THE OLYMPIAD CORNER

No. 216

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As a first set of problems for this issue, we give the problems of the 15th Balkan Mathematical Society, held in Nicosia, Cyprus. My thanks go to the Competitions Committee of the Greek Mathematical Society for making them available for our use.

15th BALKAN MATHEMATICAL OLYMPIAD
Nicosia, 3–9 May 1998, Cyprus

1. Consider the terms of the finite sequence $\left\lfloor \frac{k^2}{1998} \right\rfloor$, $k = 1, 2, \ldots, 1997$, where $\lfloor x \rfloor$ denotes the integral part of x. How many of the terms of this sequence are different?

2. Let n be an integer, $n \geq 2$, and $0 < a_1 < a_2 < \cdots < a_{2n+1}$ be real numbers. Prove that the following inequality holds:

$$\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \cdots - \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}\,.$$

3. Denote by S the set of all points of $\triangle ABC$ except one interior point T. Show that S can be represented as a union of disjoint (line) segments.

4. Prove that the equation $y^2 = x^5 - 4$ has no integer solutions.

As a second set we give the problems of the 1st Mediterranean Mathematical Olympiad, April 22, 1998. Thanks again go to the Competitions Committee of the Greek Mathematical Society.

1st MEDITERRANEAN MATHEMATICAL OLYMPIAD
April 22, 1998

1. [*Greece*]

Let $ABCD$ be a square inscribed in a circle. If M is a point on the arc AB show that $MC \cdot MD > 3\sqrt{3} \cdot MA \cdot MB$.

2. [*Croatia*]

(a) Prove that the polynomial $z^{2n} + z^n + 1$, $n \in \mathbb{N}$, is divisible by the polynomial $z^2 + z + 1$ if and only if n is not a multiple of 3.

(b) Find the necessary and sufficient condition that the natural numbers p, q must satisfy for the polynomial $z^p + z^q + 1$ to be divisible by $z^2 + z + 1$.

3. [*Spain*]

In a triangle ABC, I is the incentre and $D \in (BC)$, $E \in (CA)$, $F \in (AB)$ are the points of tangency of the incircle with the sides of the triangle. Let $M \in (BC)$ be the foot of the interior bisector of $\angle BIC$ and $\{P\} = FE \cap AM$. Prove that DP is the interior bisector of the angle $\angle FDE$.

As a third set provided by the Competitions Committee of the Greek Mathematical Society, we give the Final National Selection Competition for the Greek Team 1998.

FINAL NATIONAL SELECTION COMPETITION 1998
for Greek Team

1. If $x, y, z > 0$, $k > 2$ and $a = x + ky + kz$, $b = kx + y + kz$, $c = kx + ky + z$, show that

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \geq \frac{3}{2k+1}\,.$$

2. Let $ABCD$ be a trapezoid ($AB \parallel CD$) and M, N be points on the lines AD and BC respectively, such that $MN \parallel AB$. Prove that

$$DC \cdot MA + AB \cdot MD = MN \cdot AD\,.$$

3. Prove that if the number $A = \underbrace{1\,1\,1\,\ldots\,1}_{n \text{ digits}}$ is prime then the number n must be prime. Is the converse true?

4. (a) A polynomial $P(x)$ with integer coefficients takes the value -2 for seven distinct integer values of x. Prove that it cannot take the value 1996.

(b) Prove that there are irrational numbers x, y such that the number x^y is rational.

5. Let I be an open interval of width $\frac{1}{n}$, $n \in \mathbb{N} - \{0\}$. Determine the maximum number of irreducible fractions $\frac{a}{b}$ with $1 \leq b \leq n$ that lie in I.

6. The sum of k different even and l different odd natural numbers is 1997. Determine the maximum value the number $3k + 4l$ can take.

Next we give the problems of grade 3 and grade 4 of the 38^{th} National Mathematical Olympiad of Slovenia 1994. [Ed: not the same as grade 3 and grade 4 in North America!!] Thanks go to Mohammed Aassila for sending them for our use.

38^{th} NATIONAL MATHEMATICAL OLYMPIAD OF SLOVENIA 1994

Final Round

Grade Three

1. Let n be a positive integer. Prove: if $2n+1$ and $3n+1$ are perfect squares, then n is divisible by 40.

2. Show that the inequality $\cos(\sin x) > \sin(\cos x)$ holds for every real number x.

3. The polynomial $p(x) = x^3 + ax^2 + bx + c$ has real roots only. Show that the polynomial $q(x) = x^3 - bx^2 + acx - c^2$ has at least one non-negative root.

4. Let the point D on the hypotenuse AC of the right triangle ABC be such that $|AB| = |CD|$. Prove that the bisector of the angle at A, the median through B, and the altitude through D of the triangle ABD have a common point.

Grade Four

1. Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$, for which $f(f(x)) = x + 1$ for every $x \in \mathbb{Z}$.

