THE ACADEMY CORNER

No. 43

Bruce Shawyer

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Once again, we are pleased to present the Bernoulli Trials 2001. Thanks to Christopher Small for forwarding this to us.

The Bernoulli Trials 2001

Ian VanderBurgh and Christopher G. Small

It was a great turnout at the University of Waterloo on the first Saturday in March 2001, as 37 contestants matched wits for the coveted title of Bernoulli Trials Champion. As always, the last person left standing in any Bernoulli Trials competition owes this status to a combination of mathematical prowess and old fashioned good luck.

Contestants in the Bernoulli Trials proceed through the trials answering each mathematical question by "true" or "false". A contestant can carry one mistake into future rounds, but after two mistakes he or she is eliminated from formal competition. Since no solutions need be given, the contestant who is truly clueless can flip a mental coin and hope it lands the right way. (No calculators are allowed, but randomisers such as coins are considered within contest regulations.)

After 11 rounds of competition there were two battle-weary mathematicians left. Sabin Cautis, a veteran of many IMO and Putnam contests had accumulated one error on a previous round. His adversary, Marshall Drew-Brook, was a relative newcomer who had shown outstanding tenacity in solving one question after another. Like Cautis, he had also accumulated one error. Therefore, it came down to a final question on the 12th round. If both answered the same way, another question would be required. If their answers were different, a winner would be declared. As fate would have it, the answers were different.

First place in the competition goes to Marshall Drew-Brook, who won 200 dollars (awarded in coins). Sabin Cautis took second place and received 100 dollars. Cash prizes were also awarded to people ranked 3rd through 5th in the Bernoulli Trials. Third place went to Joel Kamnitzer, fourth place to Mark Mann, and fifth place to Masoud Kamgarpour.

Here are the questions.

The Bernoulli Trials 2001

1. To win a contest, a contestant must answer questions in order from a list of 2001 questions.

   - Questions are chosen at random from the list, and continue until he gives an incorrect answer. Of course, no question is ever asked twice.
   - It is known that 501 of the questions are too difficult for him. Thus, he will answer these questions incorrectly if any is asked. He will answer the remaining questions correctly.

   TRUE or FALSE? The probability that he misses on the 501^{st} question is

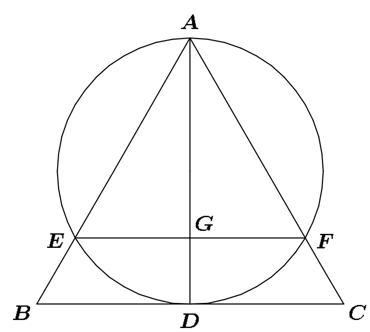
$$\frac{\binom{1500}{1000}}{\binom{2001}{1500}}.$$

2. Tony and Maria are training for a race by running all the way up and down a 700 metre long slope.

   - They run up the slope at constant speeds, but their speeds differ.
   - Coming down the slope, each runs at double his or her uphill speed.
   - Maria reaches the top first, and immediately starts running back down, meeting Tony 70 m from the top.

   TRUE or FALSE? When Maria reaches the bottom of the hill, Tony is 240 metres behind.

3. Altitude AD of triangle $\triangle ABC$ is a diameter of the circle shown. $\triangle ABC$ is equilateral. The circle intersects AB and AC at E and F respectively.



   TRUE or FALSE? $\dfrac{EF}{BC} = \dfrac{3}{4}$.

4. Define

$$H_n = \frac{1}{n} + \frac{1}{n+3} + \cdots + \frac{1}{n+3(n-1)}.$$

TRUE or FALSE? $\lim_{n\to\infty} H_n \geq \frac{1}{2}$.

5. Let $d(m)$ denote the number of divisors of m including 1 and m itself.

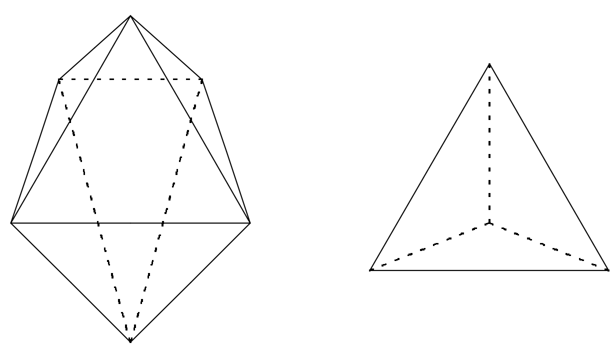
TRUE or FALSE? There exist infinitely many positive integers k such that $d\left(2001^n\right) = 2kn + 1$ has a solution in positive integers n.

6. Alice and Barbara play the following game.

- Alice and Barbara alternate choosing numbers from 1 to 2001, inclusive.
- A number, once chosen cannot be chosen again.
- Play continues until exactly two numbers are left.
- Alice wins if the remaining numbers are relatively prime. Barbara wins if the two numbers are not relatively prime.
- Alice goes first.

TRUE or FALSE? With best play by both sides, Barbara wins.

7. A tetrahedron and an octahedron are built from a common stock of equilateral triangles. The volume of the tetrahedron is 1.



TRUE or FALSE? The volume of the octahedron is 4.

8. Let f be a positive continuous real-valued function defined on the positive real numbers, satisfying

$$\frac{1}{a-b}\int_b^a f(x)\,dx = \sqrt{f(a)\,f(b)}$$

for all $a > b > 0$. Suppose also that $f(1) = 2001$.

TRUE or FALSE? $f(x) = 2001$ for all $x > 0$.

9. In the following equation, each letter stands for a numeral in base 10:

$$6 \times \mathtt{FORWAT} = 7 \times \mathtt{WATFOR}$$

TRUE or FALSE? $\mathtt{W} = 4$.

10. TRUE or FALSE? There exists a triangle ABC with the tangents of the internal angles satisfying $\tan A = x$, $\tan B = 1 + x$, $\tan C = 1 - x$ for some real value x.

11. Suppose that $x^{x^{2001}} = 2001$ where $x > 0$.

TRUE or FALSE? $x > 12.5^{0.0014}$.

12. TRUE or FALSE?

$$2^{1001} \;\Big|\; \lfloor (1+\sqrt{3})^{2001} \rfloor .$$

Who wrote the following?

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere
In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.

THE OLYMPIAD CORNER

No. 216

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a first set of problems for this issue, we give the problems of the 15^{th} Balkan Mathematical Society, held in Nicosia, Cyprus. My thanks go to the Competitions Committee of the Greek Mathematical Society for making them available for our use.

15^{th} BALKAN MATHEMATICAL OLYMPIAD
Nicosia, 3–9 May 1998, Cyprus

1. Consider the terms of the finite sequence $\left\lfloor \frac{k^2}{1998} \right\rfloor$, $k = 1, 2, \ldots, 1997$, where $\lfloor x \rfloor$ denotes the integral part of x. How many of the terms of this sequence are different?

2. Let n be an integer, $n \geq 2$, and $0 < a_1 < a_2 < \cdots < a_{2n+1}$ be real numbers. Prove that the following inequality holds:

$$\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \cdots - \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}\,.$$

3. Denote by S the set of all points of $\triangle ABC$ except one interior point T. Show that S can be represented as a union of disjoint (line) segments.

4. Prove that the equation $y^2 = x^5 - 4$ has no integer solutions.

As a second set we give the problems of the 1^{st} Mediterranean Mathematical Olympiad, April 22, 1998. Thanks again go to the Competitions Committee of the Greek Mathematical Society.

1^{st} MEDITERRANEAN MATHEMATICAL OLYMPIAD
April 22, 1998

1. [*Greece*]

Let $ABCD$ be a square inscribed in a circle. If M is a point on the arc AB show that $MC \cdot MD > 3\sqrt{3} \cdot MA \cdot MB$.

2. [*Croatia*]

(a) Prove that the polynomial $z^{2n} + z^n + 1$, $n \in \mathbb{N}$, is divisible by the polynomial $z^2 + z + 1$ if and only if n is not a multiple of 3.

(b) Find the necessary and sufficient condition that the natural numbers p, q must satisfy for the polynomial $z^p + z^q + 1$ to be divisible by $z^2 + z + 1$.

3. [*Spain*]

In a triangle ABC, I is the incentre and $D \in (BC)$, $E \in (CA)$, $F \in (AB)$ are the points of tangency of the incircle with the sides of the triangle. Let $M \in (BC)$ be the foot of the interior bisector of $\angle BIC$ and $\{P\} = FE \cap AM$. Prove that DP is the interior bisector of the angle $\angle FDE$.

As a third set provided by the Competitions Committee of the Greek Mathematical Society, we give the Final National Selection Competition for the Greek Team 1998.

FINAL NATIONAL SELECTION COMPETITION 1998
for Greek Team

1. If $x, y, z > 0$, $k > 2$ and $a = x + ky + kz$, $b = kx + y + kz$, $c = kx + ky + z$, show that

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \geq \frac{3}{2k+1}.$$

2. Let $ABCD$ be a trapezoid ($AB \parallel CD$) and M, N be points on the lines AD and BC respectively, such that $MN \parallel AB$. Prove that

$$DC \cdot MA + AB \cdot MD = MN \cdot AD.$$

3. Prove that if the number $A = \underbrace{1\,1\,1\,\ldots\,1}_{n \text{ digits}}$ is prime then the number n must be prime. Is the converse true?

4. (a) A polynomial $P(x)$ with integer coefficients takes the value -2 for seven distinct integer values of x. Prove that it cannot take the value 1996.

(b) Prove that there are irrational numbers x, y such that the number x^y is rational.

5. Let I be an open interval of width $\frac{1}{n}$, $n \in \mathbb{N} - \{0\}$. Determine the maximum number of irreducible fractions $\frac{a}{b}$ with $1 \leq b \leq n$ that lie in I.

6. The sum of k different even and l different odd natural numbers is 1997. Determine the maximum value the number $3k + 4l$ can take.

Next we give the problems of grade 3 and grade 4 of the 38^{th} National Mathematical Olympiad of Slovenia 1994. [Ed: not the same as grade 3 and grade 4 in North America!!] Thanks go to Mohammed Aassila for sending them for our use.

38^{th} NATIONAL MATHEMATICAL OLYMPIAD OF SLOVENIA 1994

Final Round

Grade Three

1. Let n be a positive integer. Prove: if $2n+1$ and $3n+1$ are perfect squares, then n is divisible by 40.

2. Show that the inequality $\cos(\sin x) > \sin(\cos x)$ holds for every real number x.

3. The polynomial $p(x) = x^3 + ax^2 + bx + c$ has real roots only. Show that the polynomial $q(x) = x^3 - bx^2 + acx - c^2$ has at least one non-negative root.

4. Let the point D on the hypotenuse AC of the right triangle ABC be such that $|AB| = |CD|$. Prove that the bisector of the angle at A, the median through B, and the altitude through D of the triangle ABD have a common point.

Grade Four

1. Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$, for which $f(f(x)) = x + 1$ for every $x \in \mathbb{Z}$.

2. Put a positive integer in every empty field of the table so that you get an arithmetic sequence in every row and every column.

	74			
				186
		103		
0				

3. The sequence 49, 4489, 444889, $44448889\ldots$ is given (in the n^{th} term there are n fours, $n-1$ eights and a nine). Prove that every term of the sequence is a perfect square.

4. Let Q be the mid-point of the side AB of the inscribed quadrilateral $ABCD$ and let S be the intersection of its diagonals. Denote by P and R the orthogonal projections of S on AD and BC respectively. Prove that $|PQ| = |QR|$.

As a final set for this number we give the problems of the Final Round of the 47th Czech and Slovak Mathematical Olympiad. My thanks go to Chris Small, Canadian Team Leader to the 38th IMO in Argentina for collecting them for our use.

47th CZECH AND SLOVAK MATHEMATICAL OLYMPIAD

March 22–25, 1998

1. Find all solutions in the real domain of the equation

$$x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor \;=\; 88\,,$$

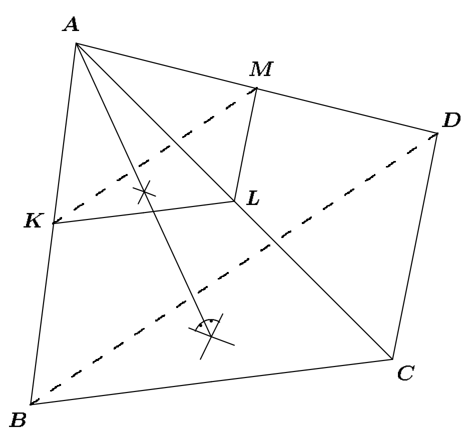
where $\lfloor a \rfloor$ is the integer part of a real number a; that is, the integer satisfying $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$. For instance, $\lfloor 3.7 \rfloor = 3$, $\lfloor -3.7 \rfloor = -4$ and $\lfloor 6 \rfloor = 6$.

2. Show that from any fourteen different natural numbers it is possible to choose, for a suitable k $(1 \leq k \leq 7)$, two disjoint k-element subsets $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_k\}$ in such a way that the sums

$$A \;=\; \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \quad \text{and} \quad B \;=\; \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_k}$$

differ by less than 0.001; that is $|A - B| < 0.001$.

3. A sphere is inscribed in a given tetrahedron $ABCD$. Its four tangent planes, which are parallel to the faces of the tetrahedron, cut four smaller tetrahedra from the tetrahedron. Prove that the sum of lengths of all their 24 edges is equal to twice the sum of the lengths of the edges of the tetrahedron $ABCD$.



4. For each date of the current year (1998) we evaluate the expression

$$\text{day}^{\text{month}} - \text{year}$$

and then find the highest power of 3 dividing it. For instance, for April 21 we obtain $21^4 - 1998 = 192\,483 = 3^3 \cdot 7129$, which is a multiple of 3^3, but not of 3^4. Find all days for which the corresponding power is the greatest.

5. In the exterior of a circle k a point A is given. Show that the diagonals of all trapezoids which are inscribed into the circle k and whose extended arms intersect at the point A intersect at the same point U.

6. Let a, b, c be positive numbers. Show that the triangle with sides a, b, c exists if and only if the system of equations

$$\frac{y}{z} + \frac{z}{y} = \frac{a}{x}, \quad \frac{z}{x} + \frac{x}{z} = \frac{b}{y}, \quad \frac{x}{y} + \frac{y}{x} = \frac{c}{z}$$

has a solution in the real domain.

Next we turn to solutions from our readers to problems of the Republic of Moldova XL Mathematical Olympiad, 1996 [1999 : 325–326].

REPUBLIC OF MOLDOVA XL MATHEMATICAL OLYMPIAD

Chişinău, 17–20 April, 1996

First Day (Time: 4 hours)

10 Form

1. Let $n = 2^{13} \cdot 3^{11} \cdot 5^7$. Find the number of divisors of n^2 which are less than n and are not divisors of n.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's write-up.

The divisors of n^2 are the integers $2^a \cdot 3^b \cdot 5^c$ where the integers a, b, c satisfy: $0 \leq a \leq 26$, $0 \leq b \leq 22$, $0 \leq c \leq 14$, so that there are $N = 27 \times 23 \times 15$ divisors of n^2 altogether.

Let $N = 2K + 1$. We can partition the set of all divisors of n^2 as follows:

$$\{n\},\ \{d_1, d_{K+1}\},\ \{d_2, d_{K+2}\},\ \ldots,\ \{d_K, d_{2K}\}$$

where, for $i = 1, 2, \ldots, K$:

$$d_i < n\,, \quad d_{K+i} > n \quad \text{and} \quad d_i \cdot d_{K+i} = n^2\,.$$

From this, we see that exactly K divisors of n^2 are less than n. Moreover, the divisors of n (n excepted) are all among these divisors. Hence, the number we seek is $K - (14 \times 12 \times 8 - 1)$; that is

$$\frac{27 \times 23 \times 15 - 1}{2} - 14 \times 12 \times 8 + 1 \;=\; 3314\,.$$

2. Distinct square trinomials $f(x)$ and $g(x)$ have leading coefficient equal to one. It is known that $f(-12) + f(2000) + f(4000) = g(-12) + g(2000) + g(4000)$. Find all the real values of x which satisfy the equation $f(x) = g(x)$.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; and by George Evagelopoulos, Athens, Greece. We give the write-up by Evagelopoulos.

Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be two distinct square trinomials.

From the given inequality we get

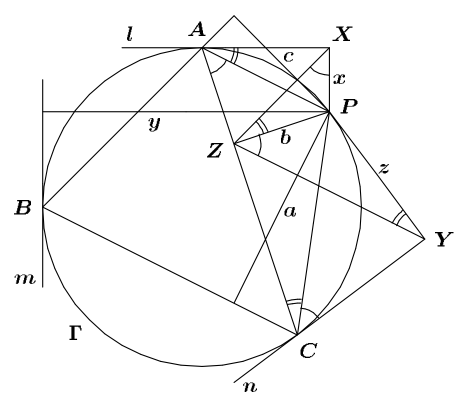
$$1996a + b \;=\; 1996c + d\,.$$

Thus, we have $a \neq c$ and $b \neq d$.

Therefore, $x = 1996$ is the single root of the equation $f(x) = g(x)$.

3. Through the vertices of a triangle tangents to the circumcircle are constructed. The distances of an arbitrary point of the circle to the straight lines containing the sides of the triangle are equal to a, b and c and to the tangents are equal to x, y and z. Prove that $a^2 + b^2 + c^2 = xy + xz + yz$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.



Let the triangle be ABC, and let Γ be the circumcircle of $\triangle ABC$, and let l, m, n be the tangents to Γ at A, B, C, respectively. Suppose that

P is a point on Γ and that a, b, c, are the distances from P to BC, CA, AB, respectively, and that x, y, z, be the distances from P to l, m, n, respectively.

Let X, Y and Z, be the feet of the perpendiculars from P to l, n and AC, respectively. Then $PX = x$, $PY = z$, and $PZ = b$.

Since $\angle PXA = \angle PZA = 90^\circ$, we have that P, X, A, Z, are concyclic, and similarly P, Y, C, Z, are concyclic. Since AX is tangent to Γ, we have

$$\angle PZX = \angle PAX = \angle PCA = \angle PCZ = \angle PYZ . \qquad (1)$$

Similarly we have

$$\angle PXZ = \angle PAZ = \angle PAC = \angle PCY = \angle PZY . \qquad (2)$$

From (1) and (2), we get $\triangle PXZ \sim \triangle PZY$, so that $PX : PZ = PZ : PY$; that is

$$PZ^2 = PX \cdot PY .$$

This implies that $b^2 = xz$. Similarly, we have $a^2 = yz$ and $c^2 = xy$. [Ed.: this is stronger than what was asked for.] Thus, we obtain

$$a^2 + b^2 + c^2 = xy + xz + yz .$$

11–12 Form

1. Prove the equality

$$\frac{1}{666}+\frac{1}{667}+\cdots+\frac{1}{1996} = 1+\frac{2}{2\cdot 3\cdot 4}+\frac{2}{5\cdot 6\cdot 7}+\cdots+\frac{2}{1994\cdot 1995\cdot 1996} .$$

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the write-up by Wang.

We prove that, in general, for all natural numbers n,

$$\begin{aligned} 1 + \frac{2}{2\cdot 3\cdot 4} + \frac{2}{5\cdot 6\cdot 7} + \cdots + \frac{2}{(3n-1)(3n)(3n+1)} \\ = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} . \end{aligned} \qquad (1)$$

The given equality is the special case when $n = 665$.

To prove (1), let S_1 and S_2 denote the left and right hand sides of (1), respectively. By simple partial fractions, we find that

$$\frac{2}{(3k-1)(3k)(3k+1)} = \frac{1}{3k-1} - \frac{2}{3k} + \frac{1}{3k+1} \quad \text{for all } k \geq 1 .$$

Hence,

$$\begin{aligned} S_1 &= 1+\sum_{k=1}^{n}\frac{2}{(3k-1)(3k)(3k+1)} \\ &= 1+\sum_{k=1}^{n}\left(\frac{1}{3k-1}-\frac{2}{3k}+\frac{1}{3k+1}\right) \\ &= 1+\sum_{k=1}^{n}\left(\frac{1}{3k-1}+\frac{1}{3k}+\frac{1}{3k+1}\right)-3\sum_{k=1}^{n}\frac{1}{3k} \\ &= 1+\sum_{k=2}^{3n+1}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k} \\ &= \sum_{k=n+1}^{3n+1}\frac{1}{k} \\ &= S_2\,. \end{aligned}$$

2. Prove that the product of the roots of the equation

$$\sqrt{1996}\cdot x^{\log_{1996} x} \;=\; x^6$$

is an integer number and find the last four digits of this number.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's write-up.

Let $a = \log_{1996} x$. Since $\log_{1996}(\sqrt{1996}) = \frac{1}{2}$, the given equation is equivalent to

$$\frac{1}{2}+a^2 \;=\; 6a \quad \text{(by applying } \log_{1996} \text{ to each side).}$$

Denote by a_1 and a_2 the two roots of this equation. Then the roots of the given equation are $x_1 = 1996^{a_1}$ and $x_2 = 1996^{a_2}$ whose product is $1996^{a_1+a_2} = 1996^6$. Thus, the product x_1x_2 is the integer 1996^6.

Now $1996^6 = (2000-4)^6 = 2^6(2-1000)^6 = 2^6(2^6-6\cdot 2^5\cdot 1000+\cdots)$ [dots represent terms all multiple of 10000, of no influence on the last four digits]. Hence,

$$\begin{aligned} 1996^6 &\equiv 2^6(2^6-192000) \pmod{10000} \\ &\equiv 64(64+8000) \pmod{10000} \\ &\equiv 6096 \pmod{10000}\,, \end{aligned}$$

so that the last four digits of 1996^6 are $6\ 0\ 9\ 6$.

3. Two disjoint circles C_1 and C_2 with centres O_1 and O_2 are given. A common exterior tangent touches C_1 and C_2 at points A and B, respectively. The segment O_1O_2 cuts C_1 and C_2 at points C and D, respectively. Prove that:

(a) the points A, B, C and D are concyclic;

(b) the straight lines (AC) and (BD) are perpendicular.

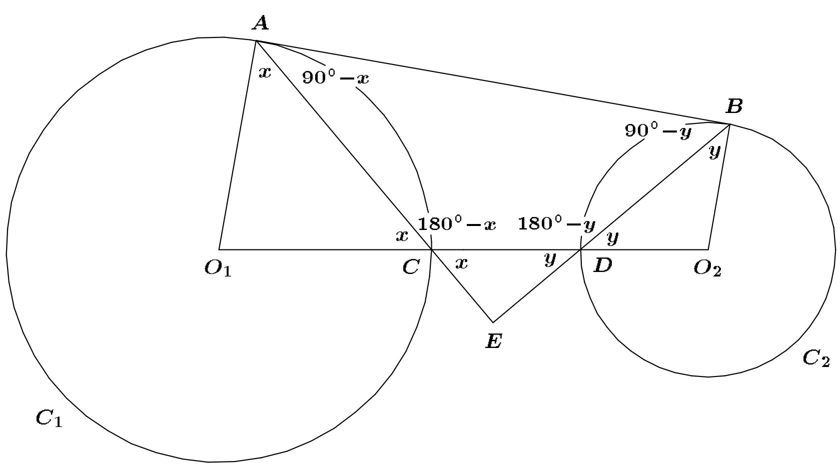
Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Amengual Covas.

(a) Letting the base angles in isosceles triangles AO_1C and BO_2D be x and y, respectively, the sum of the angles in quadrilateral $ABDC$ is

$$(90^\circ - x) + (90^\circ - y) + (180^\circ - y) + (180^\circ - x) \;=\; 360^\circ\,,$$

and we have

$$x + y \;=\; 90^\circ\,. \qquad (1)$$



Hence, in $ABDC$, the angles at A and D add up to

$$(90^\circ - x) + (180^\circ - y) \;=\; 270^\circ - (x+y) \;=\; 270^\circ - 90^\circ \;=\; 180^\circ\,,$$

and thus, $ABDC$ is cyclic. This proves (a).

(b) Let $E = AC \cap CD$. It follows from equation (1) that in triangle CED the angles at C and D add up to 90°. Thus, CED is a right-angled triangle with the right angle at E and AC and BD are in fact perpendicular.

We continue the Moldova XL set with readers' solutions to problems of Day 2 of the Republic of Moldova XL Mathematical Olympiad, 10 and 11-12 forms [1999 : 326–327].

REPUBLIC OF MOLDOVA XL MATHEMATICAL OLYMPIAD

Chişinău, 17–20 April, 1996

Second Day (Time: 4 hours)

10 Form

5. Prove that for all natural numbers $m \geq 2$ and $n \geq 2$ the smallest among the numbers $\sqrt[n]{m}$ and $\sqrt[m]{n}$ does not exceed the number $\sqrt[3]{3}$.

Solutions by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Bornsztein.

For positive integers k, let $x_k = k^{1/k}$. First note that

$$\sqrt[3]{3} \; > \; 1.44 \, . \tag{1}$$

Case 1. $m = n$.

From (1), we have $x_1 < x_2 < x_3$.

Suppose that, for a fixed integer $p \geq 3$, we have $x_p \leq x_3$. Then $p^3 \leq 3^p$ (since $(x \mapsto x^{3p})$ is non-decreasing). It follows that:

$$\begin{aligned} 3^{p+1} &\geq 3p^3 \\ &= p^3 + 3p^2 + 3p + (p-3)p^2 + (p^2-3)p \\ &> p^3 + 3p^2 + 3p + 1 \quad (\text{since } p \geq 3) \\ &= (p+1)^3 \, . \end{aligned}$$

Then, we have $x_{p+1} \leq x_3$.

By induction, we then have $x_p \leq x_3$ for all $p \geq 3$. Thus,

$$x_p \; \leq \; x_3 \quad \text{for all positive integers } p \, .$$

Case 2. $m \neq n$.

With no loss of generality, we may suppose that $m < n$.

Then

$$\begin{aligned} \sqrt[n]{m} &< \sqrt[n]{n} \quad ((x \mapsto \sqrt[n]{x}) \text{ is increasing}) \\ &\leq \sqrt[3]{3} \quad \text{from case 1.} \end{aligned}$$

Then, for all positive integers m, n, the smallest among $\sqrt[m]{n}$ and $\sqrt[n]{m}$ does not exceed the number $\sqrt[3]{3}$.

Remark. Equality occurs if and only if $m = n = 3$.

6. Prove the inequality $2^{a_1}+2^{a_2}+\cdots+2^{a_{1996}} \leq 1995+2^{a_1+a_2+\cdots+a_{1996}}$ for any real non-positive numbers a_1, a_2, $\ldots$, a_{1996}.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Lee.

First we prove the following lemma.

Lemma. Let $f : X \to \mathbb{R}$ be a function for $X \subset \mathbb{R}$.

Suppose that $f(x+y) = f(x)f(y)$ for all x, $y \in X$. For a_1, a_2, $\ldots$, $a_n \in X$, let $s_i = \sum_{j=1}^{i} a_j$ for $1 \leq i \leq n$. Then we have

$$f(s_n) + n - 1 \;=\; \sum_{i=1}^{n-1} (f(s_i)-1)(f(a_{i+1})-1) + \sum_{i=1}^{n} f(a_i)$$

for $n \geq 2$.

Proof. We have

$$\begin{aligned}
&\sum_{i=1}^{n-1} (f(s_i)-1)(f(a_{i+1})-1) + \sum_{i=1}^{n} f(a_i) \\
&\quad= \sum_{i=1}^{n-1} f(s_i)f(a_{i+1}) - \sum_{i=1}^{n-1} f(s_i) - \sum_{i=1}^{n-1} f(a_{i+1}) + \sum_{i=1}^{n-1} 1 + \sum_{i=1}^{n} f(a_i) \\
&\quad= \sum_{i=1}^{n-1} f(s_i + a_{i+1}) - \sum_{i=1}^{n-1} f(s_i) - \sum_{i=2}^{n} f(a_i) + \sum_{i=1}^{n} f(a_i) + n - 1 \\
&\quad= \sum_{i=1}^{n-1} f(s_{i+1}) - \sum_{i=1}^{n-1} f(s_i) - \sum_{i=2}^{n} f(a_i) + \sum_{i=1}^{n} f(a_i) + (n-1) \\
&\quad= \sum_{i=2}^{n} f(s_i) - \sum_{i=1}^{n-1} f(s_i) - \sum_{i=2}^{n} f(a_i) + \sum_{i=1}^{n} f(a_i) + (n-1) \\
&\quad= f(s_n) - f(s_1) + f(a_1) + (n-1) \;=\; f(s_n) + (n-1)\,,
\end{aligned}$$

since $s_1 = a_1$. ■

Since $2^{x+y} = 2^x \cdot 2^y$ for all x, $y \in \mathbb{R}$, we have

$$2^{a_1+\cdots+a_{1996}} + 1995 \;=\; \sum_{i=1}^{1995} \left(2^{\sum_{j=1}^{i} a_j} - 1\right)\left(2^{a_{i+1}} - 1\right) + \sum_{i=1}^{1996} 2^{a_i}$$

from the lemma. Since $2^\alpha \leq 1$ for all $\alpha \leq 0$, we easily deduce that

$$\sum_{i=1}^{1996} 2^{a_i} \;\leq\; 2^{a_1+\cdots+a_{1996}} + 1995\,, \qquad \text{as desired.}$$

Remark. Bornsztein notes that the inequality is strict unless at most one of the a_i is non-zero.

7. The perpendicular bisector to the side $[BC]$ of a triangle ABC intersects the straight line (AC) at a point M and the perpendicular bisector to the side $[AC]$ intersects the straight line (BC) at a point N. Let O be the centre of the circumcircle to the triangle ABC. Prove that:

(a) points A, B, M, N and O lie on a circle S;

(b) the radius of S equals the radius of the circumcircle of the triangle MNC.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give the argument of Amengual Covas.

(a) If $\angle C = 90^\circ$ then O is the mid-point of AB, and if we consider the line AB as a degenerate circle, it also contains the (infinite) points M and N.

Suppose that in $\triangle ABC$, we have $\angle C < 90^\circ$.

Since M lies on the perpendicular bisector to the side BC, $\triangle BMC$ is isosceles and its exterior angle at M is $\angle AMB = 2\angle C$.

Similarly, $\angle ANB = 2\angle C$.

Now, AB subtends at the centre O twice the angle it subtends at C on the circumcircle, implying that

$$\angle AOB \;=\; 2\angle C\,.$$

Consequently, we deduce that the three points O, M, N all lie on the arc of a circle S on the chord AB which contains the angle $2\angle C$.

This proves (a).

(b) Since quadrilateral $AMNB$ is cyclic, we immediately have that $\angle CAB = 180^\circ - \angle MNB = \angle MNC$, so that triangles MNC and BAC are similar.

Hence,

$$\frac{MN}{AB} \;=\; \frac{CM}{BC} \;=\; \frac{CM}{2\cdot CA'}\,,$$

where A' is the mid-point of the side BC.

