THE SKOLIAD CORNER

No. 56

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Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 February 2002*. Look for prizes for solutions in the new year.

This issue's items come from the Mandelbrot competition. The competition is made up of four rounds at different times of the year. Each round consists of an individual part and a team part. A school's score is made up of the top four individual scores and the score of the team of four (chosen by the school) on the team part. This round's team part has a twist: a little essay was provided to the students a couple of days prior to the test. The test was then based on the essay.

My thanks go to Sam Vandervelde at Greater Testing Concepts for providing the contest material. For more information about the contest you can visit the website

http://www.mandelbrot.org or email info@mandelbrot.org

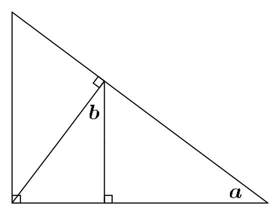
The Mandelbrot Competition
Division B Round Two Individual Test
December 1997

1. If a group of positive integers has a sum of 8, what is the greatest product the group can have? (**1 point**)

2. There is one two-digit number such that if we add 1 to the number and reverse the digits of the result, we obtain a divisor of the number. What is the number? (**1 point**)

3. Ten slips of paper, numbered 1 through 10, are placed in a hat. Three numbers are drawn out, one after another. What is the probability that the three numbers are drawn in increasing order? (**2 points**)

4. The three marked angles are right angles. If $\angle a = 20°$, then what is the measure of $\angle b$? (**2 points**)



5. Vicky asks Charlene to identify all non-congruent triangles $\triangle ABC$ given:

1. the value of $\angle A$,
2. $AB = 10$, and
3. length BC equals either 5 or 15.

Charlene responds that there are only two triangles meeting the given conditions. What is the value of $\angle A$? (**2 points**)

6. Five pirates find a cache of 5 gold coins. They decide that the shortest pirate will become bursar and distribute the coins — if half or more of the pirates (including the bursar) agree to the distribution, it will be accepted; otherwise, the bursar will walk the plank and the next shortest pirate will become bursar. This process will continue until a distribution of coins is agreed upon. If each pirate always acts so as to stay aboard if possible and maximize his wealth, and would rather see another pirate walk the plank than not (all else being equal), then how many coins will the shortest pirate keep for himself? (**3 points**)

7. The twelve positive integers $a_1 \le a_2 \le \cdots \le a_{12}$ have the property that no three of them can be the side lengths of a non-degenerate triangle. Find the smallest possible value of $\frac{a_{12}}{a_1}$. (**3 points**)

Mandelbrot Morsels
An Interpretation of Interpolation

One of the most fascinating capabilities that mathematical science offers is the ability to make predictions. Imagine a simple experiment in which we place a bowling ball on a slope and release it at the same instant that we start a stopwatch. The distance in centimetres that the ball has rolled after t seconds would resemble Table 1. With a bit of experimentation, one could discover the rule $d \approx 0.38t^2$ and predict that the ball should roll about 85.5 cm in 15 seconds. It is quite satisfying to check this figure experimentally and find that it is correct.

Table 1.

Time	Position
0	0.00
2	1.55
4	6.05
8	24.35
10	37.95

In an actual experiment the bowling ball may have some initial velocity by the time we begin our readings, and will start a ways from our distance recorder, so that the formula for distance will be a general second degree polynomial: $d \approx at^2 + bt + c$. For example, we may obtain the data shown in Table 2. Here it is not at all clear what coefficients a, b, and c we should use in order to accurately model our data.

Table 2.

Time	Position
2	7.00
3	9.45
6	21.00
7	26.25

Considerations such as these led mathematicians to develop a general method for finding equations of polynomials which pass through given points. The method, known as Lagrange Interpolation, is attributed to the French mathematician Joseph-Louis Lagrange (1736-1813). It stems from the following

FACT: Let $n+1$ data points (x_0, y_0), (x_1, y_1), ... , (x_n, y_n) be given. Then there is exactly one polynomial $p(x)$ of degree n or less passing through all $n+1$ points.