2. Put a positive integer in every empty field of the table so that you get an arithmetic sequence in every row and every column.

	74			
				186
		103		
0				

3. The sequence 49, 4489, 444889, $44448889\ldots$ is given (in the n^{th} term there are n fours, $n-1$ eights and a nine). Prove that every term of the sequence is a perfect square.

4. Let Q be the mid-point of the side AB of the inscribed quadrilateral $ABCD$ and let S be the intersection of its diagonals. Denote by P and R the orthogonal projections of S on AD and BC respectively. Prove that $|PQ| = |QR|$.

As a final set for this number we give the problems of the Final Round of the 47th Czech and Slovak Mathematical Olympiad. My thanks go to Chris Small, Canadian Team Leader to the 38th IMO in Argentina for collecting them for our use.

47th CZECH AND SLOVAK MATHEMATICAL OLYMPIAD
March 22–25, 1998

1. Find all solutions in the real domain of the equation

$$x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor = 88,$$

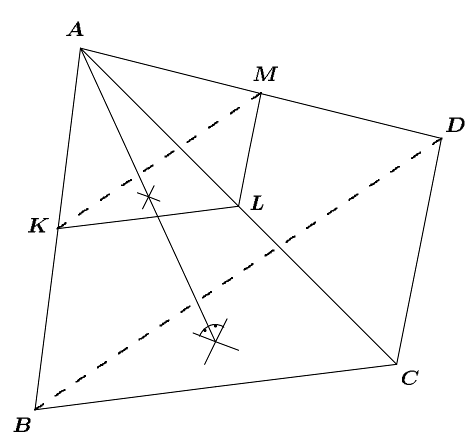
where $\lfloor a \rfloor$ is the integer part of a real number a; that is, the integer satisfying $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$. For instance, $\lfloor 3.7 \rfloor = 3$, $\lfloor -3.7 \rfloor = -4$ and $\lfloor 6 \rfloor = 6$.

2. Show that from any fourteen different natural numbers it is possible to choose, for a suitable k ($1 \leq k \leq 7$), two disjoint k-element subsets $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_k\}$ in such a way that the sums

$$A = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \quad \text{and} \quad B = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_k}$$

differ by less than 0.001; that is $|A - B| < 0.001$.

3. A sphere is inscribed in a given tetrahedron $ABCD$. Its four tangent planes, which are parallel to the faces of the tetrahedron, cut four smaller tetrahedra from the tetrahedron. Prove that the sum of lengths of all their 24 edges is equal to twice the sum of the lengths of the edges of the tetrahedron $ABCD$.



4. For each date of the current year (1998) we evaluate the expression

$$\text{day}^{\text{month}} - \text{year}$$

and then find the highest power of 3 dividing it. For instance, for April 21 we obtain $21^4 - 1998 = 192\,483 = 3^3 \cdot 7129$, which is a multiple of 3^3, but not of 3^4. Find all days for which the corresponding power is the greatest.

5. In the exterior of a circle k a point A is given. Show that the diagonals of all trapezoids which are inscribed into the circle k and whose extended arms intersect at the point A intersect at the same point U.

6. Let a, b, c be positive numbers. Show that the triangle with sides a, b, c exists if and only if the system of equations

$$\frac{y}{z} + \frac{z}{y} = \frac{a}{x}, \quad \frac{z}{x} + \frac{x}{z} = \frac{b}{y}, \quad \frac{x}{y} + \frac{y}{x} = \frac{c}{z}$$

has a solution in the real domain.

Next we turn to solutions from our readers to problems of the Republic of Moldova XL Mathematical Olympiad, 1996 [1999 : 325–326].

REPUBLIC OF MOLDOVA XL MATHEMATICAL OLYMPIAD

Chişinău, 17–20 April, 1996

First Day (Time: 4 hours)

10 Form

1. Let $n = 2^{13} \cdot 3^{11} \cdot 5^7$. Find the number of divisors of n^2 which are less than n and are not divisors of n.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's write-up.

The divisors of n^2 are the integers $2^a \cdot 3^b \cdot 5^c$ where the integers a, b, c satisfy: $0 \leq a \leq 26$, $0 \leq b \leq 22$, $0 \leq c \leq 14$, so that there are $N = 27 \times 23 \times 15$ divisors of n^2 altogether.

Let $N = 2K + 1$. We can partition the set of all divisors of n^2 as follows:

$$\{n\}, \{d_1, d_{K+1}\}, \{d_2, d_{K+2}\}, \ldots, \{d_K, d_{2K}\}$$

where, for $i = 1, 2, \ldots, K$:

$$d_i < n, \quad d_{K+i} > n \quad \text{and} \quad d_i \cdot d_{K+i} = n^2 .$$

From this, we see that exactly K divisors of n^2 are less than n. Moreover, the divisors of n (n excepted) are all among these divisors. Hence, the number we seek is $K - (14 \times 12 \times 8 - 1)$; that is

$$\frac{27 \times 23 \times 15 - 1}{2} - 14 \times 12 \times 8 + 1 \;=\; 3314\,.$$

2. Distinct square trinomials $f(x)$ and $g(x)$ have leading coefficient equal to one. It is known that $f(-12) + f(2000) + f(4000) = g(-12) + g(2000) + g(4000)$. Find all the real values of x which satisfy the equation $f(x) = g(x)$.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by George Evagelopoulos, Athens, Greece. We give the write-up by Evagelopoulos.

Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be two distinct square trinomials.

From the given inequality we get

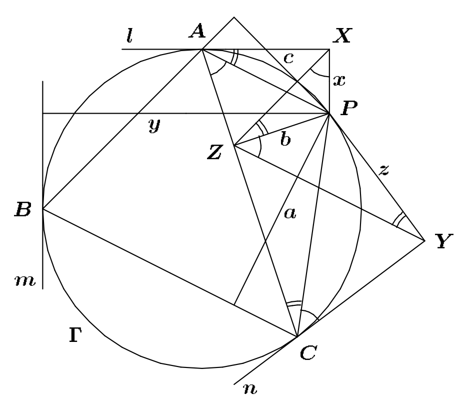
$$1996a + b \;=\; 1996c + d\,.$$

Thus, we have $a \neq c$ and $b \neq d$.

Therefore, $x = 1996$ is the single root of the equation $f(x) = g(x)$.

3. Through the vertices of a triangle tangents to the circumcircle are constructed. The distances of an arbitrary point of the circle to the straight lines containing the sides of the triangle are equal to a, b and c and to the tangents are equal to x, y and z. Prove that $a^2 + b^2 + c^2 = xy + xz + yz$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.



Let the triangle be ABC, and let Γ be the circumcircle of $\triangle ABC$, and let l, m, n be the tangents to Γ at A, B, C, respectively. Suppose that

P is a point on Γ and that a, b, c, are the distances from P to BC, CA, AB, respectively, and that x, y, z, be the distances from P to l, m, n, respectively.

Let X, Y and Z, be the feet of the perpendiculars from P to l, n and AC, respectively. Then $PX = x$, $PY = z$, and $PZ = b$.

Since $\angle PXA = \angle PZA = 90^\circ$, we have that P, X, A, Z, are concyclic, and similarly P, Y, C, Z, are concyclic. Since AX is tangent to Γ, we have

$$\angle PZX \;=\; \angle PAX \;=\; \angle PCA \;=\; \angle PCZ \;=\; \angle PYZ\,. \qquad (1)$$

Similarly we have

$$\angle PXZ \;=\; \angle PAZ \;=\; \angle PAC \;=\; \angle PCY \;=\; \angle PZY\,. \qquad (2)$$

From (1) and (2), we get $\triangle PXZ \sim \triangle PZY$, so that $PX : PZ = PZ : PY$; that is

$$PZ^2 \;=\; PX \cdot PY\,.$$

This implies that $b^2 = xz$. Similarly, we have $a^2 = yz$ and $c^2 = xy$. [Ed.: this is stronger than what was asked for.] Thus, we obtain

$$a^2 + b^2 + c^2 \;=\; xy + xz + yz\,.$$

11–12 Form

1. Prove the equality

$$\frac{1}{666}+\frac{1}{667}+\cdots+\frac{1}{1996} \;=\; 1+\frac{2}{2\cdot3\cdot4}+\frac{2}{5\cdot6\cdot7}+\cdots+\frac{2}{1994\cdot1995\cdot1996}\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the write-up by Wang.

We prove that, in general, for all natural numbers n,

$$\begin{aligned} 1+\frac{2}{2\cdot3\cdot4}+\frac{2}{5\cdot6\cdot7}+\cdots+\frac{2}{(3n-1)(3n)(3n+1)} \\ = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}\,. \end{aligned} \qquad (1)$$

The given equality is the special case when $n = 665$.

To prove (1), let S_1 and S_2 denote the left and right hand sides of (1), respectively. By simple partial fractions, we find that

$$\frac{2}{(3k-1)(3k)(3k+1)} \;=\; \frac{1}{3k-1}-\frac{2}{3k}+\frac{1}{3k+1} \quad \text{for all } k \ge 1\,.$$

Hence,

$$\begin{aligned} S_1 &= 1+\sum_{k=1}^{n}\frac{2}{(3k-1)(3k)(3k+1)} \\ &= 1+\sum_{k=1}^{n}\left(\frac{1}{3k-1}-\frac{2}{3k}+\frac{1}{3k+1}\right) \\ &= 1+\sum_{k=1}^{n}\left(\frac{1}{3k-1}+\frac{1}{3k}+\frac{1}{3k+1}\right)-3\sum_{k=1}^{n}\frac{1}{3k} \\ &= 1+\sum_{k=2}^{3n+1}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k} \\ &= \sum_{k=n+1}^{3n+1}\frac{1}{k} \\ &= S_2\,. \end{aligned}$$

2. Prove that the product of the roots of the equation

$$\sqrt{1996}\cdot x^{\log_{1996} x} \;=\; x^6$$

is an integer number and find the last four digits of this number.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's write-up.