Now, in right triangle $CA'M$,

$$CA' \;=\; CM\cdot\cos C\,,$$

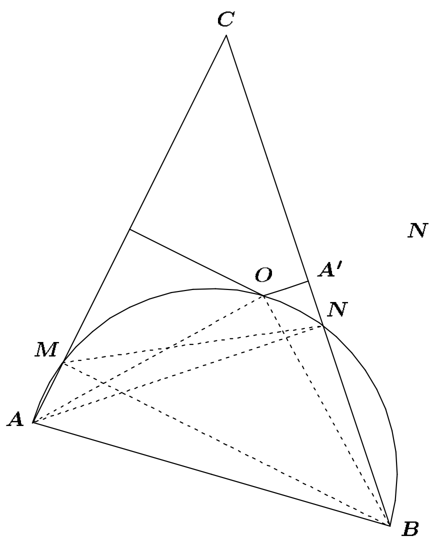
and therefore,

$$\frac{MN}{AB} \;=\; \frac{1}{2\cos C}\,.$$

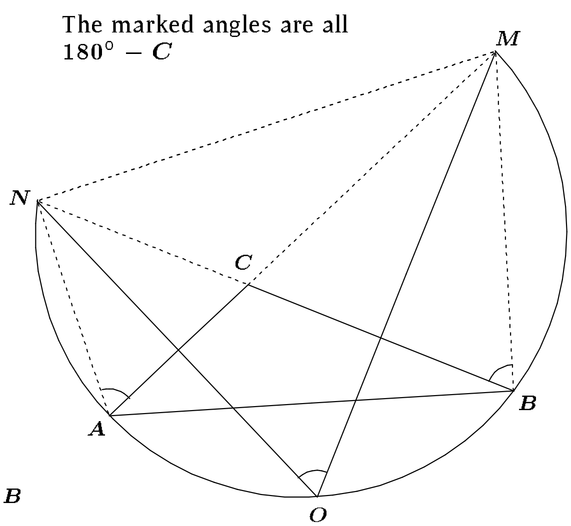
Hence, the ratio between the corresponding circumradii of $\triangle MNC$ and $\triangle ABC$ will be $1 : 2\cos C$, yielding

$$\begin{aligned}
\text{Circumradius of } \triangle MNC &= \frac{\text{circumradius of } \triangle ABC}{2\cos C} \\
&= \frac{\frac{c}{2\sin C}}{2\cos C} \\
&= \frac{c}{2\sin 2C} \\
&= \text{circumradius of } \triangle AMB \\
&= \text{radius of } S\,.
\end{aligned}$$

The present solution can be applied with minor modifications to the case $\angle C > 90°$ as well.



The case $C < 90°$



The case $C > 90°$

11–12 Form

5. Let p be the number of functions defined on the set $\{1, 2, \ldots, m\}$, $m \in N^*$, with values in the set $\{1, 2, \ldots, 35, 36\}$ and q be the number of functions defined on the set $\{1, 2, \ldots, n\}$, $n \in N^*$, with values in the set $\{1, 2, 3, 4, 5\}$. Find the least possible value for the expression $|p - q|$.

Solution by Pierre Bornsztein, Pontoise, France.

Let $m, n \in N^*$. We have $p = 36^m$ and $q = 5^n$. The problem is then to find the least possible value of $|36^m - 5^n|$ over all $m, n \in N^*$.

For $m, n \in N^*$,

$$\begin{aligned} 36^m &= a \pmod{100} \quad \text{where } a \in \{36, 96, 56, 16, 76\}, \\ 5^n &= 25 \pmod{100} \quad \text{for } n \geq 2\,. \end{aligned}$$

Since $36 - 5^2 = 11$, the least possible value is then 9 or 11. But

$$36^m - 5^n \;=\; -5^n \pmod 9 \;\;\neq\;\; 0 \pmod 9\,,$$

so that $36^m - 5^n = \pm 9$ is impossible.

It follows that the least possible value of $|36^m - 5^n|$ is 11.

6. Solve in real numbers the equation

$$2x^2 - 3x \;=\; 1 + 2x\sqrt{x^2 - 3x}\,.$$

Solutions by Pierre Bornsztein, Pontoise, France; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Evageloupolos.

The given equation

$$2x^2 - 3x \;=\; 1 + 2x\sqrt{x^2 - 3x}$$

is equivalent to

$$\begin{aligned} x^2 - 2x\sqrt{x^2-3x} + x^2 - 3x &= 1\,, \\ \text{or} \quad (x - \sqrt{x^2-3x})^2 - 1 &= 0\,, \\ \text{or} \quad (x - \sqrt{x^2-3x} + 1)(x - \sqrt{x^2-3x} - 1) &= 0\,. \end{aligned}$$

The equation $x - \sqrt{x^2-3x} - 1 = 0$ has no roots, whereas the equation $x - \sqrt{x^2-3x} + 1 = 0$ has the root $x = -\frac{1}{5} = -0.2$.

7. On a sphere distinct points A, B, C and D are chosen so that segments $[AB]$ and $[CD]$ cut each other at point F, and points A, C and F are equidistant to a point E. Prove that the straight lines (BD) and (EF) are perpendicular

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give the solution of Bataille.

Points A, C, F are on a sphere S with centre E, and the plane P determined by these points cuts S along a circle Γ. The centre I of Γ is the projection of E onto P so that $(EI) \perp P$. Hence, $(EI) \perp (BD)$, since points B, D are in P.

Now, P cuts the given sphere along a circle γ containing the points A, B, C, D and, in the plane P, the circles γ and Γ intersect at A and C.

In P, consider the inversion with centre F and power $\overline{FA} \cdot \overline{FB} = \overline{FC} \cdot \overline{FD}$ = power of F with respect to γ. Under this inversion, the line

(BD) is transformed into Γ, so that the centre I of Γ is on the perpendicular to (BD) through F. Hence, $(IF) \perp (BD)$.

Since (BD) is perpendicular to (EI) and to (IF), it is perpendicular to the plane (EIF) and consequently to the line (EF).

8. 20 children attend a rural elementary school. Every two children have a grandfather in common. Prove that some grandfather has not less than 14 grandchildren in this school.

Solution by Pierre Bornsztein, Pontoise, France.

Let c_1, c_2, $\ldots$, c_{20} be the children. For a grandfather g_k we denote by $G_k = \{c_i \mid c_i \text{ is a grandchild of } g_k\}$. Since there is a finite number of children, there is also a finite number of grandfathers.

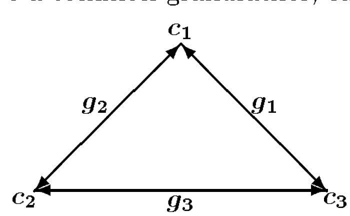
Let g_1 be a grandfather such that $n_1 = \mathrm{Card}\, G_1$ is maximal. It is clear that $n_1 \geq 2$.

Suppose, for a contradiction, that $n_1 \leq 13$.

Say $c_1 \in G_1$. There exists a child, say c_2, such that $c_2 \notin G_1$. c_1 and c_2 have a common grandfather, say g_2, and $g_1 \neq g_2$.

If each grandchild of g_1 has grandfather g_2 too, then $G_1 \cup \{c_2\} \subset G_2$. Thus, $n_2 = \text{Card}\,(G_2) \geq n + 1$, which contradicts that n_1 is maximal.

Thus, there exists a grandchild of g_1, say c_3, that is not a grandchild of g_2. Then, c_2 and c_3 have a common grandfather, say g_3, and $g_3 \notin \{g_1, g_2\}$. Then, we have



For $i \geq 4$, c_i has a common grandfather with each of the children c_1, c_2, c_3. Then, the two grandfathers of c_i are in the set $\{g_1, g_2, g_3\}$.

It follows that the 40 grandfathers of the c_i's form the set $\{g_1, g_2, g_3\}$ since $40 = 3 \times 13 + 1$, from the Pigeonhole Principle, at least one of g_1, g_2, g_3 has at least 14 grandchildren.

Thus, $n_1 \geq 14$, a contradiction.

Thus, $n_1 \geq 14$, and we are done.

Remark. 14 cannot be improved. If $c_1, \ldots, c_{14}$ have g_1 for a grandfather, $c_1, \ldots, c_7$ have g_2, $c_8, \ldots, c_{14}$ have g_3, $c_{15}, \ldots, c_{20}$ have g_2 and g_3. Thus, $n_1 = 14$ and $n_2 = n_3 = 13$.

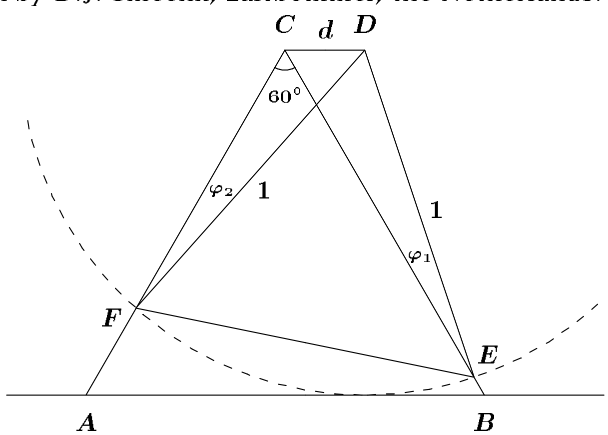
For n children ($n \geq 3$), with the same reasoning we obtain:

$$n_1 \geq \left\lfloor \frac{2n+2}{3} \right\rfloor + 1\,, \quad \text{where } \lfloor \ldots \rfloor \text{ denotes the integer part.}$$

Next we turn to the November 1999 number and reader solutions and comments about problems of the 31^{st} Canadian Mathematical Olympiad [1999 : 387–388].

2. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and centre on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.



The rolling circle intersects CB at E and CA at F. D is the centre of the rolling circle.

$$\angle CED = \varphi_1\,, \quad \angle CFD = \varphi_2\,, \quad \angle DCE = 60^\circ\,, \quad \angle DCF = 120^\circ\,,$$

$$\overline{CD} = d\,, \quad DE = DF = 1\,.$$

From the Sine Law in $\triangle CDE$, we have

$$d : \sin\varphi_1 = 1 : \sin 60^\circ \Longrightarrow \sin\varphi_1 = d\sin 60^\circ\,.$$

From the Sine Law in $\triangle CDF$, we have

$$\begin{aligned} d : \sin\varphi_2 &= 1 : \sin 120^\circ \Longrightarrow \sin\varphi_2 = d\sin 120^\circ \\ &\Longrightarrow \sin\varphi_1 = \sin\varphi_2 \Longrightarrow \varphi_1 = \varphi_2 = \varphi\,. \\ &\Longrightarrow \text{Quad}CDEF \text{ can be inscribed in a circle} \\ &\Longrightarrow \angle EDF = \angle ECF = \angle BCA = 60^\circ = \text{ constant.} \end{aligned}$$

5. Let x, y and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

$$x^2y + y^2z + z^2x \le \frac{4}{27}\,,$$

and find when equality occurs.

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

In my proposed problem #1292 (Math. Mag., 62 (1989) 137), one was to determine the maximum value of

$$x_1^2x_2 + x_2^2x_3 + \cdots + x_n^2x_1$$

given that $x_1 + x_2 + \cdots + x_n = 1$, $x_1, x_2, \ldots, x_n \geq 0$ and $n \geq 3$.

The published elegant solution was by the 1988 USA Olympiad Math Team. First it was shown that the maximum would occur if one took $x_i = 0$ for $i > 3$. Then, without loss of generality, the indices were cycled so that $(x_2 - x_1)(x_2 - x_3) \leq 0 \leq x_1x_2$, so that

$$x_1^2x_2 + (x_2 - x_1)(x_2 - x_3)x_3 \;\leq\; x_1^2x_2 + x_1x_2x_3\,.$$

These terms were then arranged to give

$$\begin{aligned} x_1^2x_2 + x_2^2x_3 + x_3^2x_1 \;&\leq\; (x_1 + x_3)^2x_2 \;=\; (1 - x_2)^2x_2 \\ &\leq\; \frac{4\left[\frac{1-x_2}{2} + \frac{1-x_2}{2} + x_2\right]^3}{27} \;=\; \frac{4}{27}\,, \end{aligned}$$

the last by the AM–GM inequality and with equality if $x_2 = \frac{1}{3}$, and then $x_1 = \frac{2}{3}$ and $x_3 = 0$, or vice-versa.

For a more direct solution but not particularly elegant, we can assume without loss of generality that $x \geq y \geq z$ (if we changed the cyclic order of these, $x^2y + y^2z + z^2x$ would be decreased). In terms of a homogeneous inequality, we want to show that

$$4(x + y + z)^3 \;\geq\; 27(x^2y + y^2z + z^2x)\,.$$

We now set $z = a$, $y = a+b$, and $x = a+b+c$ where $a, b, c \geq 0$. Expanding out, we get

$$27a^3 + 54ab + 27a^2c + 36ab^2 + 9ac^2 + 36abc + (5b + 4c)(b - c)^2 \;\geq\; 0$$

and which is now evident. There is equality, if and only if, $a = 0$ and $b = c$.

That completes the *Corner* for this issue. Send me your Olympiad contests and your nice solutions and generalizations.

BOOK REVIEWS

ALAN LAW

Mathematical Chestnuts from Around the World, by Ross Honsberger, published by the Mathematical Association of America, 2001,
ISBN: 0-88385-330-2, softcover, 310+ pages, $32.95 (U.S.).
Reviewed by **Mohammed Aassila**, *University of Strasbourg, Strasbourg, France.*

This book is No. 24 of the Dolciani Mathematical Expositions, a series including many books with challenging problems for lovers of mathematics, which often approach a subject without formally introducing the reader to it. This is another in a series of books by the author on problems and solutions taken from various national and international competitions, and from various mathematical journals.

Altogether the book has 150 problems collected into 26 sections, with no discernible reason for the order they are put in, and a chapter devoted to exercises and their solutions. At the end, the problems are classified under three subjects (roughly: Combinatorics and Combinatorial Geometry, Geometry, Algebra, Number Theory, Probability, and Calculus).

The following is just a small sample of some of the problems discussed in the book:

1. S is any set of n points in the plane, $n \geq 2$. Let D and d, respectively, denote the greatest and least distances determined by two points of S. Prove that

$$D \;>\; \frac{\sqrt{3}}{2}(\sqrt{n}-1)d\,.$$

2. Let $\alpha = \{a_1, a_2, \ldots, a_k\}$ be a subset of the first n positive integers which has been arranged in increasing order:

$$1 \;\leq\; a_1 \;<\; a_2 \;<\; \cdots \;<\; a_k \;\leq\; n\,.$$

(i) How many such α are there that begin with an odd number and thereafter in parity:

$$\alpha \;=\; \{\text{odd, even, odd, even,}\ldots\}\,?$$

For convenience, include the empty set $\emptyset$ in the count.

(ii) How many α are there of size k?

The author has an efficient and clear approach to proofs and explanations that allows the book to be read easily. This book is a welcome addition to the shelves of anyone associated with mathematics problem solving in general and mathematics competitions in particular.

Teaching First: A Guide for New Mathematicians, by Thomas W. Rishel, published by the Mathematical Association of America, 2000,
ISBN: 0-88385-165-2, softcover, 161+ pages, $19.00 (U.S.).
Reviewed by ***Ruby Kocurko****, Memorial University of Newfoundland, St. John's, Newfoundland.*

How does one assemble all the helpful hints and valuable lessons gleaned from years of university teaching experience and present them in a digestible package for the novice teacher of university mathematics? In his book, Thomas Rishel has accomplished this daunting task in a concise, well-organized manner that addresses an abundance of concerns for the beginner. In one section, he addresses the importance of knowing the interests of your class audience. It is evident in this book that the author knows his intended audience and shares his wisdom and experience in a skilful way that encourages the reader to turn the pages.

