PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 April 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in* LaTeX *format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

In the September issue, [2001 : 337], problems 2660–2663 were incorrectly numbered 2560–2563. How could all of our proof-readers have missed that!!

Professor Murray Klamkin has pointed out that proposal 2636 is essentially the same as proposal 2609. Solvers should therefore consider 2636 as withdrawn.

2664. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Let a, b and c be positive real numbers such that $a+b+c=abc$. Prove that $a^5(bc-1)+b^5(ca-1)+c^5(ab-1) \geq 54\sqrt{3}$.

2665. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In $\triangle ABC$, we have $\angle ACB = 90^\circ$ and sides $AB=c$, $BC=a$ and $CA=b$. In $\triangle DEF$, we have $\angle EFD = 90^\circ$, $EF=(a+c)\sin\left(\frac{B}{2}\right)$ and $FD=(b+c)\sin\left(\frac{A}{2}\right)$. Show that $DE \geq c$.

2666. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Two circles, Γ with diameter AB, and Δ with centre A, intersect at points C and D. The point M (distinct from C and D) lies on Δ. The lines BM, CM and DM intersect Γ again at N, P and Q respectively. Prove that

1. the quadrilateral $MPBQ$ is a parallelogram;
2. MN is equal to the geometric mean of NC and ND.

2667. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

You are given a circle Γ and two points A and B outside of Γ such that the line through A and B does not intersect Γ. Let X be any point on Γ.

Determine at which point X on Γ the sum $AX + XB$ attains its minimum value.

2668$\star$. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Suppose that $0 < r < q < 1$ and that $0 < m < \infty$. Show that

$$\begin{aligned}(1-q)(q+r-qr)\sqrt{1+m^2} + q(1-r)\sqrt{(q-2)^2+m^2q^2} \\ > (1-r)(q+r-qr)\sqrt{1+m^2} + r(1-q)\sqrt{(r-2)^2+m^2r^2}\,.\end{aligned}$$

2669$\star$. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* (Professor Klamkin offers a prize of $50 for the first correct solution received by the Editor-in-Chief.)

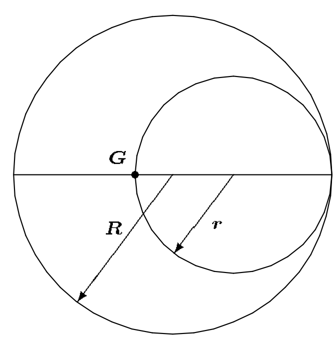
Let A_1, A_2, $\ldots$, A_{2n}, be any $2n$ points in $\mathbb{E}^m$. Determine the largest k_n such that

$$A_1A_2^2 + A_2A_3^2 + \cdots + A_{2n}A_1^2 \;\geq\; k_n\left(A_1A_{n+1}^2 + A_2A_{n+2}^2 + \cdots + A_nA_{2n}^2\right)\,.$$

For $n = 2$, it is easily shown that $k_2 = 1$. That $k_3 = \frac{1}{2}$ is an Armenian Olympiad problem [2001 : 9].

2670. *Proposed by Robert C.H. Schmidt, Minnetonka, MN, USA.*

A disk of uniform thickness and composition has radius R. From this disk, a smaller disk of radius r is cut that is internally tangent to the original disk. If the centre of gravity, G, of the resulting object lies on the common diameter of the two disks, at the point (other than the point of tangency) where this diameter intersects the circumference of the smaller disk, determine the ratio $R : r$.



2671. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given $\triangle ABC$ with circumcentre O and incentre I, let P be the point dividing OI in the ratio $OP : PI = -1 : 4$. Let X, Y and Z be the feet of the perpendiculars from P to the sides BC, CA and AB respectively.

(a) Show that $AY + AZ = BZ + BX = CX + CY$.

(b) Show that P is the centroid of the triangle whose vertices are the excentres of $\triangle ABC$.

2672. *Proposed by Vedula N. Murty, Dover, PA, USA.*

(a) Suppose that $\alpha > 0$. Prove that $\sum_{k=1}^{n} k^{\alpha} < \frac{(n+1)^{\alpha+1}-1}{\alpha+1}$.

(b) Suppose that $-1 < \alpha < 0$. Prove that $\frac{(n+1)^{\alpha+1}-1}{\alpha+1} < \sum_{k=1}^{n} k^{\alpha}$.

[These two inequalities appear differently in "*Analytic Inequalities*" by Nicholas D. Kazarinoff, Holt Rinehart and Winston, p. 24. The term "-1" is missing from the numerators.]

2673. *Proposed by George Baloglou, SUNY Oswego, Oswego, NY, USA.*

Let $n \geq 2$ be an integer.

(a) Show that

$$(1+a_1\cdots a_n)^n \geq (a_1\cdots a_n)\left(1+a_1^{n-2}\right)\left(1+a_2^{n-2}\right)\cdots\left(1+a_n^{n-2}\right)$$

for all $a_1 \geq 1$, $a_2 \geq 1$, ..., $a_n \geq 1$, if and only if $n \leq 4$.

(b) Show that

$$\frac{1}{a_1\left(1+a_2^{n-2}\right)} + \frac{1}{a_2\left(1+a_3^{n-2}\right)} + \cdots + \frac{1}{a_n\left(1+a_1^{n-2}\right)} \geq \frac{n}{1+a_1\cdots a_n}$$

for all $a_1 > 0$, $a_2 > 0$, ..., $a_n > 0$, if and only if $n \leq 3$.

(c) Show that

$$\frac{1}{a_1\left(1+a_1^{n-2}\right)} + \frac{1}{a_2\left(1+a_2^{n-2}\right)} + \cdots + \frac{1}{a_n\left(1+a_n^{n-2}\right)} \geq \frac{n}{1+a_1\cdots a_n}$$

for all $a_1 > 0$, $a_2 > 0$, ..., $a_n > 0$, if and only if $n \leq 8$.

(d)$\star$ Show that

$$\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\left(\frac{1}{1+a_1^{n-2}}+\frac{1}{1+a_2^{n-2}}+\cdots+\frac{1}{1+a_n^{n-2}}\right) \geq \frac{n^2}{1+a_1\cdots a_n}$$

for all $a_1 > 0$, $a_2 > 0$, ..., $a_n > 0$, if and only if $n \leq 5$.

2674. *Proposed by Mohammed Aassila, Strasbourg, France.*

Find an explicit formula for the least number $f(n)$ of distinct points in the plane such that, for each $k = 1, 2, \ldots, n$, there exists a straight line containing exactly k of these points.

2675. *Proposed by Joe Howard, Portales, NM, USA.*

Show that

$$\cos^2\left(\frac{\pi}{7}\right)+\cos^2\left(\frac{2\pi}{7}\right)+\cos^2\left(\frac{3\pi}{7}\right) = 10\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right).$$