Let $a = \log_{1996} x$. Since $\log_{1996}(\sqrt{1996}) = \frac{1}{2}$, the given equation is equivalent to

$$\frac{1}{2}+a^2 \;=\; 6a \quad \text{(by applying } \log_{1996} \text{ to each side).}$$

Denote by a_1 and a_2 the two roots of this equation. Then the roots of the given equation are $x_1 = 1996^{a_1}$ and $x_2 = 1996^{a_2}$ whose product is $1996^{a_1+a_2} = 1996^6$. Thus, the product x_1x_2 is the integer 1996^6.

Now $1996^6 = (2000-4)^6 = 2^6(2-1000)^6 = 2^6(2^6-6\cdot 2^5\cdot 1000+\cdots)$ [dots represent terms all multiple of 10000, of no influence on the last four digits]. Hence,

$$\begin{aligned} 1996^6 &\equiv 2^6(2^6-192000) \pmod{10000} \\ &\equiv 64(64+8000) \pmod{10000} \\ &\equiv 6096 \pmod{10000}\,, \end{aligned}$$

so that the last four digits of 1996^6 are 6 0 9 6.

3. Two disjoint circles C_1 and C_2 with centres O_1 and O_2 are given. A common exterior tangent touches C_1 and C_2 at points A and B, respectively. The segment O_1O_2 cuts C_1 and C_2 at points C and D, respectively. Prove that:

(a) the points A, B, C and D are concyclic;

(b) the straight lines (AC) and (BD) are perpendicular.

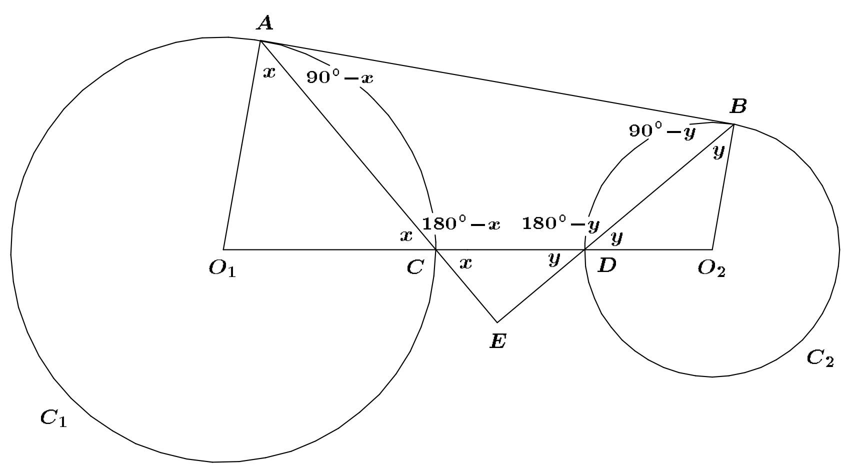
Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Amengual Covas.

(a) Letting the base angles in isosceles triangles AO_1C and BO_2D be x and y, respectively, the sum of the angles in quadrilateral $ABDC$ is

$$(90^\circ - x) + (90^\circ - y) + (180^\circ - y) + (180^\circ - x) \;=\; 360^\circ\,,$$

and we have

$$x + y \;=\; 90^\circ\,. \qquad (1)$$



Hence, in $ABDC$, the angles at A and D add up to

$$(90^\circ - x) + (180^\circ - y) \;=\; 270^\circ - (x + y) \;=\; 270^\circ - 90^\circ \;=\; 180^\circ\,,$$

and thus, $ABDC$ is cyclic. This proves (a).

(b) Let $E = AC \cap CD$. It follows from equation (1) that in triangle CED the angles at C and D add up to 90°. Thus, CED is a right-angled triangle with the right angle at E and AC and BD are in fact perpendicular.

We continue the Moldova XL set with readers' solutions to problems of Day 2 of the Republic of Moldova XL Mathematical Olympiad, 10 and 11-12 forms [1999 : 326–327].

REPUBLIC OF MOLDOVA XL MATHEMATICAL OLYMPIAD

Chişinău, 17–20 April, 1996

Second Day (Time: 4 hours)

10 Form

5. Prove that for all natural numbers $m \geq 2$ and $n \geq 2$ the smallest among the numbers $\sqrt[n]{m}$ and $\sqrt[m]{n}$ does not exceed the number $\sqrt[3]{3}$.

Solutions by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Bornsztein.

For positive integers k, let $x_k = k^{1/k}$. First note that

$$\sqrt[3]{3} \;>\; 1.44\,. \tag{1}$$

Case 1. $m = n$.