The book is organized in three main sections followed by case studies, references and appendices. The first section deals with the day to day activity and encounters of a typical semester. Included are standard topics: lecture preparation, use of technology, grading issues, motivation of students, classroom atmosphere as well as less conventional: "how to get fired", and "getting along with colleagues". Each topic is done briefly (2–5 pages), giving essential ideas and advice in a conversational manner with hints on what to say or do in situations that may arise. At times, sample classroom scenarios, summary lists, and additional references for a topic are given. In reading this first section, I found my own thirty years in teaching university mathematics reflected in the pages. I especially liked the comment "Maybe it might help to think of the 'standing in front of the class' part of teaching as the fun part, and grading as the part you get paid for." The situations presented are both relevant, current and familiar in the university environment. As an example of the author's technique, I will quote the headings in the chapter "What Was That Question Again?":

The standard question,
The question that makes no sense,
The silly question,
The unintelligible question,
The "challenge to your authority" question,
The "good question",
The question you don't have any answer for.

Careful consideration is given to each type in an honest, open, sometimes humorous way drawing on past experience. The author puts a personal touch in his answers so that you get a sense that here is a person willing to share ideas and time in helping a young colleague or T.A. succeed. The last chapter in this section deals with conducting oneself in a professional manner and outlines the responsibilities involved in teaching mathematics.

In the second section, entitled "More Advanced Topics", the author discusses more theoretical aspects of teaching and learning. Examples drawn from his own teaching of mathematics enhance the material presented. This section also considers the more difficult aspects of dealing with students and colleagues and gives advice for international students.

Since this book is aimed at Teaching Assistants, it is appropriate that the third section contains pertinent information on job applications, giving talks, teaching evaluations, and for the new faculty member – the role of research and scholarship, teaching and service. It is fitting that the closing topic is the essence of good teaching. The author claims that "good" teaching can be taught, and if one reads and follows his suggestions, one should be able to give a competent, coherent lecture and be a contributing departmental colleague. Over the years I have helped new faculty, graduate students, student markers, faculty whose first language was not English; I have written recommendations and class evaluations, given advice and mentored. In each of these cases this book would have been extremely useful, and it would be of benefit to anyone starting a university career in mathematics. The author does not claim to have all the answers but does give a comprehensive, down-to-earth approach. The collection of case studies would certainly elicit lots of opinion at our coffee room, and it is interesting that no opinion is offered, but the author has invited response from the reader.

I enjoyed this book, and only wish I had read it at the start of my career instead of at the end. It would make a valuable reference and a reassuring guide to beginning Teaching Assistants and new faculty. While it is a guide to "good" teaching, perhaps our aim should be "great" teaching. To quote the author " ...great teaching comes in all forms, but mainly it comes from the delicate interaction between two personalities: that of the instructor who somehow conveys a love of learning, and the student who comes ready to absorb and apply what the instructor has to give". My suspicion is that Thomas W. Rishel belongs in the "great" category.

A Message to all — in code

048 018 504 528 195 002 180 152 015 273 013

Eight Proofs of One Theorem

Jooyong Ahn, Hojoo Lee and Choongyup Sung

The old saying "*there is more than one way to skin a cat*" is certainly true in geometry; there is no unique way to prove a theorem or to solve a problem.

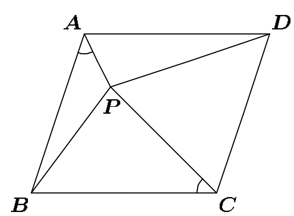
Howard Eves, *Fundamentals of Modern Geometry*

Introduction.

In an article in this journal [1998 : 81], Georg Gunther presented one problem with six different solutions (see [1]). And Jimmy Chui [1999 : 235] gave four different proofs of a combinatorial identity (see [2]). It is pleasant to see different solutions of a problem. It is our purpose in this article to examine various different solutions to a geometry problem. We also hope to emphasize the importance of studying a problem with different solutions.

Theorem 1.

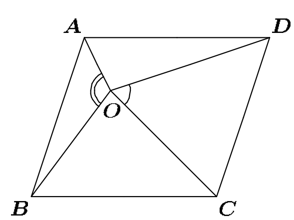
If P is a point in the interior of a parallelogram $ABCD$ such that the angles at A and C with chord BP are congruent, then the angle at D and B with chord AP are also congruent; if $\angle PAB = \angle PCB$, then $\angle PDA = \angle PBA$.



First, let us examine the following Canadian Mathematical Olympiad problem and Theorem 2 before we give eight proofs of Theorem 1.

1997 Canadian Mathematical Olympiad Problem 4.

If the point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$, prove that $\angle OBC = \angle ODC$.



Theorem 1 and the above CMO problem yield the following interesting result:

Theorem 2.

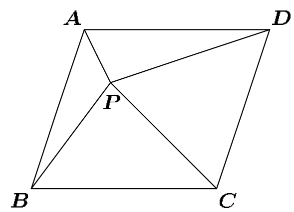
Let P be a point in the interior of parallelogram $ABCD$. Then the following are equivalent:

(1) $\angle PAB = \angle PCB$,

(2) $\angle PDA = \angle PBA$,

(3) $\angle APB + \angle CPD = 180°$,

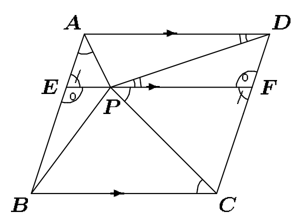
(4) $\angle BPC + \angle DPA = 180°$.



Now, we offer eight proofs of Theorem 1, all requiring no more than the ordinary geometry and trigonometry background of a high school student. Enjoy our proofs!

Proof 1. (Similarity)

Let the line though P, parallel to BC, meet AB and CD at E and F, respectively. We note that $\triangle APE \sim \triangle PCF$ since $\angle AEP = \angle PFC$ and $\angle PAE = \angle PCB = \angle CPF$.



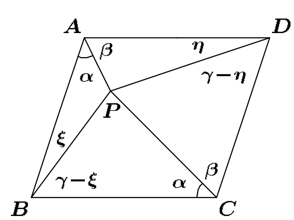
From $AD \parallel EF \parallel BC$, it follows that
$\dfrac{AE}{EB} = \dfrac{DF}{FC}$,

and from $\triangle APE \sim \triangle PCF$, it follows that $\dfrac{EP}{AE} = \dfrac{FC}{PF}$, so that $\dfrac{EP}{EB} = \dfrac{DF}{FP}$.

This result, and $\angle PEB = \angle DFP$, imply that $\triangle PEB \sim \triangle DFP$, so that $\angle ABP = \angle PBE = \angle DPF = \angle ADP$, as desired.

Proof 2. (Law of Sines)

Let $\angle PAB = \angle PCB = \alpha$, $\angle PAD = \angle PCD = \beta$, $\angle ABP = \xi$, $\angle ADP = \eta$, and $\angle ABC = \angle ADC = \gamma$.



By the Law of Sines, we obtain that

$$\frac{\sin\xi}{\sin\alpha} = \frac{PA}{PB}, \qquad \frac{\sin\alpha}{\sin(\gamma-\xi)} = \frac{PB}{PC},$$
$$\frac{\sin(\gamma-\eta)}{\sin\beta} = \frac{PC}{PD}, \qquad \frac{\sin\beta}{\sin\eta} = \frac{PD}{PA}.$$

Multiplying all of these gives $\dfrac{\sin\xi\,\sin(\gamma-\eta)}{\sin\eta\,\sin(\gamma-\xi)} \;=\; 1$, which means that

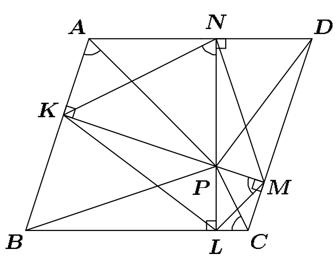
$$\sin\xi\,(\sin\gamma\cos\eta - \cos\gamma\sin\eta) \;=\; \sin\eta\,(\sin\gamma\cos\xi - \cos\gamma\sin\xi)\ , \qquad \text{or}$$

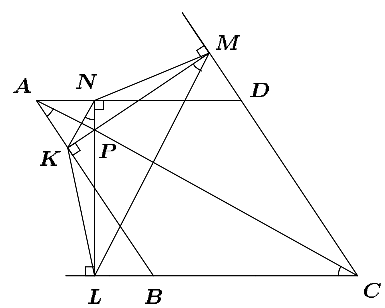
$$\sin\gamma\,(\sin\xi\cos\eta - \cos\xi\sin\eta) \;=\; 0\,.$$

This is equivalent to $\sin\gamma\sin(\xi-\eta) = 0$.

This shows that $\xi = \eta$ since $0 < \gamma,\ \xi,\ \eta < \pi$. We conclude that $\angle PBA = \angle PDA$.

Proof 3. (Cyclic Quadrilaterals)





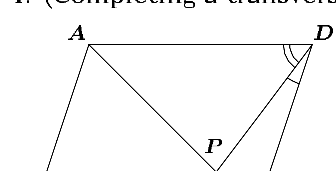
Project P onto the lines AB, BC, CD, and DA to obtain the points K, L, M, and N.

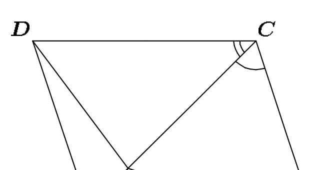
Because $AB \parallel CD$, the points K, P, and M are collinear, and because $AD \parallel BC$, the points L, P, and N are collinear.

Note that $AKPN$, $BLPK$, $CMPL$, and $DNPM$ are cyclic quadrilaterals, which may not be convex if not all of K, L, M, and N lie on the side-segments of the quadrilateral.

Hence $\angle PNK = \angle PAB$, $\angle PLK = \angle PBA$, $\angle PML = \angle PCB$, and $\angle PMN = \angle PDA$. Therefore, $KLMN$ is also a cyclic quadrilateral. Thus, we obtain that $\angle PBA = \angle KLN = \angle KMN = \angle PDA$.

Proof 4. (Completing a transversal)





We use the 2nd diagram. Let R be the intersection of the line AB and line CP.

Then we obtain that $\angle PRB = \angle PCD = \angle PAD$, because $AB \parallel CD$ and $\angle PAB = \angle PCB$. The Sine Law applied to $\triangle ARP$, and then to $\triangle CRB$

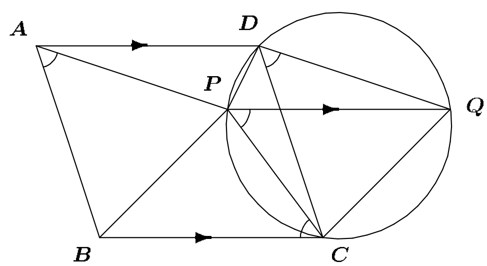
gives

$$\frac{PR}{PA} = \frac{\sin\angle PAB}{\sin\angle PRA} = \frac{\sin\angle PCB}{\sin\angle PRB} = \frac{RB}{BC} = \frac{RB}{AD},$$

so that $\dfrac{PA}{AD} = \dfrac{RB}{PR}$.

This, and $\angle PAD = \angle PRB$, imply that $\triangle PAD \sim \triangle PRB$. As a consequence, $\angle PBA = \angle PBR = \angle PDA$, and the problem is solved.

Proof 5. (Parallelograms and Cyclic quadrilaterals)



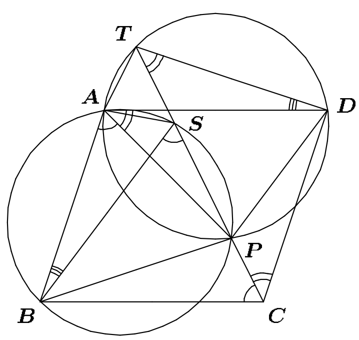
First, we complete the parallelogram $PBCQ$.

Note that $APQD$ is also a parallelogram, since $AD \parallel BC \parallel PQ$ and $AD = BC = PQ$. Now, $\triangle PAB$ and $\triangle QDC$ are congruent triangles, since $PA = QD$, $AB = DC$, and $BP = CQ$.

Since $\angle CPQ = \angle PCB = \angle PAB = \angle QDC$, or $\angle CPQ = \angle QDC$, we have that $PCQD$ is a cyclic quadrilateral.

Thus, we have that $\angle ADP = \angle QPD = \angle QCD = \angle PBA$, or $\angle ADP = \angle PBA$.

Proof 6. (Three isosceles triangles)



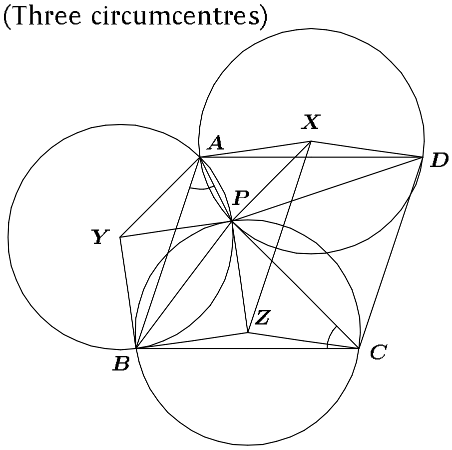
Suppose that CP meets the circle ABP again at S, and suppose that CP meets the circle ADP again at T.

Since $\angle BCS = \angle PAB = \angle PSB$, $\triangle BCS$ is isosceles with legs $BC = BS$. Similarly, $\triangle DCT$ is isosceles with legs $DT = DC$.

Thus, we obtain $AD = SB$ and $DT = BA$, since $AD = BC$ and $CD = BA$. Also, $\angle SBA = \angle SPA = \angle TPA = \angle TDA$, since $ABPS$ and $APDT$ are cyclic.

Combining results, $\triangle ADT \equiv \triangle SBA$. Therefore, $\triangle AST$ is isosceles with legs $AS = AT$. Thus, $\angle ABP = \angle AST = \angle ATS = \angle ATP = \angle ADP$, as we wanted to show.

Proof 7. (Three circumcentres)



Let Y be the circumcentre of $\triangle ABP$ and let $YA = YB = YP = r_1$. Let Z be the circumcentre of $\triangle BCP$ and let $ZB = ZC = ZP = r_2$.

We complete the parallelogram $ABZX$. Note that $XA = ZB = ZC = XD$.

From

$$\begin{aligned} r_1 \sin\angle PAB = r_1 \sin\left(\frac{\angle PYB}{2}\right) &= \frac{BP}{2} = r_1 \sin\left(\frac{\angle PZB}{2}\right) \\ &= r_1 \sin\angle PCB\,, \end{aligned}$$

and

$$\angle PAB = \angle PCB\,,$$

it follows that $r_1 = r_2$, and thus, $YP = YB = ZB = ZP$. Consequently, $YBZP$ is a rhombus.

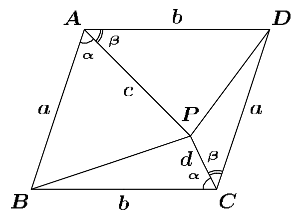
We also note that $AX \parallel BZ \parallel YP$ and $AX = BZ = YP$ which implies $AYPX$ is a parallelogram, in fact also a rhombus, because $YA = YP$. As a result, $XD = XA = XP$ which makes X the circumcentre of $\triangle DAP$.

As above, $YP \sin\angle ABP = \frac{AP}{2} = XP \sin\angle ADP = YP \sin\angle ADP$, and $\sin\angle ABP = \sin\angle ADP$.

Since $\angle ABP + \angle ADP < \angle ABC + \angle ADC = \pi$, we obtain that $\angle ADP = \angle PBA$.

Proof 8. (The Cosine Law)

We put $AB = CD = a$ and $BC = AD = b$, $AP = c$, $CP = d$, $\angle PAB = \angle PCB = \alpha$, and $\angle PAD = \angle PCD = \beta$.



