Eight Proofs of One Theorem

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The old saying "*there is more than one way to skin a cat*" is certainly true in geometry; there is no unique way to prove a theorem or to solve a problem.

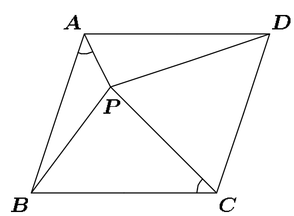
Howard Eves, *Fundamentals of Modern Geometry*

Introduction.

In an article in this journal [1998 : 81], Georg Gunther presented one problem with six different solutions (see [1]). And Jimmy Chui [1999 : 235] gave four different proofs of a combinatorial identity (see [2]). It is pleasant to see different solutions of a problem. It is our purpose in this article to examine various different solutions to a geometry problem. We also hope to emphasize the importance of studying a problem with different solutions.

Theorem 1.

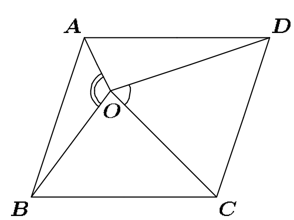
If P is a point in the interior of a parallelogram $ABCD$ such that the angles at A and C with chord BP are congruent, then the angle at D and B with chord AP are also congruent; if $\angle PAB = \angle PCB$, then $\angle PDA = \angle PBA$.



First, let us examine the following Canadian Mathematical Olympiad problem and Theorem 2 before we give eight proofs of Theorem 1.

1997 Canadian Mathematical Olympiad Problem 4.

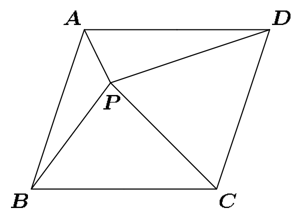
If the point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$, prove that $\angle OBC = \angle ODC$.



Theorem 1 and the above CMO problem yield the following interesting result:

Theorem 2.

Let P be a point in the interior of parallelogram $ABCD$. Then the following are equivalent:



(1) $\angle PAB = \angle PCB$,

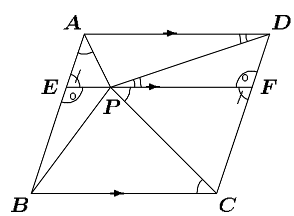
(2) $\angle PDA = \angle PBA$,

(3) $\angle APB + \angle CPD = 180°$,

(4) $\angle BPC + \angle DPA = 180°$.

Now, we offer eight proofs of Theorem 1, all requiring no more than the ordinary geometry and trigonometry background of a high school student. Enjoy our proofs!

Proof 1. (Similarity)



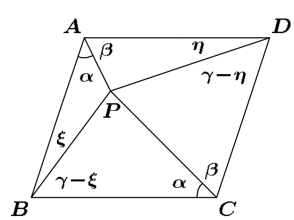
Let the line though P, parallel to BC, meet AB and CD at E and F, respectively. We note that $\triangle APE \sim \triangle PCF$ since $\angle AEP = \angle PFC$ and $\angle PAE = \angle PCB = \angle CPF$.

From $AD \parallel EF \parallel BC$, it follows that $\dfrac{AE}{EB} = \dfrac{DF}{FC}$,

and from $\triangle APE \sim \triangle PCF$, it follows that $\dfrac{EP}{AE} = \dfrac{FC}{PF}$, so that $\dfrac{EP}{EB} = \dfrac{DF}{FP}$.

This result, and $\angle PEB = \angle DFP$, imply that $\triangle PEB \sim \triangle DFP$, so that $\angle ABP = \angle PBE = \angle DPF = \angle ADP$, as desired.

Proof 2. (Law of Sines)



Let $\angle PAB = \angle PCB = \alpha$, $\angle PAD = \angle PCD = \beta$, $\angle ABP = \xi$, $\angle ADP = \eta$, and $\angle ABC = \angle ADC = \gamma$.