From (1), we have $x_1 < x_2 < x_3$.

Suppose that, for a fixed integer $p \geq 3$, we have $x_p \leq x_3$. Then $p^3 \leq 3^p$ (since $(x \mapsto x^{3p})$ is non-decreasing). It follows that:

$$\begin{aligned} 3^{p+1} &\geq 3p^3 \\ &= p^3 + 3p^2 + 3p + (p-3)p^2 + (p^2-3)p \\ &> p^3 + 3p^2 + 3p + 1 \quad (\text{since } p \geq 3) \\ &= (p+1)^3\,. \end{aligned}$$

Then, we have $x_{p+1} \leq x_3$.

By induction, we then have $x_p \leq x_3$ for all $p \geq 3$. Thus,

$$x_p \;\leq\; x_3 \quad \text{for all positive integers } p\,.$$

Case 2. $m \neq n$.

With no loss of generality, we may suppose that $m < n$.

Then

$$\begin{aligned} \sqrt[n]{m} &< \sqrt[n]{n} \quad ((x \mapsto \sqrt[n]{x}) \text{ is increasing}) \\ &\leq \sqrt[3]{3} \quad \text{from case 1.} \end{aligned}$$

Then, for all positive integers m, n, the smallest among $\sqrt[m]{n}$ and $\sqrt[n]{m}$ does not exceed the number $\sqrt[3]{3}$.

Remark. Equality occurs if and only if $m = n = 3$.

6. Prove the inequality $2^{a_1}+2^{a_2}+\cdots+2^{a_{1996}} \leq 1995+2^{a_1+a_2+\cdots+a_{1996}}$ for any real non-positive numbers $a_1, a_2, \ldots, a_{1996}$.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Lee.

First we prove the following lemma.

Lemma. Let $f : X \to \mathbb{R}$ be a function for $X \subset \mathbb{R}$.

Suppose that $f(x+y) = f(x)f(y)$ for all $x, y \in X$. For $a_1, a_2, \ldots, a_n \in X$, let $s_i = \sum_{j=1}^{i} a_j$ for $1 \leq i \leq n$. Then we have

$$f(s_n) + n - 1 = \sum_{i=1}^{n-1}(f(s_i)-1)(f(a_{i+1})-1) + \sum_{i=1}^{n} f(a_i)$$

for $n \geq 2$.

Proof. We have

$$\begin{aligned}
&\sum_{i=1}^{n-1}(f(s_i)-1)(f(a_{i+1})-1) + \sum_{i=1}^{n} f(a_i) \\
&= \sum_{i=1}^{n-1} f(s_i)f(a_{i+1}) - \sum_{i=1}^{n-1} f(s_i) - \sum_{i=1}^{n-1} f(a_{i+1}) + \sum_{i=1}^{n-1} 1 + \sum_{i=1}^{n} f(a_i) \\
&= \sum_{i=1}^{n-1} f(s_i + a_{i+1}) - \sum_{i=1}^{n-1} f(s_i) - \sum_{i=2}^{n} f(a_i) + \sum_{i=1}^{n} f(a_i) + n - 1 \\
&= \sum_{i=1}^{n-1} f(s_{i+1}) - \sum_{i=1}^{n-1} f(s_i) - \sum_{i=2}^{n} f(a_i) + \sum_{i=1}^{n} f(a_i) + (n-1) \\
&= \sum_{i=2}^{n} f(s_i) - \sum_{i=1}^{n-1} f(s_i) - \sum_{i=2}^{n} f(a_i) + \sum_{i=1}^{n} f(a_i) + (n-1) \\
&= f(s_n) - f(s_1) + f(a_1) + (n-1) = f(s_n) + (n-1),
\end{aligned}$$

since $s_1 = a_1$. ■

Since $2^{x+y} = 2^x \cdot 2^y$ for all $x, y \in \mathbb{R}$, we have

$$2^{a_1+\cdots+a_{1996}} + 1995 = \sum_{i=1}^{1995}\left(2^{\sum_{j=1}^{i} a_j} - 1\right)(2^{a_{i+1}} - 1) + \sum_{i=1}^{1996} 2^{a_i}$$

from the lemma. Since $2^{\alpha} \leq 1$ for all $\alpha \leq 0$, we easily deduce that

$$\sum_{i=1}^{1996} 2^{a_i} \leq 2^{a_1+\cdots+a_{1996}} + 1995, \qquad \text{as desired.}$$

Remark. Bornsztein notes that the inequality is strict unless at most one of the a_i is non-zero.

7. The perpendicular bisector to the side $[BC]$ of a triangle ABC intersects the straight line (AC) at a point M and the perpendicular bisector to the side $[AC]$ intersects the straight line (BC) at a point N. Let O be the centre of the circumcircle to the triangle ABC. Prove that:

(a) points A, B, M, N and O lie on a circle S;

(b) the radius of S equals the radius of the circumcircle of the triangle MNC.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give the argument of Amengual Covas.