The Cosine Law applied to $\triangle ABP$, and then to $\triangle BCP$, gives

$$\cos\alpha = \frac{a^2+c^2-BP^2}{2ac} = \frac{b^2+d^2-BP^2}{2bd}$$

or

$$(bd-ac)BP^2 = bd(a^2+c^2) - ac(b^2+d^2) = (ad-bc)(ab-cd)\,. \quad (1)$$

Analogously, we have

$$(ad-bc)DP^2 = (bd-ac)(ab-cd)\,. \quad (2)$$

If $bd - ac = 0$ or $ad - bc = 0$, then we obtain $a = b$, $c = d$; that is, $ABCD$ is a rhombus and P is on BD, in which case $\angle ABP = \angle ADP$ is obvious. Therefore, we assume that $bd - ac \neq 0$ and $ad - bc \neq 0$.

From (1) and (2), it then follows that $ab - cd \neq 0$ and that $BP : DP = |ad - bc| : |bd - ac|$.

If $ad - bc > 0$, we obtain $[\triangle PCD] > [\triangle PAD]$, and because

$$\begin{aligned}[\triangle PCD] + [\triangle PAB] &= [\triangle PAD] + [\triangle PBC]\,,\\ [\triangle PAB] &< [\triangle PBC]\,;\end{aligned}$$

that is, $bd - ac > 0$.

Similarly, if $ad - bc < 0$, we obtain that $bd - ac < 0$.

Combining these results, we have

$$BP : DP = ad - bc : bd - ac\,. \quad (3)$$

From (1) and (2), it follows that

$$\cos\angle ABP = \frac{a^2+BP^2-c^2}{2aBP} = \frac{2abd - c(a^2+b^2-c^2+d^2)}{2BP(bd-ac)} \quad \text{and}$$

$$\cos\angle ADP = \frac{2abd - c(a^2+b^2-c^2+d^2)}{2DP(ad-bc)}\,.$$

which, together with (3), imply that $\cos\angle ABP = \cos\angle ADP$, so that $\angle ABP = \angle ADP$, as desired.

ACKNOWLEDGEMENT

The authors wish to thank the referee for suggestions and detailed comments.

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1. Georg Gunther, *One Problem — Six Solutions*, CRUX with MAYHEM **24** : 2 (1998), 81–97.

2. Jimmy Chui, *Four ways to Count*, CRUX with MAYHEM **25** : 7 (1999), 235–237.

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THE SKOLIAD CORNER

No. 56

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mayhem-editors@cms.math.ca.

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 February 2002*. Look for prizes for solutions in the new year.

This issue's items come from the Mandelbrot competition. The competition is made up of four rounds at different times of the year. Each round consists of an individual part and a team part. A school's score is made up of the top four individual scores and the score of the team of four (chosen by the school) on the team part. This round's team part has a twist: a little essay was provided to the students a couple of days prior to the test. The test was then based on the essay.

My thanks go to Sam Vandervelde at Greater Testing Concepts for providing the contest material. For more information about the contest you can visit the website

http://www.mandelbrot.org or email info@mandelbrot.org

The Mandelbrot Competition
Division B Round Two Individual Test
December 1997

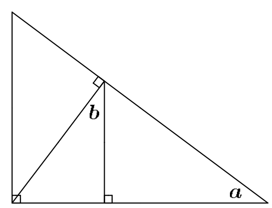
1. If a group of positive integers has a sum of 8, what is the greatest product the group can have? (**1 point**)

2. There is one two-digit number such that if we add 1 to the number and reverse the digits of the result, we obtain a divisor of the number. What is the number? (**1 point**)

3. Ten slips of paper, numbered 1 through 10, are placed in a hat. Three numbers are drawn out, one after another. What is the probability that the three numbers are drawn in increasing order? (**2 points**)

4. The three marked angles are right angles. If $\angle a = 20°$, then what is the measure of $\angle b$?

(**2 points**)



5. Vicky asks Charlene to identify all non-congruent triangles $\triangle ABC$ given:

1. the value of $\angle A$,
2. $AB = 10$, and
3. length BC equals either 5 or 15.

Charlene responds that there are only two triangles meeting the given conditions. What is the value of $\angle A$? (**2 points**)

6. Five pirates find a cache of 5 gold coins. They decide that the shortest pirate will become bursar and distribute the coins — if half or more of the pirates (including the bursar) agree to the distribution, it will be accepted; otherwise, the bursar will walk the plank and the next shortest pirate will become bursar. This process will continue until a distribution of coins is agreed upon. If each pirate always acts so as to stay aboard if possible and maximize his wealth, and would rather see another pirate walk the plank than not (all else being equal), then how many coins will the shortest pirate keep for himself? (**3 points**)

7. The twelve positive integers $a_1 \leq a_2 \leq \cdots \leq a_{12}$ have the property that no three of them can be the side lengths of a non-degenerate triangle. Find the smallest possible value of $\frac{a_{12}}{a_1}$. (**3 points**)

Mandelbrot Morsels
An Interpretation of Interpolation

One of the most fascinating capabilities that mathematical science offers is the ability to make predictions. Imagine a simple experiment in which we place a bowling ball on a slope and release it at the same instant that we start a stopwatch. The distance in centimetres that the ball has rolled after t seconds would resemble Table 1. With a bit of experimentation, one could discover the rule $d \approx 0.38t^2$ and predict that the ball should roll about 85.5 cm in 15 seconds. It is quite satisfying to check this figure experimentally and find that it is correct.

Table 1.

Time	Position
0	0.00
2	1.55
4	6.05
8	24.35
10	37.95

In an actual experiment the bowling ball may have some initial velocity by the time we begin our readings, and will start a ways from our distance recorder, so that the formula for distance will be a general second degree polynomial: $d \approx at^2 + bt + c$. For example, we may obtain the data shown in Table 2. Here it is not at all clear what coefficients a, b, and c we should use in order to accurately model our data.

Table 2.

Time	Position
2	7.00
3	9.45
6	21.00
7	26.25

Considerations such as these led mathematicians to develop a general method for finding equations of polynomials which pass through given points. The method, known as Lagrange Interpolation, is attributed to the French mathematician Joseph-Louis Lagrange (1736-1813). It stems from the following

FACT: Let $n+1$ data points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) be given. Then there is exactly one polynomial $p(x)$ of degree n or less passing through all $n+1$ points.

The difficulty, naturally, is figuring out a formula for $p(x)$ based on the data points. A very clever technique handles this dilemma, which we will present by way of an example: find the third degree polynomial which satisfies $p(0) = -2$, $p(1) = 1$, $p(3) = 1$ and $p(4) = -14$. An alternate way of phrasing this is to ask for a cubic polynomial which passes through the points $(0, -2)$, $(1, 1)$, $(3, 1)$, and $(4, -14)$.

The key is to build our polynomial from the functions such as

$$\chi_0 \;=\; -2\,\frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \;=\; \frac{1}{6}\,(x-1)(x-3)(x-4)\,.$$

Notice that when we plug in $x = 0$ we obtain -2, as desired (mentally check this). However, substituting $x = 1$, $x = 3$, or $x = 4$ results in a value of 0. Now create χ_1, χ_3, and χ_4 in the same way (try it!), and add all four functions together to obtain

$$\begin{aligned} p(x) \;&=\; \frac{1}{6}\,(x-1)(x-3)(x-4) + \frac{1}{6}\,x(x-3)(x-4) \\ &\quad - \frac{1}{6}\,x(x-1)(x-4) - \frac{7}{6}\,x(x-1)(x-3)\,. \end{aligned}$$

We claim this is precisely the polynomial we are after. For example, when $x = 4$ the first three terms equal 0, while the fourth is $-\frac{7}{6}(12) = -14$, so $p(4) = -14$, just as we wanted. This method generalizes to more than four points in exactly the way one would expect it to.

EXERCISE: Use Lagrange Interpolation to find a second degree polynomial passing through the first three data points of Table 2. Check your formula correctly predicts the distance for the final point, when $t = 7$.

Now for the "interpretation" part. Let us examine this business of Lagrange Interpolation from a slightly different angle. Suppose we are given the values of a third degree polynomial $p(x)$ at $x = 0, 1, 2, 3$. What is $p(4)$ in terms of $p(0)$ through $p(3)$? Let us call $p(0) = Y_0, p(1) = Y_1, p(2) = Y_2$, and $p(3) = Y_3$. Then by Lagrange Interpolation $p(x)$ equals

$$\begin{aligned} &\frac{Y_0(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + \frac{Y_1 x(x-2)(x-3)}{(1-0)(1-2)(1-3)} \\ &\quad + \frac{Y_2 x(x-1)(x-3)}{(2-0)(2-1)(2-3)} + \frac{Y_3 x(x-1)(x-2)}{(3-0)(3-1)(3-2)}\,. \end{aligned}$$

Hence, $p(4)$ can be found by substituting $x = 4$ in the above expression. We obtain

$$p(4) \;=\; -Y_0 + 4Y_1 - 6Y_2 + 4Y_3\,.$$

There are two important observations to be made at this point. First, $p(4)$ is a *linear combination* of $p(0)$ through $p(3)$. In other words, $p(4)$ can be obtained by multiplying each of $p(0)$ through $p(3)$ by certain constants and adding up the results. Once we have found those constants (in this case -1, 4, -6 and 4), we can always predict $p(4)$ given the values of $p(x)$ at $x = 0, 1, 2, 3$.

As for the second observation, if you noticed that the above constants looked like the fourth row of Pascal's triangle, give yourself a pat on the back. Pascal's triangle arises from binomial expansions, so it is not surprising that there is a nifty function that generates the constants -1, 4, -6, and 4. It is $x^4 - (x-1)^4$, which equals $4x^3 - 6x^2 + 4x - 1$. These findings are true for any degree polynomial; we summarize them in our first

THEOREM: Let $p(x)$ be a polynomial of degree n for which we know the values of $p(0)$, $p(1)$, ..., $p(n)$. Then $p(x)$ can be calculated using Lagrange Interpolation, and $p(n+1)$ is a linear combination of $p(0)$ through $p(n)$. In other words,

$$p(n+1) \;=\; A_0 p(0) + A_1 p(1) + \cdots + A_n p(n)\,,$$

for some constants A_0, A_1, ..., A_n. These constants appear in row $n+1$ of Pascal's triangle, and are generated by the function

$$x^{n+1} - (x-1)^{n+1} \;=\; A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0\,. \qquad (1)$$

The final equation is quite versatile. For instance, it answers the following popular olympiad level problem from the 1980's: if $p(x)$ is a degree n polynomial such that $p(0) = 1$, $p(1) = 2$, $p(2) = 4$, $p(3) = 8$, ..., $p(n) = 2^n$, then find $p(n+1)$. From our theorem we know that $p(n+1)$ is a linear combination of $p(0)$ through $p(n)$; in particular

$$p(n+1) \;=\; A_0 \cdot 1 + A_1 \cdot 2 + A_2 \cdot 4 + \cdots + A_n \cdot 2^n\,.$$

Substituting $x = 2$ in equation (1) magically produces $2^{n+1} - 1$ as our answer. Now try this

EXERCISE: Suppose $p(x)$ is a degree n polynomial such that $p(0) = 1$, $p(1) = -1$, $p(2) = 1$, $p(3) = -1$, ..., $p(n) = (-1)^n$. Calculate $p(n+1)$.

If done correctly you should obtain an answer very similar to $2^{n+1} - 1$.

Hopefully you have made it this far and appreciate some of the interesting and clever mathematics embedded within Lagrange Interpolation. If not, do not worry; there are still a few more days before the team test. Put these crazy theorems away, clear your head, and come back tomorrow, at which point everything will make perfect sense. The team test will be based on these ideas, especially the last few paragraphs. Happy interpolating.

The Mandelbrot Competition
Division B Round Two Team Test
December 1997

FACTS: A polynomial $p(x)$ of degree n or less is determined by its value at $n+1$ x–coordinates. For $n = 1$ this is a familiar statement; a line (degree one polynomial) is determined by two points. Moreover, the value of $p(x)$ at any other x–value can be computed in a particularly nice way using Lagrange interpolation, as outlined in the essay *An Interpretation of Interpolation*.

We will also need a result from linear algebra which states that a system of n "different" linear equations in n variables has exactly one solution. For example, there is only one choice for x, y, and z which satisfies the equations $x + y + z = 1$, $x + 2y + 3z = 4$, and $x + 4y + 9z = 16$.

SETUP: Let $p(x)$ be a degree three polynomial for which we know the values of $p(1)$, $p(2)$, $p(4)$, and $p(8)$. By the facts section there is exactly one such polynomial. According to Lagrange interpolation the number $p(16)$ can be deduced; it equals

$$p(16) \;=\; A_0 p(1) + A_1 p(2) + A_2 p(4) + A_3 p(8)\,,$$

for some constants A_0 through A_3. The goal of this team test will be to compute the A_i and use them to find information about $p(16)$ *without ever finding an explicit formula for* $p(x)$.

PROBLEMS:

Part i: (4 points) We claim that the A_i can be found by subtracting

$$x^4 - (x-1)(x-2)(x-4)(x-8) \;=\; A_3x^3 + A_2x^2 + A_1x + A_0\,. \tag{2}$$

Implement this claim to compute A_0 through A_3.

Part ii: (4 points) To show that these A_i are in fact the correct numbers we must show that they correctly predict $p(16)$ for four "different" polynomials. We begin with the case $p(x) = x$. Show that the value of $p(16)$ agrees with the prediction $A_0p(1) + A_1p(2) + A_2p(4) + A_3p(8)$. (HINT: try $x = 2$ in (2).)

Part iii: (5 points) Continuing the previous part, show that the A_i correctly predict $p(16)$ for the three other polynomials $p(x) = 1$, $p(x) = x^2$ and $p(x) = x^3$.

Part iv: (4 points) Suppose that $p(x)$ is a third degree polynomial with $p(1) = 0$, $p(2) = 1$, and $p(4) = 3$. What value should $p(8)$ have to guarantee that $p(x)$ has a root at $x = 16$?

Part v: (4 points) Let $p(x)$ be a degree three polynomial with $p(1) = 1$, $p(2) = 3$, $p(4) = 9$, and $p(8) = 27$. Calculate $p(16)$. How close does it come to the natural guess of 81?

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5**, or to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario. N2L 3G1**. The electronic address is

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Mayhem Problems

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Please include on any correspondence your name, school, grade, city, province
or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 February 2002*. Look for prizes for solutions in the new year.

M8. *Proposed by the Mayhem staff.*

Find all right-angled triangles with integer sides if one of the sides is 2001 units long.

M9. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

Find integers a, b, and c (not all equal) with $a + b + c = 2001$, such that a, b, and c form an arithmetic sequence (in that order) and $a + b$, $b + c$, and $c + a$ form a geometric sequence (in that order).

M10. *Proposed by Nicolae Gustia, North York, Ontario.*

(a) Factor fully $2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$.

(b) Find the geometric interpretation of the above expression if a, b, and c are sides of a non-degenerate triangle.

M11. *Proposed by the Mayhem staff.*

Two sequences a_1, a_2, …, a_{2001} and b_1, b_2, …, b_{2001} are formed by the following rules:

- $a_1 = 5$ and $a_2 = 3$,
- $b_1 = 9$ and $b_2 = 7$,
- $\dfrac{a_n}{b_n} = \dfrac{a_{n-1} + a_{n-2}}{b_{n-1} + b_{n-2}}$ for $n > 2$ such that each $\dfrac{a_n}{b_n}$ is in lowest terms.

What is the smallest fraction of the form $\dfrac{a_n}{b_n}$?

M12. *Proposed by Richard Hoshino, Dalhousie University, Halifax, Nova Scotia.*

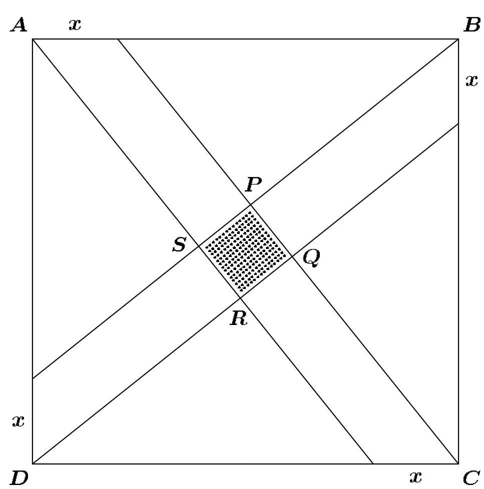
Determine all ordered pairs (x, y) with $\gcd(x, y) = 1$, and $x < y$ such that $2000\left(\frac{x}{y} + \frac{y}{x}\right)$ is an odd integer.