By the Law of Sines, we obtain that

$$\frac{\sin\xi}{\sin\alpha} = \frac{PA}{PB}, \qquad \frac{\sin\alpha}{\sin(\gamma-\xi)} = \frac{PB}{PC},$$
$$\frac{\sin(\gamma-\eta)}{\sin\beta} = \frac{PC}{PD}, \qquad \frac{\sin\beta}{\sin\eta} = \frac{PD}{PA}.$$

Multiplying all of these gives $\dfrac{\sin\xi\sin(\gamma-\eta)}{\sin\eta\sin(\gamma-\xi)} = 1$, which means that

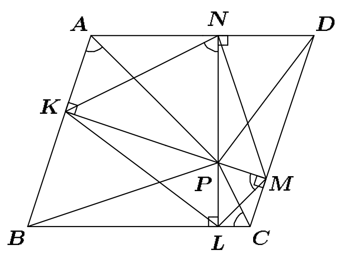
$$\sin\xi\,(\sin\gamma\cos\eta - \cos\gamma\sin\eta) \;=\; \sin\eta\,(\sin\gamma\cos\xi - \cos\gamma\sin\xi)\ , \qquad \text{or}$$

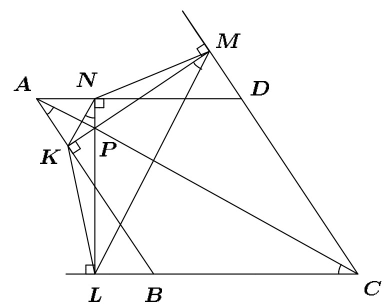
$$\sin\gamma\,(\sin\xi\cos\eta - \cos\xi\sin\eta) \;=\; 0\,.$$

This is equivalent to $\sin\gamma\sin(\xi-\eta) = 0$.

This shows that $\xi = \eta$ since $0 < \gamma,\ \xi,\ \eta < \pi$. We conclude that $\angle PBA = \angle PDA$.

Proof 3. (Cyclic Quadrilaterals)





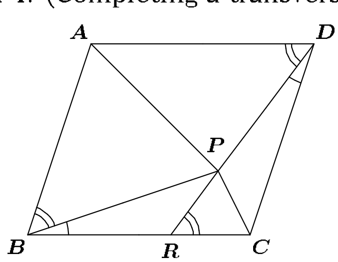
Project P onto the lines AB, BC, CD, and DA to obtain the points K, L, M, and N.

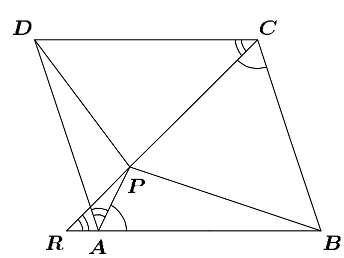
Because $AB \parallel CD$, the points K, P, and M are collinear, and because $AD \parallel BC$, the points L, P, and N are collinear.

Note that $AKPN$, $BLPK$, $CMPL$, and $DNPM$ are cyclic quadrilaterals, which may not be convex if not all of K, L, M, and N lie on the side-segments of the quadrilateral.

Hence $\angle PNK = \angle PAB$, $\angle PLK = \angle PBA$, $\angle PML = \angle PCB$, and $\angle PMN = \angle PDA$. Therefore, $KLMN$ is also a cyclic quadrilateral. Thus, we obtain that $\angle PBA = \angle KLN = \angle KMN = \angle PDA$.

Proof 4. (Completing a transversal)





We use the 2$^{\text{nd}}$ diagram. Let R be the intersection of the line AB and line CP.