The difficulty, naturally, is figuring out a formula for $p(x)$ based on the data points. A very clever technique handles this dilemma, which we will present by way of an example: find the third degree polynomial which satisfies $p(0) = -2$, $p(1) = 1$, $p(3) = 1$ and $p(4) = -14$. An alternate way of phrasing this is to ask for a cubic polynomial which passes through the points $(0,-2)$, $(1,1)$, $(3,1)$, and $(4,-14)$.

The key is to build our polynomial from the functions such as

$$\chi_0 \;=\; -2\,\frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \;=\; \frac{1}{6}\,(x-1)(x-3)(x-4)\,.$$

Notice that when we plug in $x = 0$ we obtain -2, as desired (mentally check this). However, substituting $x = 1$, $x = 3$, or $x = 4$ results in a value of 0. Now create χ_1, χ_3, and χ_4 in the same way (try it!), and add all four functions together to obtain

$$\begin{aligned} p(x) \;&=\; \frac{1}{6}\,(x-1)(x-3)(x-4) + \frac{1}{6}\,x(x-3)(x-4) \\ &\quad - \frac{1}{6}\,x(x-1)(x-4) - \frac{7}{6}\,x(x-1)(x-3)\,. \end{aligned}$$

We claim this is precisely the polynomial we are after. For example, when $x = 4$ the first three terms equal 0, while the fourth is $-\frac{7}{6}(12) = -14$, so $p(4) = -14$, just as we wanted. This method generalizes to more than four points in exactly the way one would expect it to.

EXERCISE: Use Lagrange Interpolation to find a second degree polynomial passing through the first three data points of Table 2. Check your formula correctly predicts the distance for the final point, when $t = 7$.

Now for the "interpretation" part. Let us examine this business of Lagrange Interpolation from a slightly different angle. Suppose we are given the values of a third degree polynomial $p(x)$ at $x = 0,\ 1,\ 2,\ 3$. What is $p(4)$ in terms of $p(0)$ through $p(3)$? Let us call $p(0) = Y_0, p(1) = Y_1, p(2) = Y_2$, and $p(3) = Y_3$. Then by Lagrange Interpolation $p(x)$ equals

$$\begin{aligned} &\frac{Y_0(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + \frac{Y_1 x(x-2)(x-3)}{(1-0)(1-2)(1-3)} \\ &\quad + \frac{Y_2 x(x-1)(x-3)}{(2-0)(2-1)(2-3)} + \frac{Y_3 x(x-1)(x-2)}{(3-0)(3-1)(3-2)}\,. \end{aligned}$$

Hence, $p(4)$ can be found by substituting $x = 4$ in the above expression. We obtain

$$p(4) \;=\; -Y_0 + 4Y_1 - 6Y_2 + 4Y_3\,.$$

There are two important observations to be made at this point. First, $p(4)$ is a *linear combination* of $p(0)$ through $p(3)$. In other words, $p(4)$ can be obtained by multiplying each of $p(0)$ through $p(3)$ by certain constants and adding up the results. Once we have found those constants (in this case -1, 4, -6 and 4), we can always predict $p(4)$ given the values of $p(x)$ at $x = 0,\ 1,\ 2,\ 3$.

As for the second observation, if you noticed that the above constants looked like the fourth row of Pascal's triangle, give yourself a pat on the back. Pascal's triangle arises from binomial expansions, so it is not surprising that there is a nifty function that generates the constants -1, 4, -6, and 4. It is $x^4 - (x-1)^4$, which equals $4x^3 - 6x^2 + 4x - 1$. These findings are true for any degree polynomial; we summarize them in our first

THEOREM: Let $p(x)$ be a polynomial of degree n for which we know the values of $p(0)$, $p(1)$, ..., $p(n)$. Then $p(x)$ can be calculated using Lagrange Interpolation, and $p(n+1)$ is a linear combination of $p(0)$ through $p(n)$. In other words,

$$p(n+1) \; = \; A_0 p(0) + A_1 p(1) + \cdots + A_n p(n)\,,$$

for some constants A_0, A_1, ..., A_n. These constants appear in row $n+1$ of Pascal's triangle, and are generated by the function

$$x^{n+1} - (x-1)^{n+1} \; = \; A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0\,. \tag{1}$$