M13. *Proposed by the Mayhem staff.*

Given $n = 9 + 99 + 999 + \cdots + \overbrace{999\cdots 99}^{2001\ 9\text{'s}}$, how many of n's digits are 1's?

M14. *Proposed by the Mayhem staff.*

Starting with a square $ABCD$ with unit area we construct four points a distance x from each vertex (located on the clockwise side). Each of these points is connected to the vertex opposite to it so that a small quadrilateral $PQRS$ is formed in the middle of the square. Find the value of x that makes the area of $PQRS$ equal to $\frac{1}{2001}$.



Polya's Paragon

Shawn Godin

Number theory is a beautiful area of mathematics that, for the most part, deals with properties of the positive integers $1, 2, 3, \ldots$. As a result, many of the toughest problems in the field can be understood by high school students.

If you study number theory, you will find that prime numbers play a central role. Sometimes you will find that breaking things down into primes can shed some light on the original problem. This is helpful because of the following theorem

The Fundamental Theorem of Arithmetic

Every integer $n > 1$ can be expressed as a product of primes in only one way (apart from rearranging the factors).

Thus, $6 = 2{\cdot}3$, $17 = 17$ and $1440 = 2{\cdot}2{\cdot}2{\cdot}3{\cdot}3{\cdot}5{\cdot}5{\cdot}5{\cdot}5 = 2^3{\cdot}3^2{\cdot}5^4$. And there will never be a case where there is more than one such decomposition.

Now let us take a look at this unique factorization in action in a couple of problems.

Example 1: How many five-digit positive integers have the property that the product of their digits is 3000?

Solution: If we break it up into primes we get $3000 = 2^3 \cdot 3 \cdot 5^3$. Now any number that divides evenly into 3000 must be of the form $2^x \cdot 3^y \cdot 5^z$, where $0 \le x \le 3$, $0 \le y \le 1$ and $0 \le z \le 3$. Since we are looking for a five-digit number, we are looking for numbers d_i with $0 < d_i < 10$ for $i = 1$, $2, \ldots, 5$ such that $d_1 \times d_2 \times \cdots \times d_5 = 3000$.

If we look at the condition that $d_i < 10$ it tells us that if we use a factor of 5 in one of the d_i's, there can be no other factors. Thus three of the digits must be 5's.

The other two digits must be made up of 2's and 3's in such a way as their product is $2^3 \cdot 3 = 24$. When it is in this form we see that the term that has a factor of 3 can only be 3 or $3 \cdot 2$. Thus the other two digits can be either 3 and 8 or 6 and 4.

Therefore, to answer the original question, the digits can be 3, 5, 5, 5, 8 (we can create $\frac{5!}{3!} = 20$ five-digit numbers from these) or 4, 5, 5, 5, 6 (similarly with 20 numbers); thus, there are 40 five-digit numbers with the product of their digits equal to 3000.

Example 2: Find the sum of all the divisors of 540000.

Solution: This is based on a classic result from number theory. Notice that $540000 = 2^5 \cdot 3^3 \cdot 5^4$. As in example 1, we note that any divisor of 540000 must be of the form $2^x \cdot 3^y \cdot 5^z$ with $0 \le x \le 5$, $0 \le y \le 3$ and $0 \le z \le 4$. The result we are after is the sum of all of such numbers. If we take a second to examine the product

$$(1 + 2 + 2^2 + 2^3 + 2^4 + 2^5)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2 + 5^3 + 5^4)\,,$$

we notice that every term in that expansion is in the required form, and that every number that we are after is there. Therefore, the result we want is then just

$$63 \cdot 40 \cdot 781 \;=\; 1968120\,.$$

The result above is just a tad easier than finding and adding up all 120 (how did I get that?) divisors of 540000, is it not?

We can summarize this result by defining $\sigma(n)$ to be the sum of all the divisors of n. Then, if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of n, we must have

$$\begin{aligned}\sigma(n) \;=\; & (1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1}) \times (1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}) \\ & \times \cdots \times (1 + p_k + p_k^2 + \cdots + p_k^{\alpha_k})\,.\end{aligned}$$

A closer examination of each of the brackets in the product reveals that they are all geometric series. Therefore, we can write

$$\sigma(n) \;=\; \left(\frac{1 - p_1^{\alpha_1+1}}{1 - p_1}\right)\left(\frac{1 - p_2^{\alpha_2+1}}{1 - p_2}\right)\cdots\left(\frac{1 - p_k^{\alpha_k+1}}{1 - p_k}\right)\,.$$

Whew! I think we had better stop there for this issue. Besides at least one problem from elsewhere in this issue, here are a couple of others to try.

1. If $n \ge 1$, the notation $n!$ (read n *factorial*) represents the product of all integers from 1 to n. So $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$. (For reasons that we will not go into here, it is useful to define $0! = 1$. Maybe some other time). Find the value of n if $n! = 2^{31} \cdot 3^{15} \cdot 5^7 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$.

2. Find the largest power of 2 that evenly divides $2001!$

3. Find the positive integer $n < 10^{10}$ that has the most divisors.

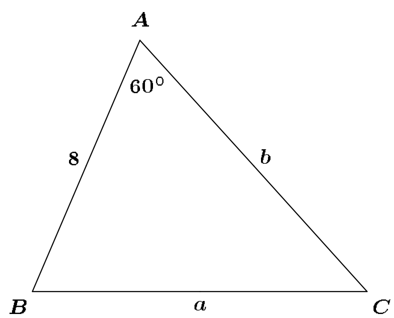
Problem of the Month

Jimmy Chui, student, University of Toronto

Problem.

Problem. In triangle ABC, $AB = 8$, and $\angle CAB = 60^\circ$. Sides BC and AC have integer lengths a and b, respectively. Find all possible values of a and b.

(1997 Euclid, Problem 9)



Comments. Because we are dealing with integer solutions to a problem, there is an extremely good chance that we must somehow form an algebraic equation from this geometric question. We want to find the relation between the three sides of a triangle, given an angle. The Cosine Law gives us this relationship!

Because we are reducing to an algebraic equation, we must be certain not to get answers that make no sense; that is, we must make sure that none of the implicit assumptions are lost in the translation from geometry to algebra. In this case, we must make sure that a and b are positive, and that a, b, and 8 form a legitimate triangle. They will form a triangle if and only if they satisfy the triangle inequality: the sum of any two sides must exceed the third.

Solution. The Cosine Law applied to $\angle CAB$ gives us the equation

$$\begin{aligned} a^2 &= b^2 + 8^2 - 2 \cdot b \cdot 8 \cdot \cos 60^\circ \\ &= b^2 - 8b + 64\,. \end{aligned}$$

Let us factor this equation with all the variables on one side. This is a standard technique for solving equations for integer solutions.

$$\begin{aligned} a^2-(b^2-8b) &= 64\\ a^2-(b^2-8b+16) &= 64-16\\ a^2-(b-4)^2 &= 48\\ (a+b-4)(a-b+4) &= 48\,. \end{aligned}$$

Now, since a and b are both integers, both factors on the left side must be integers as well. Hence, we can equate them to pairs of factors of 48. For example, $48 = 6 \times 8$ and so we solve for:

$$\begin{cases} a+b-4 = 6\\ a-b+4 = 8 \end{cases}$$

and

$$\begin{cases} a+b-4 = 8\\ a-b+4 = 6 \end{cases}$$

We would have to do this for every pair of integers, positive and negative, that multiply to 48, and disregard any answers that do not conform to the triangle inequality. But instead of trying all the pairs, why not make some more calculations to get rid of possible pairs?

First, let us assume that we let the two factors $(a+b-4)$ and $(a-b+4)$ be both negative. Then $(a+b-4)+(a-b+4) = 2a$ which would be negative. But a is a measurement, and must be positive! $2a$ must be positive as well. $2a$ cannot be both negative and positive; hence, we can throw out the pairs of negative numbers that multiply to 48.

Second, notice that $(a+b-4)+(a-b+4) = 2a$. The sum of the two factors is even. What does this tell us? The factors must be either both odd or both even! (We say that the factors have the same parity.) Therefore, the couple 1 and 48 can be thrown out, as well as 3 and 16.

This leaves us with the couples 2 and 48, 4 and 12, 6 and 8. Now that we have a lesser number of cases to test, why not just solve for a and b for each pair?

We will get 6 pairs of (a, b) answers from these couples.

$$\begin{cases} a+b-4 = 2\\ a-b+4 = 24 \end{cases}$$

gives us $a = 13$ and $b = -7$.

$$\begin{cases} a+b-4 = 24\\ a-b+4 = 2 \end{cases}$$

gives us $a = 13$ and $b = 15$.

$$\begin{cases} a+b-4 = 4\\ a-b+4 = 12 \end{cases}$$

gives us $a = 8$ and $b = 0$.

$$\begin{cases} a+b-4 &= 12 \\ a-b+4 &= 4 \end{cases}$$

gives us $a = 8$ and $b = 8$.

$$\begin{cases} a+b-4 &= 6 \\ a-b+4 &= 8 \end{cases}$$

gives us $a = 7$ and $b = 3$.

$$\begin{cases} a+b-4 &= 8 \\ a-b+4 &= 6 \end{cases}$$

gives us $a = 7$ and $b = 5$.

We can throw out $(a, b) = (13, -7)$ since b must be positive. We can also throw out $(a, b) = (8, 0)$ since b must be positive. (We should mention that this latter solution is a degenerate solution.)

Now we are left with the other four pairs of solutions. Consider the pair $(a, b) = (13, 15)$. Can we have a triangle with sides 13, 15, and 8? From the triangle inequality, $13 + 15 - 8 = 20 > 0$, $13 + 8 - 15 = 6 > 0$, $15 + 8 - 13 = 10 > 0$. Therefore, the sum of any two sides exceeds the third.

We could do this to the other three pairs of solutions, but we will find that they also satisfy the triangle inequality.

The final check we might want to do would be to verify that $\angle CAB = 60^\circ$. However, we arrived at all these solutions from the Cosine Law, and so it is obvious that these solutions will satisfy the Cosine Law. Hence it is not necessary to do this check.

We conclude that the only values for a and b are $(a, b) = (7, 3)$, $(7, 5)$, $(8, 8)$, and $(13, 15)$.

High School Solutions

H273. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, and c be complex numbers such that $a + b + c = 0$. Prove that

$$\begin{vmatrix} 2ab - c^2 & b^2 & a^2 \\ b^2 & 2bc - a^2 & c^2 \\ a^2 & c^2 & 2ac - b^2 \end{vmatrix} = 0\,.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let D denote the given determinant. Adding the second and third column to the first column, we have:

$$D = \begin{vmatrix} (a+b)^2 - c^2 & b^2 & a^2 \\ (b+c)^2 - a^2 & 2bc - a^2 & c^2 \\ (c+a)^2 - b^2 & c^2 & 2ca - b^2 \end{vmatrix}$$

Since $(a+b)^2 - c^2 = (a+b+c)(a+b-c) = 0$, $(b+c)^2 - a^2 = (b+c+a)(b+c-a) = 0$ and $(c+a)^2 - b^2 = (c+a+b)(c+a-b) = 0$, it follows that $D = 0$.

H274. Find a simplified expression for

$$\sum_{i=1}^{\infty} \frac{i}{k^i}$$

in terms of a real number $k > 1$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let S_k denote the sum of the given series, which is clearly convergent by the Ratio Test. Then,

$$\begin{aligned} kS_k &= \sum_{i=1}^{\infty} \frac{i}{k^{i-1}} = \sum_{i=0}^{\infty} \frac{i+1}{k^i} \\ &= 1 + S_k + \sum_{i=1}^{\infty} \frac{1}{k^i} = 1 + S_k + \frac{\frac{1}{k}}{1 - \frac{1}{k}} = 1 + S_k + \frac{1}{k-1}, \end{aligned}$$

from which we get $(k-1)S_k = \frac{k}{k-1}$ and thus, $S_k = \frac{k}{(k-1)^2}$.

H275. How many non-negative integers less than 10^n are there whose digits are in non-increasing order?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The answer is $\binom{n+10}{10} - n$.

Proof. Let $S = \{t \in \mathbb{Z} \mid 0 \leq t < 10^n$ such that the digits of t are in non-increasing order $\}$. Note first that any $t \in S$ has k digits for some $k = 1, 2, \ldots, n$. These k digits are all from $\{0, 1, 2, \ldots, 9\}$ with possible repetitions. Conversely, any k digits chosen from $\{0, 1, 2, \ldots, 9\}$ with repetitions permitted, can be arranged in non-increasing order, in only one way, to yield some $t \in S$, except when $k \geq 2$ and all the digits chosen are 0's. From a well known result in combinatorics, the number of such choices, for each fixed k, is $\binom{10+k-1}{k}$ or $\binom{k+9}{9}$. Discarding the cases when all the k digits chosen are 0's and adding 1 to account for the choice of 0 when $k = 1$, we then have

$$\begin{aligned} |S| &= 1 + \sum_{k=1}^{n} \left(\binom{k+9}{9} - 1 \right) = -n + 1 + \sum_{k=1}^{n} \binom{k+9}{9} \\ &= -n + \sum_{k=0}^{n} \binom{k+9}{9} \end{aligned}$$

Since it is well known that $\binom{a}{a} + \binom{a+1}{a} + \cdots + \binom{b}{a} = \binom{b+1}{a+1}$ for all a, $b \in \mathbb{N}$ with $a \leq b$ (see, for example, p. 217 of Applied Combinatorics by Alan Tucker, 3rd ed.), we finally obtain $|S| = -n + \binom{n+10}{10}$ as claimed.

H276. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

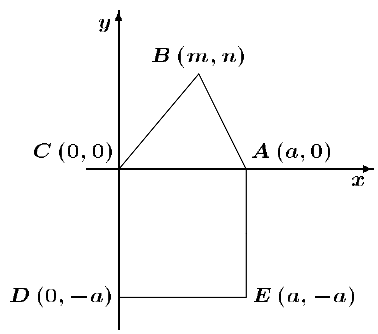
Let $ABCDE$ be a convex pentagon such that $ACDE$ is a square, and

$$\cot \angle BDE + \cot \angle DEB + \cot \angle EBD = 2 .$$

Show that $\triangle ABC$ is a right triangle.

Solution by the proposer.

Attach our figure to a grid and label the points as in the diagram below.



It is known that

$$\cot(\angle BDE) + \cot(\angle DEB) + \cot(\angle EBD) = \frac{\overline{BD}^2 + \overline{DE}^2 + \overline{EB}^2}{4[BDE]} ,$$

where $[BDE]$ is the area of the figure BDE. It follows that

$$\begin{aligned} &\cot(\angle BDE) + \cot(\angle DEB) + \cot(\angle EBD) \\ &= \frac{m^2 + (n+a)^2 + a^2 + (m-a)^2 + (n+a)^2}{4 \cdot \frac{1}{2} a(a+n)} \\ &= \frac{2m^2 - 2ma + 2n^2 + 4an + 4a^2}{2a(a+n)} = \frac{2m^2 - 2ma + 2n^2}{2a(a+n)} + 2 . \end{aligned}$$

By the hypothesis, we get $0 = \frac{2m^2-2ma+2n^2}{2a(a+n)}$ or $2m^2-2ma+2n^2=0$. This implies that $(m^2+n^2)+\{(m-a)^2+n^2\}=a^2$, which means that $\overline{BC}^2+\overline{BA}^2=\overline{CA}^2$. It follows that $\angle ABC = 90°$.

