and the proposer. There was one incorrect solution.

Janous commented that it was a neat problem, and that he intended to pose it to his gifted students.

2567. [2000 : 373] *Proposed by K.R.S. Sastry, Dodballapur, India.*

In triangle ABC, points P and Q are on the line segment BC such that AP and AQ are trisectors of $\angle BAC$ and $BQ = QC$. If $AC = \sqrt{2}\,AQ$, find the measure of $\angle BAC$.

Solution by Panos E. Tsaoussoglou, Athens, Greece.

If the sides of $\triangle ABC$ are a, b, and c (in the usual notation), then $AC = b$, $AQ = \dfrac{b\sqrt{2}}{2}$, and $BQ = QC = \dfrac{a}{2}$. Let $\theta = \dfrac{A}{3}$. The Sine Law applied to $\triangle ABQ$ gives

$$\frac{a}{2\sin 2\theta} = \frac{b\sqrt{2}}{2\sin B}\,,$$

and to $\triangle ABC$ gives

$$\frac{a}{\sin 3\theta} = \frac{b}{\sin B}\,.$$

Consequently, θ satisfies

$$\frac{\sin 3\theta}{\sin 2\theta} = \sqrt{2}\,.$$

Using the identities

$$\sin 3\theta = \sin\theta(3 - 4\sin^2\theta) = \sin\theta(4\cos^2\theta - 1)\,,$$

and

$$\sin 2\theta = 2\sin\theta\cos\theta\,,$$

this becomes

$$4\cos^2\theta - 2\sqrt{2}\cos\theta - 1 = 0\,.$$

Since the negative root is extraneous, $\cos\theta = \dfrac{(\sqrt{2}+\sqrt{6})}{4}$; hence, $\theta = 15°$, so that $A = 3\theta = 45°$.

Editor's remarks. The majority of the submitted solutions ordered the points B, P, Q, C along BC as in the featured solution; however, there is no reason why the order could not be B, Q, P, C. In the latter case, the above method leads to the equation

$$\frac{\sin 3\theta}{\sin\theta} \;=\; \sqrt{2}\,,$$

in which case $4\sin^2\theta = 3 - \sqrt{2}$, and

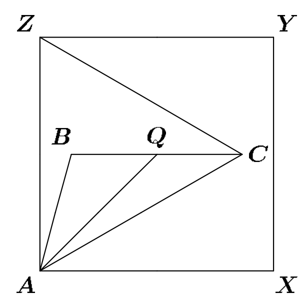
$$A \;=\; 3\sin^{-1}\frac{\sqrt{3-\sqrt{2}}}{2} \;\approx\; 117.07°\,.$$

Only Loeffler included both interpretations.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, Bucharest, Romania; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Manila, Philippines; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; CHALLA K.S.N.M. SANKAR, Andhra Pradesk, India; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Bataille and Yiu both showed how to construct $\triangle ABC$ in the case where Q lies between P and C. Here is Yiu's construction:

> *Inside square $AXYZ$ construct the equilateral triangle AZC on side AZ. Then Q is the centre of the square (because $\angle CAQ = 60° - 45° = 15°$, while $AQ = \frac{1}{\sqrt{2}}AZ = \frac{1}{\sqrt{2}}AC$). Finally, B is the point of CQ for which $\angle CAB = 3\angle CAQ$ [and the half-turn about Q interchanges B with C so that Q is the mid-point as desired]. It is clear from Yiu's figure that $\angle ACB = 30°$ and $\angle ABC = 105°$.*



Several of the submitted solutions exploited a variant of this figure to avoid the use of any trigonometry. Konečný's solution in particular was based on such a figure; the idea was suggested to him by his published solution to Mathematics Magazine *problem* #825 [46 pp. 45–46].

2568. [2000 : 373] *Proposed by K.R.S. Sastry, Bangalore, India.*

The sides a, b and c of a non-degenerate triangle ABC satisfy the relations $b^2 = ca + a^2$ and $c^2 = ab + b^2$. Find the measures of the angles of triangle ABC.

Solution by Václav Konečný, Ferris State University, Big Rapids, MI, USA.

Let a, b and c be the sides and A, B and C be the angles of $\triangle ABC$. Clearly, $a < b < c$, so that $A < B < C$. Consider the first condition, $b^2 = ca + a^2$. By the Law of Cosines, $b^2 = c^2 + a^2 - 2ca\cos B$.

Hence $ca = c^2 - 2ca\cos B$, so that $\frac{c}{a} - 2\cos B = 1$. By the Law of Sines, $\frac{c}{a} = \frac{\sin C}{\sin A}$. Then $\frac{\sin C}{\sin A} - 2\cos B = 1$, which gives $\sin A\cos B + \cos A\sin B - 2\cos B\sin A = \sin A$, or, $\sin(B - A) = \sin A$. Consequently, $B = 2A$. Similarly, the condition $c^2 = ab + b^2$ implies $C = 2B$. Therefore, $C = 2B = 4A$, so that $A = \pi/7$, $B = 2\pi/7$ and $C = 4\pi/7$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, IMAR, Bucharest, Romania; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD EDEN, Ateneo de Manila University, Philippines; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; HENRY LIU, Trinity College, Cambridge, England (2 solutions); DAVID LOEFFLER, student, Cotham School, Bristol, UK; GERRY LEVERSHA, St. Paul's School, London, England; VEDULA N. MURTY, Visakhapatnam, India; CHALLA SANKAR, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.