(a) If $\angle C = 90°$ then O is the mid-point of AB, and if we consider the line AB as a degenerate circle, it also contains the (infinite) points M and N.

Suppose that in $\triangle ABC$, we have $\angle C < 90°$.

Since M lies on the perpendicular bisector to the side BC, $\triangle BMC$ is isosceles and its exterior angle at M is $\angle AMB = 2\angle C$.

Similarly, $\angle ANB = 2\angle C$.

Now, AB subtends at the centre O twice the angle it subtends at C on the circumcircle, implying that

$$\angle AOB \;=\; 2\angle C\,.$$

Consequently, we deduce that the three points O, M, N all lie on the arc of a circle S on the chord AB which contains the angle $2\angle C$.

This proves (a).

(b) Since quadrilateral $AMNB$ is cyclic, we immediately have that $\angle CAB = 180° - \angle MNB = \angle MNC$, so that triangles MNC and BAC are similar.

Hence,

$$\frac{MN}{AB} \;=\; \frac{CM}{BC} \;=\; \frac{CM}{2\cdot CA'}\,,$$

where A' is the mid-point of the side BC.

Now, in right triangle $CA'M$,

$$CA' \;=\; CM\cdot\cos C\,,$$

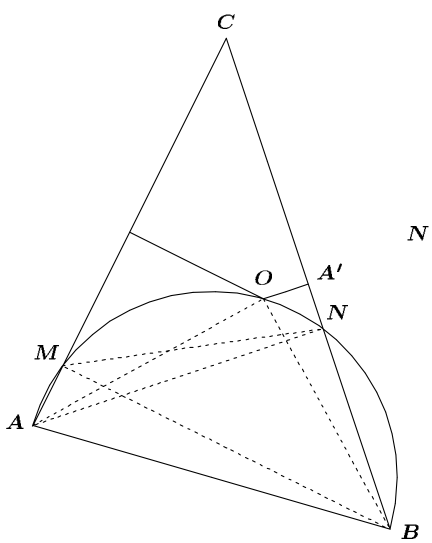
and therefore,

$$\frac{MN}{AB} \;=\; \frac{1}{2\cos C}\,.$$

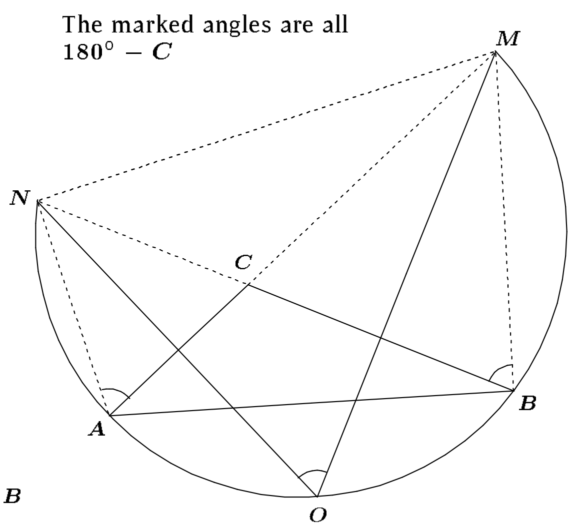
Hence, the ratio between the corresponding circumradii of $\triangle MNC$ and $\triangle ABC$ will be $1 : 2\cos C$, yielding

$$\begin{aligned}\text{Circumradius of } \triangle MNC &= \frac{\text{circumradius of } \triangle ABC}{2\cos C} \\ &= \frac{\frac{c}{2\sin C}}{2\cos C} \\ &= \frac{c}{2\sin 2C} \\ &= \text{circumradius of } \triangle AMB \\ &= \text{radius of } S\,.\end{aligned}$$

The present solution can be applied with minor modifications to the case $\angle C > 90°$ as well.



The case $C < 90°$



The case $C > 90°$

11–12 Form

5. Let p be the number of functions defined on the set $\{1, 2, \ldots, m\}$, $m \in N^*$, with values in the set $\{1, 2, \ldots, 35, 36\}$ and q be the number of functions defined on the set $\{1, 2, \ldots, n\}$, $n \in N^*$, with values in the set $\{1, 2, 3, 4, 5\}$. Find the least possible value for the expression $|p - q|$.

Solution by Pierre Bornsztein, Pontoise, France.

Let $m, n \in N^*$. We have $p = 36^m$ and $q = 5^n$. The problem is then to find the least possible value of $|36^m - 5^n|$ over all $m, n \in N^*$.

For $m, n \in N^*$,

$$\begin{aligned} 36^m &= a \pmod{100} \quad \text{where } a \in \{36, 96, 56, 16, 76\}, \\ 5^n &= 25 \pmod{100} \quad \text{for } n \geq 2. \end{aligned}$$

Since $36 - 5^2 = 11$, the least possible value is then 9 or 11. But

$$36^m - 5^n = -5^n \pmod 9 \neq 0 \pmod 9,$$

so that $36^m - 5^n = \pm 9$ is impossible.