H277.

(a) Find all right triangles with integer sides with perimeter 60.

(b) Find all right triangles with integer sides with area 600.

Solution by Mihály Bencze, Brasov, Romania.

(a) If we let a, b and c be the sides of the triangle, we have a, b, $c \in \mathbb{Z}^+$ with $a+b+c=60$ and $a^2=b^2+c^2$. Thus we must have

$$\begin{array}{rclcrcl} a & = & x^2+y^2 & \quad & b & = & 2xy \\ c & = & x^2-y^2 & \quad & x & > & y \end{array}$$

Combining these gives $x(x+y)=30$ with solution $x=5$, $y=1$, which yields $a=26$, $b=10$, $c=24$.

(b) Using a, b, c, x and y as defined in (a), we get $xy(x^2-y^2)=600$.

If $y=1$ we get $(x-1)x(x+1)=600$, which has no solutions. Similarly,

$$\begin{aligned} y=2 &\implies (x-2)x(x+2)=300 \\ y=3 &\implies (x-3)x(x+3)=200 \\ y=4 &\implies (x-4)x(x+4)=150 \\ y=5 &\implies (x-5)x(x+5)=120 \\ y=6 &\implies (x-6)x(x+6)=100 \\ y=8 &\implies (x-8)x(x+8)=75 \end{aligned}$$

[Ed. Since $x>y$ we have $xy(x^2-y^2)=xy(x+y)(x-y) > y\cdot y\cdot y\cdot 1 = y^3$, so that $y^3<600$, and $y<9$.]

Since none of these yield solutions, there are no solutions.

H278. Consider the time as seen on a digital clock in 24–hour mode. (24–hour mode is representing the time relative to 12 midnight. For example, $6{:}25$ am is $06{:}25$, but $6{:}25$ pm is $18{:}25$. Also, $12{:}45$ am counts as $00{:}45$.) Let n be the number we get when we remove the colon from the time T as seen on a digital clock in 24 hour mode. Find all times T such that:

(i) n is a palindrome, [Ed. reads the same backwards as forwards.]

(ii) m, the number of minutes that T is after midnight, is a palindrome, and

(iii) $n=m$.

Solution by the editors.

Since n is a palindrome, T must be of the form:

(a) $AB : BA$ if $0 < A \leq 2, 0 \leq B \leq 5$
(b) $0C : DC$ if $0 < C \leq 9, 0 \leq D \leq 5$
(c) $00 : EE$ if $0 < E \leq 5$
(d) $00 : 0F$ if $0 \leq F \leq 9$

CASE A. $n = 1000A + 100B + 10B + A = 1001A + 110B$ and $m = 60(10A + B) + (10B + A) = 601A + 70B$. If $n = m$, then $400A + 40B = 0$. Since $A > 0$ and $B \geq 0$, there are no solutions for this case.

CASE B. $n = 100C + 10D + C = 101C + 10D$ and $m = 60C + 10D + C = 61C + 10D$. If $n = m$, $40C = 0$, a contradiction, so that there are no solutions for this case.

CASE C. All values of E from 1 to 5 inclusive work.

CASE D. All values of F from 0 to 9 inclusive work.

All times are 00:00, 00:01, 00:02, 00:03, 00:04, 00:05, 00:06, 00:07, 00:08, 00:09, 00:11, 00:22, 00:33, 00:44, 00:55.

H279. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Let a and b be integers such that $a \equiv b \pmod 3$. Prove that

$$\tfrac{2}{3}\,\left(a^2 + ab + b^2\right)$$

can be expressed as a sum of three non-negative squares.

I. *Solution by the proposer.*

We first note that $\frac{a+2b}{3}$, $\frac{2a+b}{3}$, and $\frac{a-b}{3}$ are integers since $a \equiv b \pmod 3$. By expanding and simplifying, we easily see that

$$\frac{2}{3}(a^2 + ab + b^2) \;=\; \left(\frac{a+2b}{3}\right)^2 + \left(\frac{2a+b}{3}\right)^2 + \left(\frac{a-b}{3}\right)^2 .$$

II. *Solutions by Mihály Bencze, Brasov, Romania and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Since $a \equiv b \pmod 3$ we have $a = 3k + b$ for some $k \in \mathbb{Z}$. Thus

$$\frac{2}{3}(a^2 + ab + b^2) \;=\; 6k^2 + 6kb + 2b^2 \;=\; (2k+b)^2 + k^2 + (k+b)^2 .$$

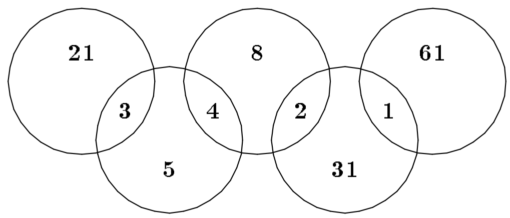
[Ed. note that the decompositions in I and II are equivalent]

H280. *Proposed by Fotifo Casablanca, Bogotá, Colombia.*

In the spirit of the Olympics: There are 9 regions inside the rings of the Olympics. Put a different positive whole number in each so that the five products of the numbers in each ring form a set of five consecutive integers.

Solution by the proposer.

The solution with the smallest possible set of consecutive numbers is as follows



H281 *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Suppose the monic polynomial $A(z) = \sum_{k=0}^{n} a_k z^k$ can be factored into $(z-z_1)(z-z_2)\cdots(z-z_n)$, where $z_1, z_2, \ldots, z_n$ are positive real numbers. Prove that $a_1 a_{n-1} \geq n^2 a_0$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We add the assumption that n is even (or equivalently, $a_0 > 0$) for otherwise it is easy to see that the conclusion is false.

By comparing the coefficients of appropriate terms, we have

$$a_0 = A(0) = (-1)^n \prod_{k=1}^{n} z_k = \prod_{k=1}^{n} z_k$$

$$a_1 = (-1)^{n-1} \sum_{k=1}^{n} \frac{z_1, z_2, \ldots, z_n}{z_k} = -\sum_{k=1}^{n} \frac{a_0}{z_k}, \quad \text{and}$$

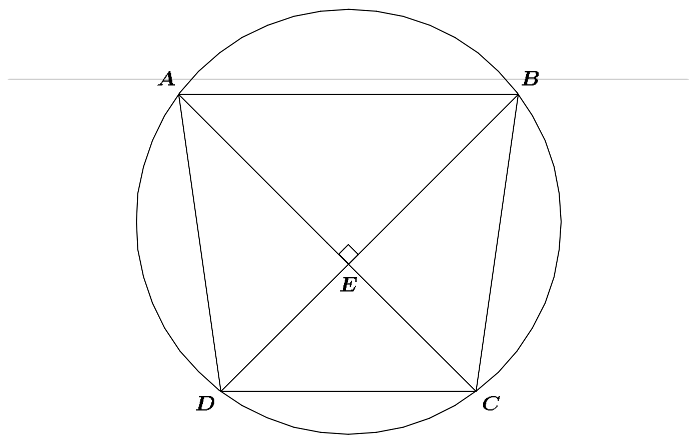
$$a_{n-1} = -\sum_{k=1}^{n} z^k .$$

Hence by the Arithmetic-Harmonic-Mean Inequality (or Cauchy–Schwarz Inequality), we have

$$a_1 a_{n-1} = a_0 \left(\sum_{k=1}^{n} \frac{1}{z_k} \right) \left(\sum_{k=1}^{n} z_k \right) \geq n^2 a_0 .$$

H282. Let $ABCD$ be a cyclic quadrilateral such that its diagonals are perpendicular. Let E be the intersection of AC and BD. It is known that $AE + ED = BE + EC$. Show that $ABCD$ is a trapezoid.

Solution by the editors.

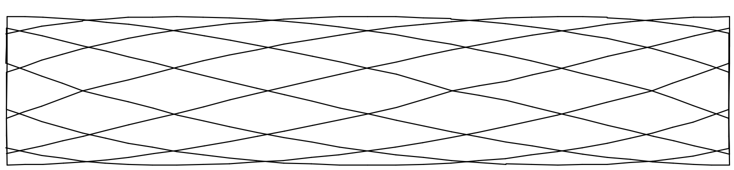


We have $AC \perp BD$ and $AE + ED = BE + EC$. Now, since the quadrilateral cyclic, we have

$$\begin{aligned} AE \cdot EC &= BE \cdot ED\,, \\ (BE + EC - ED) \cdot EC &= BE \cdot ED\,, \\ BE \cdot EC + EC^2 - DE \cdot CE - BE \cdot ED &= 0\,, \\ (EC - ED)(EC + BC) &= 0\,. \end{aligned}$$

Thus, $EC = ED$. If $EC = ED$, then $\angle DCE = \angle CDE = 45°$. But $\angle ABE = \angle DCE = 45° = \angle CDE$, so that $AB \parallel CD$.

Similarly, $ED = AE$. Then, $AD \parallel BC$. Thus, $ABCD$ is a trapezoid.



PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 April 2002**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in* LaTeX *format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

In the September issue, [2001 : 337], problems 2660–2663 were incorrectly numbered 2560–2563. How could all of our proof-readers have missed that!!

Professor Murray Klamkin has pointed out that proposal 2636 is essentially the same as proposal 2609. Solvers should therefore consider 2636 as withdrawn.

2664. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Let a, b and c be positive real numbers such that $a+b+c=abc$. Prove that $a^5(bc-1)+b^5(ca-1)+c^5(ab-1) \geq 54\sqrt{3}$.

2665. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

In $\triangle ABC$, we have $\angle ACB = 90^\circ$ and sides $AB=c$, $BC=a$ and $CA=b$. In $\triangle DEF$, we have $\angle EFD = 90^\circ$, $EF=(a+c)\sin\left(\frac{B}{2}\right)$ and $FD=(b+c)\sin\left(\frac{A}{2}\right)$. Show that $DE \geq c$.

2666. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Two circles, Γ with diameter AB, and Δ with centre A, intersect at points C and D. The point M (distinct from C and D) lies on Δ. The lines BM, CM and DM intersect Γ again at N, P and Q respectively. Prove that

1. the quadrilateral $MPBQ$ is a parallelogram;
2. MN is equal to the geometric mean of NC and ND.

2667. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

You are given a circle Γ and two points A and B outside of Γ such that the line through A and B does not intersect Γ. Let X be any point on Γ.

Determine at which point X on Γ the sum $AX + XB$ attains its minimum value.

2668$\star$. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Suppose that $0 < r < q < 1$ and that $0 < m < \infty$. Show that

$$\begin{aligned}(1-q)(q+r-qr)\sqrt{1+m^2} &+ q(1-r)\sqrt{(q-2)^2+m^2q^2} \\ &> (1-r)(q+r-qr)\sqrt{1+m^2} + r(1-q)\sqrt{(r-2)^2+m^2r^2}\,.\end{aligned}$$

2669$\star$. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* (Professor Klamkin offers a prize of $50 for the first correct solution received by the Editor-in-Chief.)

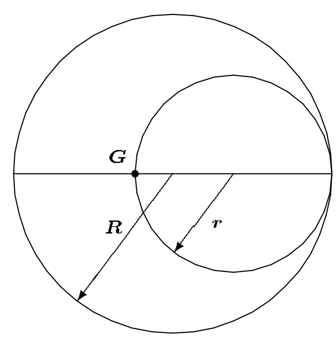
Let A_1, A_2, ..., A_{2n}, be any $2n$ points in $\mathbb{E}^m$. Determine the largest k_n such that

$$A_1A_2^2+A_2A_3^2+\cdots+A_{2n}A_1^2 \;\geq\; k_n\left(A_1A_{n+1}^2 + A_2A_{n+2}^2 + \cdots + A_nA_{2n}^2\right)\,.$$

For $n = 2$, it is easily shown that $k_2 = 1$. That $k_3 = \frac{1}{2}$ is an Armenian Olympiad problem [2001 : 9].

2670. *Proposed by Robert C.H. Schmidt, Minnetonka, MN, USA.*

A disk of uniform thickness and composition has radius R. From this disk, a smaller disk of radius r is cut that is internally tangent to the original disk. If the centre of gravity, G, of the resulting object lies on the common diameter of the two disks, at the point (other than the point of tangency) where this diameter intersects the circumference of the smaller disk, determine the ratio $R : r$.



2671. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given $\triangle ABC$ with circumcentre O and incentre I, let P be the point dividing OI in the ratio $OP : PI = -1 : 4$. Let X, Y and Z be the feet of the perpendiculars from P to the sides BC, CA and AB respectively.

(a) Show that $AY + AZ = BZ + BX = CX + CY$.

(b) Show that P is the centroid of the triangle whose vertices are the excentres of $\triangle ABC$.

2672. *Proposed by Vedula N. Murty, Dover, PA, USA.*

(a) Suppose that $\alpha > 0$. Prove that $\sum\limits_{k=1}^{n} k^{\alpha} \;<\; \dfrac{(n+1)^{\alpha+1}-1}{\alpha+1}$.

(b) Suppose that $-1 < \alpha < 0$. Prove that $\dfrac{(n+1)^{\alpha+1}-1}{\alpha+1} \;<\; \sum\limits_{k=1}^{n} k^{\alpha}$.

[These two inequalities appear differently in "*Analytic Inequalities*" by Nicholas D. Kazarinoff, Holt Rinehart and Winston, p. 24. The term "-1" is missing from the numerators.]

2673. *Proposed by George Baloglou, SUNY Oswego, Oswego, NY, USA.*

Let $n \geq 2$ be an integer.

(a) Show that

$$(1 + a_1 \cdots a_n)^n \;\geq\; (a_1 \cdots a_n)\left(1 + a_1^{n-2}\right)\left(1 + a_2^{n-2}\right) \cdots \left(1 + a_n^{n-2}\right)$$

for all $a_1 \geq 1$, $a_2 \geq 1$, ..., $a_n \geq 1$, if and only if $n \leq 4$.

(b) Show that

$$\frac{1}{a_1\left(1 + a_2^{n-2}\right)} + \frac{1}{a_2\left(1 + a_3^{n-2}\right)} + \cdots + \frac{1}{a_n\left(1 + a_1^{n-2}\right)} \;\geq\; \frac{n}{1 + a_1 \cdots a_n}$$

for all $a_1 > 0$, $a_2 > 0$, ..., $a_n > 0$, if and only if $n \leq 3$.

(c) Show that

$$\frac{1}{a_1\left(1 + a_1^{n-2}\right)} + \frac{1}{a_2\left(1 + a_2^{n-2}\right)} + \cdots + \frac{1}{a_n\left(1 + a_n^{n-2}\right)} \;\geq\; \frac{n}{1 + a_1 \cdots a_n}$$

for all $a_1 > 0$, $a_2 > 0$, ..., $a_n > 0$, if and only if $n \leq 8$.

(d)$\star$ Show that

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\left(\frac{1}{1 + a_1^{n-2}} + \frac{1}{1 + a_2^{n-2}} + \cdots + \frac{1}{1 + a_n^{n-2}}\right)$$
$$\geq\; \frac{n^2}{1 + a_1 \cdots a_n}$$

for all $a_1 > 0$, $a_2 > 0$, ..., $a_n > 0$, if and only if $n \leq 5$.

2674. *Proposed by Mohammed Aassila, Strasbourg, France.*

Find an explicit formula for the least number $f(n)$ of distinct points in the plane such that, for each $k = 1, 2, \ldots, n$, there exists a straight line containing exactly k of these points.