Then we obtain that $\angle PRB = \angle PCD = \angle PAD$, because $AB \parallel CD$ and $\angle PAB = \angle PCB$. The Sine Law applied to $\triangle ARP$, and then to $\triangle CRB$

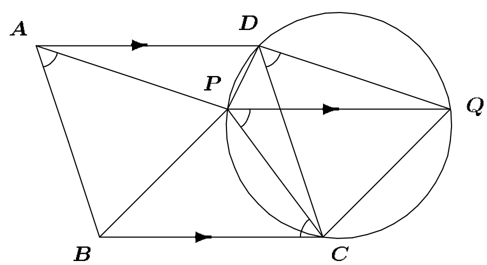
gives

$$\frac{PR}{PA} = \frac{\sin\angle PAB}{\sin\angle PRA} = \frac{\sin\angle PCB}{\sin\angle PRB} = \frac{RB}{BC} = \frac{RB}{AD},$$

so that $\dfrac{PA}{AD} = \dfrac{RB}{PR}$.

This, and $\angle PAD = \angle PRB$, imply that $\triangle PAD \sim \triangle PRB$. As a consequence, $\angle PBA = \angle PBR = \angle PDA$, and the problem is solved.

Proof 5. (Parallelograms and Cyclic quadrilaterals)



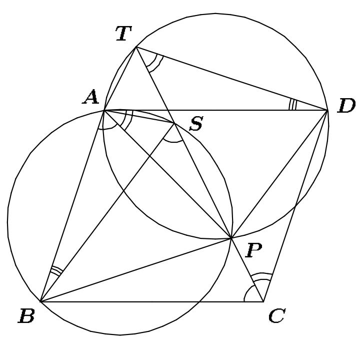
First, we complete the parallelogram $PBCQ$.

Note that $APQD$ is also a parallelogram, since $AD \parallel BC \parallel PQ$ and $AD = BC = PQ$. Now, $\triangle PAB$ and $\triangle QDC$ are congruent triangles, since $PA = QD$, $AB = DC$, and $BP = CQ$.

Since $\angle CPQ = \angle PCB = \angle PAB = \angle QDC$, or $\angle CPQ = \angle QDC$, we have that $PCQD$ is a cyclic quadrilateral.

Thus, we have that $\angle ADP = \angle QPD = \angle QCD = \angle PBA$, or $\angle ADP = \angle PBA$.

Proof 6. (Three isosceles triangles)



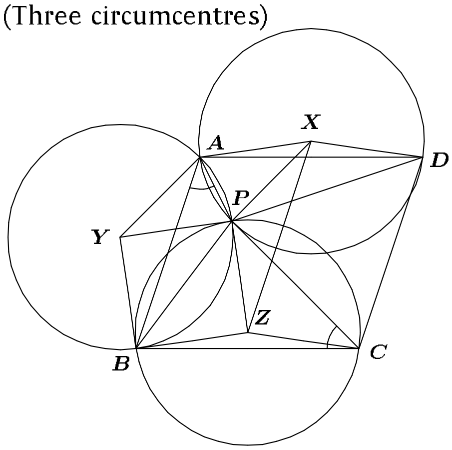
Suppose that CP meets the circle ABP again at S, and suppose that CP meets the circle ADP again at T.

Since $\angle BCS = \angle PAB = \angle PSB$, $\triangle BCS$ is isosceles with legs $BC = BS$. Similarly, $\triangle DCT$ is isosceles with legs $DT = DC$.

Thus, we obtain $AD = SB$ and $DT = BA$, since $AD = BC$ and $CD = BA$. Also, $\angle SBA = \angle SPA = \angle TPA = \angle TDA$, since $ABPS$ and $APDT$ are cyclic.

Combining results, $\triangle ADT \equiv \triangle SBA$. Therefore, $\triangle AST$ is isosceles with legs $AS = AT$. Thus, $\angle ABP = \angle AST = \angle ATS = \angle ATP = \angle ADP$, as we wanted to show.

Proof 7. (Three circumcentres)



Let Y be the circumcentre of $\triangle ABP$ and let $YA = YB = YP = r_1$. Let Z be the circumcentre of $\triangle BCP$ and let $ZB = ZC = ZP = r_2$.

We complete the parallelogram $ABZX$. Note that $XA = ZB = ZC = XD$.