The final equation is quite versatile. For instance, it answers the following popular olympiad level problem from the 1980's: if $p(x)$ is a degree n polynomial such that $p(0) = 1$, $p(1) = 2$, $p(2) = 4$, $p(3) = 8$, ..., $p(n) = 2^n$, then find $p(n+1)$. From our theorem we know that $p(n+1)$ is a linear combination of $p(0)$ through $p(n)$; in particular

$$p(n+1) \; = \; A_0 \cdot 1 + A_1 \cdot 2 + A_2 \cdot 4 + \cdots + A_n \cdot 2^n\,.$$

Substituting $x = 2$ in equation (1) magically produces $2^{n+1} - 1$ as our answer. Now try this

EXERCISE: Suppose $p(x)$ is a degree n polynomial such that $p(0) = 1$, $p(1) = -1$, $p(2) = 1$, $p(3) = -1$, ..., $p(n) = (-1)^n$. Calculate $p(n+1)$.

If done correctly you should obtain an answer very similar to $2^{n+1} - 1$.

Hopefully you have made it this far and appreciate some of the interesting and clever mathematics embedded within Lagrange Interpolation. If not, do not worry; there are still a few more days before the team test. Put these crazy theorems away, clear your head, and come back tomorrow, at which point everything will make perfect sense. The team test will be based on these ideas, especially the last few paragraphs. Happy interpolating.

The Mandelbrot Competition
Division B Round Two Team Test
December 1997

FACTS: A polynomial $p(x)$ of degree n or less is determined by its value at $n+1$ x–coordinates. For $n = 1$ this is a familiar statement; a line (degree one polynomial) is determined by two points. Moreover, the value of $p(x)$ at any other x–value can be computed in a particularly nice way using Lagrange interpolation, as outlined in the essay *An Interpretation of Interpolation*.

We will also need a result from linear algebra which states that a system of n "different" linear equations in n variables has exactly one solution. For example, there is only one choice for x, y, and z which satisfies the equations $x + y + z = 1$, $x + 2y + 3z = 4$, and $x + 4y + 9z = 16$.

SETUP: Let $p(x)$ be a degree three polynomial for which we know the values of $p(1)$, $p(2)$, $p(4)$, and $p(8)$. By the facts section there is exactly one such polynomial. According to Lagrange interpolation the number $p(16)$ can be deduced; it equals

$$p(16) \;=\; A_0p(1) + A_1p(2) + A_2p(4) + A_3p(8)\,,$$

for some constants A_0 through A_3. The goal of this team test will be to compute the A_i and use them to find information about $p(16)$ *without ever finding an explicit formula for* $p(x)$.

PROBLEMS:

Part i: (4 points) We claim that the A_i can be found by subtracting

$$x^4 - (x-1)(x-2)(x-4)(x-8) \;=\; A_3x^3 + A_2x^2 + A_1x + A_0\,. \tag{2}$$

Implement this claim to compute A_0 through A_3.

Part ii: (4 points) To show that these A_i are in fact the correct numbers we must show that they correctly predict $p(16)$ for four "different" polynomials. We begin with the case $p(x) = x$. Show that the value of $p(16)$ agrees with the prediction $A_0p(1) + A_1p(2) + A_2p(4) + A_3p(8)$. (HINT: try $x = 2$ in (2).)

Part iii: (5 points) Continuing the previous part, show that the A_i correctly predict $p(16)$ for the three other polynomials $p(x) = 1$, $p(x) = x^2$ and $p(x) = x^3$.

Part iv: (4 points) Suppose that $p(x)$ is a third degree polynomial with $p(1) = 0$, $p(2) = 1$, and $p(4) = 3$. What value should $p(8)$ have to guarantee that $p(x)$ has a root at $x = 16$?

Part v: (4 points) Let $p(x)$ be a degree three polynomial with $p(1) = 1$, $p(2) = 3$, $p(4) = 9$, and $p(8) = 27$. Calculate $p(16)$. How close does it come to the natural guess of 81?