It follows that the least possible value of $|36^m - 5^n|$ is 11.

6. Solve in real numbers the equation

$$2x^2 - 3x = 1 + 2x\sqrt{x^2 - 3x}.$$

Solutions by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Evageloupolos.

The given equation

$$2x^2 - 3x = 1 + 2x\sqrt{x^2 - 3x}$$

is equivalent to

$$\begin{aligned} x^2 - 2x\sqrt{x^2 - 3x} + x^2 - 3x &= 1, \\ \text{or} \quad (x - \sqrt{x^2 - 3x})^2 - 1 &= 0, \\ \text{or} \quad (x - \sqrt{x^2 - 3x} + 1)(x - \sqrt{x^2 - 3x} - 1) &= 0. \end{aligned}$$

The equation $x - \sqrt{x^2 - 3x} - 1 = 0$ has no roots, whereas the equation $x - \sqrt{x^2 - 3x} + 1 = 0$ has the root $x = -\frac{1}{5} = -0.2$.

7. On a sphere distinct points A, B, C and D are chosen so that segments $[AB]$ and $[CD]$ cut each other at point F, and points A, C and F are equidistant to a point E. Prove that the straight lines (BD) and (EF) are perpendicular

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give the solution of Bataille.

Points A, C, F are on a sphere S with centre E, and the plane P determined by these points cuts S along a circle Γ. The centre I of Γ is the projection of E onto P so that $(EI) \perp P$. Hence, $(EI) \perp (BD)$, since points B, D are in P.

Now, P cuts the given sphere along a circle γ containing the points A, B, C, D and, in the plane P, the circles γ and Γ intersect at A and C.

In P, consider the inversion with centre F and power $\overline{FA} \cdot \overline{FB} = \overline{FC} \cdot \overline{FD}$ = power of F with respect to γ. Under this inversion, the line

(BD) is transformed into Γ, so that the centre I of Γ is on the perpendicular to (BD) through F. Hence, $(IF) \perp (BD)$.

Since (BD) is perpendicular to (EI) and to (IF), it is perpendicular to the plane (EIF) and consequently to the line (EF).

8. 20 children attend a rural elementary school. Every two children have a grandfather in common. Prove that some grandfather has not less than 14 grandchildren in this school.

Solution by Pierre Bornsztein, Pontoise, France.

Let $c_1, c_2, \ldots, c_{20}$ be the children. For a grandfather g_k we denote by $G_k = \{c_i \mid c_i \text{ is a grandchild of } g_k\}$. Since there is a finite number of children, there is also a finite number of grandfathers.

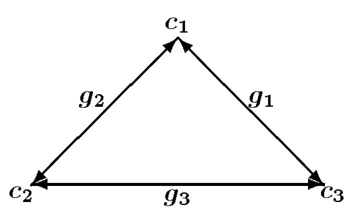
Let g_1 be a grandfather such that $n_1 = \text{Card}\, G_1$ is maximal. It is clear that $n_1 \geq 2$.

Suppose, for a contradiction, that $n_1 \leq 13$.

Say $c_1 \in G_1$. There exists a child, say c_2, such that $c_2 \notin G_1$. c_1 and c_2 have a common grandfather, say g_2, and $g_1 \neq g_2$.

If each grandchild of g_1 has grandfather g_2 too, then $G_1 \cup \{c_2\} \subset G_2$. Thus, $n_2 = \text{Card}\,(G_2) \geq n + 1$, which contradicts that n_1 is maximal.

Thus, there exists a grandchild of g_1, say c_3, that is not a grandchild of g_2. Then, c_2 and c_3 have a common grandfather, say g_3, and $g_3 \notin \{g_1, g_2\}$. Then, we have



For $i \geq 4$, c_i has a common grandfather with each of the children c_1, c_2, c_3. Then, the two grandfathers of c_i are in the set $\{g_1, g_2, g_3\}$.

It follows that the 40 grandfathers of the c_i's form the set $\{g_1, g_2, g_3\}$ since $40 = 3 \times 13 + 1$, from the Pigeonhole Principle, at least one of g_1, g_2, g_3 has at least 14 grandchildren.

Thus, $n_1 \geq 14$, a contradiction.

Thus, $n_1 \geq 14$, and we are done.

Remark. 14 cannot be improved. If $c_1, \ldots, c_{14}$ have g_1 for a grandfather, $c_1, \ldots, c_7$ have g_2, $c_8, \ldots, c_{14}$ have g_3, $c_{15}, \ldots, c_{20}$ have g_2 and g_3. Thus, $n_1 = 14$ and $n_2 = n_3 = 13$.