From

$$r_1 \sin \angle PAB = r_1 \sin\left(\frac{\angle PYB}{2}\right) = \frac{BP}{2} = r_1 \sin\left(\frac{\angle PZB}{2}\right)$$
$$= r_1 \sin \angle PCB\,,$$

and

$$\angle PAB = \angle PCB\,,$$

it follows that $r_1 = r_2$, and thus, $YP = YB = ZB = ZP$. Consequently, $YBZP$ is a rhombus.

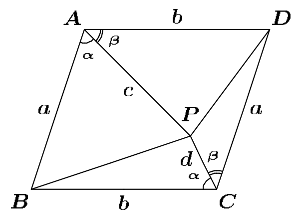
We also note that $AX \parallel BZ \parallel YP$ and $AX = BZ = YP$ which implies $AYPX$ is a parallelogram, in fact also a rhombus, because $YA = YP$. As a result, $XD = XA = XP$ which makes X the circumcentre of $\triangle DAP$.

As above, $YP \sin \angle ABP = \frac{AP}{2} = XP \sin \angle ADP = YP \sin \angle ADP$, and $\sin \angle ABP = \sin \angle ADP$.

Since $\angle ABP + \angle ADP < \angle ABC + \angle ADC = \pi$, we obtain that $\angle ADP = \angle PBA$.

Proof 8. (The Cosine Law)

We put $AB = CD = a$ and $BC = AD = b$, $AP = c$, $CP = d$, $\angle PAB = \angle PCB = \alpha$, and $\angle PAD = \angle PCD = \beta$.



The Cosine Law applied to $\triangle ABP$, and then to $\triangle BCP$, gives

$$\cos\alpha = \frac{a^2+c^2-BP^2}{2ac} = \frac{b^2+d^2-BP^2}{2bd}$$

or

$$(bd-ac)BP^2 = bd(a^2+c^2) - ac(b^2+d^2) = (ad-bc)(ab-cd)\,. \quad (1)$$

Analogously, we have

$$(ad-bc)DP^2 = (bd-ac)(ab-cd)\,. \quad (2)$$

If $bd - ac = 0$ or $ad - bc = 0$, then we obtain $a = b$, $c = d$; that is, $ABCD$ is a rhombus and P is on BD, in which case $\angle ABP = \angle ADP$ is obvious. Therefore, we assume that $bd - ac \neq 0$ and $ad - bc \neq 0$.

From (1) and (2), it then follows that $ab - cd \neq 0$ and that $BP : DP = |ad - bc| : |bd - ac|$.

If $ad - bc > 0$, we obtain $[\triangle PCD] > [\triangle PAD]$, and because

$$\begin{aligned}[\triangle PCD] + [\triangle PAB] &= [\triangle PAD] + [\triangle PBC]\,,\\ [\triangle PAB] &< [\triangle PBC]\,;\end{aligned}$$

that is, $bd - ac > 0$.

Similarly, if $ad - bc < 0$, we obtain that $bd - ac < 0$.

Combining these results, we have

$$BP : DP = ad - bc : bd - ac\,. \quad (3)$$

From (1) and (2), it follows that

$$\cos\angle ABP = \frac{a^2+BP^2-c^2}{2aBP} = \frac{2abd - c(a^2+b^2-c^2+d^2)}{2BP(bd-ac)} \quad \text{and}$$

$$\cos\angle ADP = \frac{2abd - c(a^2+b^2-c^2+d^2)}{2DP(ad-bc)}\,.$$

which, together with (3), imply that $\cos\angle ABP = \cos\angle ADP$, so that $\angle ABP = \angle ADP$, as desired.

ACKNOWLEDGEMENT

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References

1. Georg Gunther, *One Problem — Six Solutions*, CRUX with MAYHEM **24** : 2 (1998), 81–97.

2. Jimmy Chui, *Four ways to Count*, CRUX with MAYHEM **25** : 7 (1999), 235–237.

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