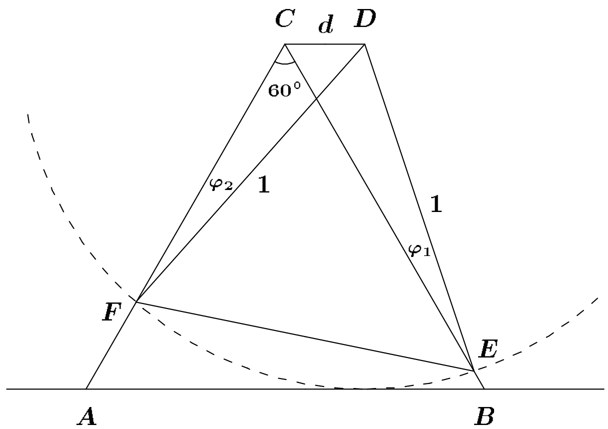
For n children ($n \geq 3$), with the same reasoning we obtain:

$$n_1 \ \geq \ \left\lfloor \frac{2n+2}{3} \right\rfloor + 1\,, \quad \text{where } \lfloor \ldots \rfloor \text{ denotes the integer part.}$$

Next we turn to the November 1999 number and reader solutions and comments about problems of the 31st Canadian Mathematical Olympiad [1999 : 387–388].

2. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and centre on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.



The rolling circle intersects CB at E and CA at F. D is the centre of the rolling circle.

$$\angle CED = \varphi_1 \,, \quad \angle CFD = \varphi_2 \,, \quad \angle DCE = 60^\circ \,, \quad \angle DCF = 120^\circ \,,$$

$$\overline{CD} = d \,, \quad DE = DF = 1 \,.$$

From the Sine Law in $\triangle CDE$, we have

$$d : \sin\varphi_1 = 1 : \sin 60^\circ \Longrightarrow \sin\varphi_1 = d\sin 60^\circ \,.$$

From the Sine Law in $\triangle CDF$, we have

$$\begin{aligned} d : \sin\varphi_2 &= 1 : \sin 120^\circ \Longrightarrow \sin\varphi_2 = d\sin 120^\circ \\ &\Longrightarrow \sin\varphi_1 = \sin\varphi_2 \Longrightarrow \varphi_1 = \varphi_2 = \varphi \,. \\ &\Longrightarrow \text{Quad}CDEF \text{ can be inscribed in a circle} \\ &\Longrightarrow \angle EDF = \angle ECF = \angle BCA = 60^\circ = \text{ constant.} \end{aligned}$$

5. Let x, y and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

$$x^2y + y^2z + z^2x \le \frac{4}{27} \,,$$

and find when equality occurs.

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

In my proposed problem #1292 (Math. Mag., 62 (1989) 137), one was to determine the maximum value of

$$x_1^2x_2 + x_2^2x_3 + \cdots + x_n^2x_1$$

given that $x_1 + x_2 + \cdots + x_n = 1$, $x_1, x_2, \ldots, x_n \geq 0$ and $n \geq 3$.

The published elegant solution was by the 1988 USA Olympiad Math Team. First it was shown that the maximum would occur if one took $x_i = 0$ for $i > 3$. Then, without loss of generality, the indices were cycled so that $(x_2 - x_1)(x_2 - x_3) \leq 0 \leq x_1x_2$, so that

$$x_1^2x_2 + (x_2 - x_1)(x_2 - x_3)x_3 \;\leq\; x_1^2x_2 + x_1x_2x_3 \,.$$

These terms were then arranged to give

$$\begin{aligned} x_1^2x_2 + x_2^2x_3 + x_3^2x_1 \;&\leq\; (x_1 + x_3)^2x_2 \;=\; (1 - x_2)^2x_2 \\ &\leq\; \frac{4\left[\frac{1-x_2}{2} + \frac{1-x_2}{2} + x_2\right]^3}{27} \;=\; \frac{4}{27}\,, \end{aligned}$$

the last by the AM–GM inequality and with equality if $x_2 = \frac{1}{3}$, and then $x_1 = \frac{2}{3}$ and $x_3 = 0$, or vice-versa.

For a more direct solution but not particularly elegant, we can assume without loss of generality that $x \geq y \geq z$ (if we changed the cyclic order of these, $x^2y + y^2z + z^2x$ would be decreased). In terms of a homogeneous inequality, we want to show that

$$4(x + y + z)^3 \;\geq\; 27(x^2y + y^2z + z^2x)\,.$$

We now set $z = a$, $y = a+b$, and $x = a+b+c$ where $a, b, c \geq 0$. Expanding out, we get

$$27a^3 + 54ab + 27a^2c + 36ab^2 + 9ac^2 + 36abc + (5b + 4c)(b - c)^2 \;\geq\; 0$$

and which is now evident. There is equality, if and only if, $a = 0$ and $b = c$.

That completes the *Corner* for this issue. Send me your Olympiad contests and your nice solutions and generalizations.