2675. *Proposed by Joe Howard, Portales, NM, USA.*

Show that

$$\cos^2\left(\frac{\pi}{7}\right) + \cos^2\left(\frac{2\pi}{7}\right) + \cos^2\left(\frac{3\pi}{7}\right) \;=\; 10\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right).$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2514. [2000 : 114, 2001 : 143] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

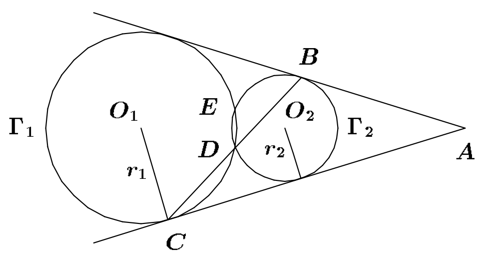
In $\triangle ABC$, the internal bisectors of $\angle ABC$ and $\angle BCA$ meet CA and AB at D and E respectively. Suppose that $AE = BD$ and that $AD = CE$. Characterize $\triangle ABC$.

Correction.

The editor's comment should have read "All of the other solvers, with one exception, also showed that the base angles of the triangle are 72°."

2560*. [2000 : 305] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Lines AB and AC are common tangents to the circles Γ_1 and Γ_2 with distinct radii r_1 and r_2 respectively, as shown.



B is a point of tangency on Γ_2 and C is a point of tangency on Γ_1. The intersection points of the circles, D and E, exist, CDB is a straight line, and $CD = DB$.

Construct such a figure using straightedge and compass.

Combination of solutions by Michel Bataille, Rouen, France and D.J. Smeenk, Zaltbommel, the Netherlands.

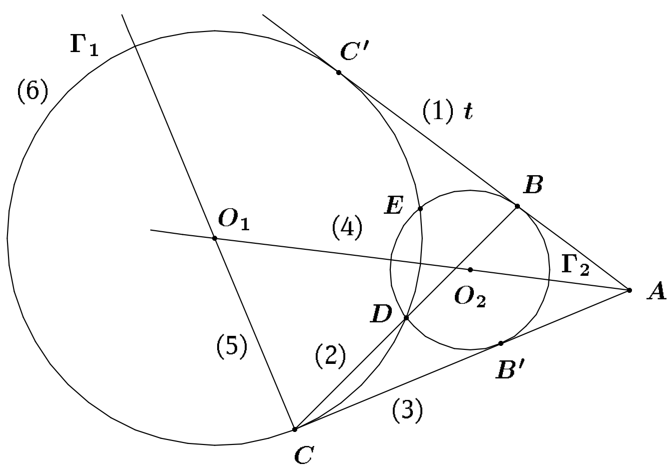
We will construct the configuration given an arbitrary circle Γ_2, its centre O_2, and any two of its points in the role of B and D.

1. Construct the tangent t to Γ_2 at B.
2. Extend the segment BD to a point C beyond D so that D is the midpoint of BC.
3. From C construct either tangent to Γ_2; call the point of tangency B' and let A be the point where it intersects t. (Note that at least one of the tangents must meet t.)

4. Construct the line AO_2.

5. Construct the perpendicular to CA at C, and let O_1 be the point where it meets AO_2. (O_1 exists since CO_1 is parallel to $B'O_2$.)

6. Draw the circle with centre O_1 and radius O_1C.



Claim: The circle in step 6 is the desired circle Γ_1, and the construction is complete.

Proof. This circle is tangent to AC (at C) by construction and to AB (at C', say) by symmetry about the line of centres AO_2O_1. It remains to prove only that Γ_1 intersects Γ_2 in D and in one other point. Assume that Γ_1 intersects the line CB again in D'; we must show that $D' = D$. Since B is exterior to Γ_1, $BD' \cdot BC = BC'^2$; since C is exterior to Γ_2, $CD \cdot CB = CB'^2$. But $CB' = BC'$ (common tangents are equal), and $BD = CD$ (since D is the mid-point of BC); thus the two equations reduce to $BD' = CD = BD$, which implies that $D' = D$ as desired. Finally, D cannot be the unique intersection point of the two circles because that would imply that it lies on the line of centres, and therefore $DB = DB' = DC$; that would force $CB'B$ to be a right angle, which contradicts the fact that $\triangle AB'B$ is isosceles.

Editor's comment. Note that the featured solution proves more than was claimed:

Theorem. Given a pair of circles that have common tangent lines AB and AC with B on one circle and C on the other, suppose BC meets one of the circles again at D; then D lies on the other circle if and only if it is the mid-point of BC.

As a consequence, in the statement of problem 2560 the fact that BCD is a straight line forces $CD = DB$.

Also solved by TOSHIO SEIMIYA, Kawasaki, Japan; Mª JESÚS VILLAR RUBIO, Santander, Spain; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Villar Rubio and Woo interpreted the problem in a different way. They showed how to construct the configuration when given the lengths of the radii r_1 and r_2. In particular, they showed that the distance between the centres satisfies $O_1O_2 = 2\sqrt{r_1r_2}$; furthermore, the construction is possible if and only if $3-\sqrt{8} < \frac{r_1}{r_2} < 3+\sqrt{8}$.

2561. [2000 : 305] *Proposed by Hassan A. ShahAli, Tehran, Iran.*

Let M disks from N different colours be placed in a row such that k_i disks are from the i^{th} colour ($i = 1, 2, \ldots, N$) and $k_1 + k_2 + \cdots + k_N = M$.

A move is an exchange of two adjacent disks.

Determine the smallest number of moves needed to rearrange the row such that all disks of the same colour are adjacent to one another.

Editor's remark: There were no solutions submitted to this problem. Indeed, it should have been starred, since the proposer did not submit a solution either. Consequently, the problem remains open.

2562. [2000 : 305] *Proposed by Bernardo Recamán Santos, Colegio Hacienda Los Alcaparros, Bogotá, Colombia.*

(a) Show that for all sufficiently large n, it is possible to find a set of n (not necessarily distinct) positive integers whose sum is the square root of their product.

(b)* Are there infinitely many n for which there is a unique set of n numbers with property (a)?

I. *Solution by Oleg Ivrii, (grade 8) student, Cummer Valley Middle School, North York, Ontario.*

For each $n \geq 3$, let $S = \{n+6, n+6, 9, 1, 1, \ldots, 1\}$ where there are $(n-3)$ 1's, and for each $n \geq 4$, let $T = \{n+2, n+2, 3, 3, 1, 1, \ldots, 1\}$ where there are $(n-4)$ 1's. Then, it is readily verified that both S and T satisfy the given condition. [Ed: Hence the answer to $(b)^\star$ is "no".]

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (slightly adapted by the editor).*

For $n = 1$, clearly the only such set is $\{1\}$. For $n = 2$, no such sets exist since $a + b = \sqrt{ab}$ is impossible for positive numbers by the AM–GM Inequality. For $n = 3$, two such sets are $\{9, 9, 9\}$ and $\{9, 36, 225\}$. For any $n \geq 4$, one such set is $\{n+2, n+2, 3, 3, 1, 1, \ldots, 1\}$ where there are $(n-4)$ 1's.

[Ed: both this set and the first set given in the $n = 3$ case are the same as the ones obtained by Ivrii.]

Furthermore, if n is a composite, $n = ab$, say, where $a \geq 2$ and $b \geq 2$, then another set satisfying the given condition would be

$$\{a+2,\, a+2,\, b+2,\, b+2,\, 1,\, 1,\, \ldots,\, 1\}\,.$$

Finally, if $n \geq 5$ is a prime, then

$$\{\tfrac{n+3}{2},\, \tfrac{n+3}{2},\, 4,\, 2,\, 2,\, 1,\, 1,\, \ldots,\, 1\}\,,$$

where there are $(n-5)$ 1's, would be another such set.

Part (a) *only was also solved by the proposer whose example is*

$$\{2n+4,\, n+2,\, 4,\, 2,\, 1,\, 1,\, \ldots,\, 1\}$$

for each $n \geq 4$.

The corresponding problem regarding a set of n positive integers with the property that their sum equals their product, and the upper bound for this common value was proposed by E.T.H. Wang in 1973 and appeared as E 2447^* (*Bounds for* k*-satisfactory sequences*) *in the* American Mathematical Monthly [80(1973), 953; 82(1975), 78-80]. *Though it is easy to find such sets, the determination of all* n *values for which such a set is unique seems to be a much harder question and, to the best knowledge of this editor, is still an open question. Back then, the editor of that problem announced that "with the aid of computer, it was discovered that up to* 10000*, the only values of* n*'s for which such a set is unique are* $n = 2, 3, 4, 6, 24, 114, 174$ *and* 444*".*

2563. [2000 : 372] *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

You are given that angle x satisfies the equation $a\sin x + b\cos x = c$.

(a) If a, b and c are real numbers, calculate angle x.

(b) Considering a, b and c as line segments, find a straightedge and compass construction for angle x.

(a) *Solution by Michel Bataille, Rouen, France.*

If $a = b = 0$, then there is no solution when $c \neq 0$, while every real number x is a solution when $c = 0$. Suppose now that $a^2 + b^2 \neq 0$. The given equation may then be written as

$$\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x \;=\; \frac{c}{\sqrt{a^2+b^2}}\,,$$

or

$$\cos(x-\alpha) \;=\; \frac{c}{\sqrt{a^2+b^2}}\,,$$

where α is determined by

$$\sin\alpha \;=\; \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos\alpha \;=\; \frac{b}{\sqrt{a^2+b^2}}\,.$$

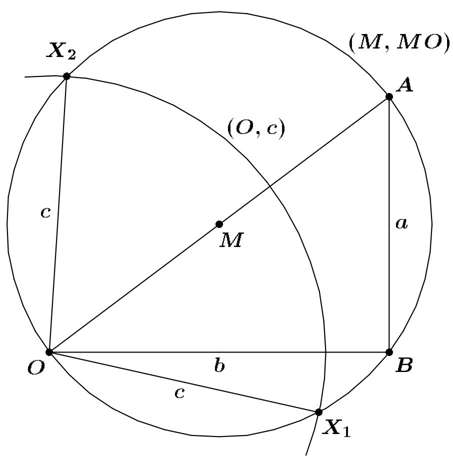
We have the following consequences:

If $c^2 > a^2 + b^2$, the equation has no solution.

If $c^2 \le a^2 + b^2$ we define β by $\cos\beta = \dfrac{c}{\sqrt{a^2+b^2}}$, and the equation becomes $\cos(x-\alpha) = \cos\beta$, whose solutions are the real numbers $\alpha+\beta+2k\pi$ and $\alpha-\beta+2k\pi$ (for all integers k).

(b) *Solution by the Angelo State Problem Group, San Angelo, TX* (*shortened by the editor*).

We are given the (directed) lengths a, b, c, and will construct the angles α and β [in the notation of solution (a)], under the assumption that $c^2 \le a^2 + b^2$; then by part (a), $x = \alpha + \beta$ or $x = \alpha - \beta$.



1. Construct right triangle AOB with right angle at B, and legs $OB = b$ and $BA = a$. (Note that $\angle BOA = \alpha$.)
2. Construct circle (O, c) with centre O and radius c.
3. Construct the mid-point M of OA.
4. Construct the circle (M, MO) with centre M and radius MO. Since $c^2 \le a^2 + b^2$, the two circles will meet at one [when $c^2 = a^2 + b^2$] or two points; call them X_1 and X_2.

Claim: $\angle BOX_1$ and $\angle BOX_2$ are the two solutions for angle x. To see this, note that because the X_i are on the circle with diameter $OA = \sqrt{a^2+b^2}$, the triangles OX_iA are congruent right triangles with leg c and hypotenuse $\sqrt{a^2+b^2}$. Hence $\angle X_1OA = \angle AOX_2 = \beta$ $\left[\text{since their cosine is } \dfrac{c}{\sqrt{a^2+b^2}}\right]$. Combining these angles with α of step 1 we see that $\angle BOX_1 = \alpha - \beta$, and $\angle BOX_2 = \alpha + \beta$, as desired.

Remarks. In the original equation, c can be interpreted as the (scalar) projection of the vector $\langle b, a\rangle$ on the unit vector $\langle \cos x, \sin x\rangle$. Thus the construction reduces to the problem of finding a point X on the circle (O, c) with the property that the line AX is perpendicular to the radius OX; that is, AX is tangent to (O, c) at X. Our construction is the standard way of constructing such a point.

Also solved by MOHAMMED AASSILA, Strasbourg, France (part (a) *only); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3 solutions); MENG DAZHE, River Valley High School, Singapore; PAUL DEIERMANN, Cape Girardeau, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria(3 solutions); VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; RENÉ LAUMEN, Deurne, Belgium; HENRY LIU, student, Cambridge, England; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.*

An alternative approach to part (a) is to replace $\sin x$ *in the given equation by* $\pm\sqrt{1-\cos^2 x}$*, then solve for* $\cos x$*, obtaining* $\cos x = \dfrac{bc \pm a\sqrt{a^2+b^2-c^2}}{a^2+b^2}$*; similarly, via* $\cos x = \pm\sqrt{1-\sin^2 x}$*, one obtains* $\sin x = \dfrac{ac \mp b\sqrt{a^2+b^2-c^2}}{a^2+b^2}$*. Note how the plus and minus signs match up, so that one obtains the same pair of solutions as in the featured approach. For an alternative construction for part (b), let* $u = \sin x$ *and* $v = \cos x$*, then plot the line* $au + bv = c$ *and the unit circle* $u^2 + v^2 = 1$ *in the* uv*–plane. If the points of intersection are denoted by* P_i $(i = 1, 2)$*, then the angles the rays* $\overrightarrow{OP_i}$ *make with the positive* u*–axis are the two solutions to the original equation.*

2564. [2000 : 373] *Proposed by Darko Veljan, University of Zagreb, Zagreb, Croatia.*

(a) Find all integer solutions (a, b, c) of the equation $\binom{a}{2} + \binom{b}{2} = \binom{c}{2}$, such that $2 \le a \le b \le c$.

(b) For each integer $n \ge 1$, find at least one integer solution (a, b, c) $(n \le a \le b \le c)$ of the equation $\binom{a}{n} + \binom{b}{n} = \binom{c}{n}$.

(c) For $n = 3$, find at least one further solution for (b).

I. *Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA (adapted by the editor).*

(a) Note first that $\dbinom{a}{2} + \dbinom{b}{2} = \dbinom{c}{2}$ is equivalent to

$$\begin{aligned} a(a-1) &+ b(b-1) = c(c-1) \quad \text{or} \\ a(a-1) &= (c-b)(c+b-1)\,. \end{aligned} \tag{1}$$

Hence, if we take any integer $a \ge 2$ and factor $a(a-1)$ into $a(a-1) = d_1 d_2$ in all possible ways in which $d_1 < d_2$, $d_1 + d_2$ is odd, and $\frac{d_2-d_1+1}{2} \ge a$, then with $b = \frac{d_2-d_1+1}{2}$ and $c = \frac{d_2+d_1+1}{2}$, one can verify immediately that (1) holds. [Ed: It is easy to see that the factorization described above is possible for all $a \ge 3$. In particular, for any $a \ge 3$, we can always take $d_1 = 1$ and $d_2 = a(a-1)$. This yields the particular solution

$\left(a, \frac{a(a-1)}{2}, \frac{a(a-1)}{2}+1\right)$. This solution was mentioned explicitly by a few solvers. Note also that this is the only solution if $a = 2^n$ such that $2^n - 1$ is a prime; that is, if $a-1$ is a Mersenne prime, since in this case it is easy to see that $d_1 = 1$ and $d_2 = a(a-1)$ is the only factorization of $a(a-1)$ satisfying all the required conditions. For example, the only solution in which $a = 8$ is $(8, 28, 29)$. On the other hand, it is easy to see that there are no solutions when $a = 2$ since (1) becomes $(c-b)(c+b-1) = 2$ which implies that $c - b = 1$ and $c + b - 1 = 2$. Solving, we get $b = 1 < a$, a contradiction.]

Conversely, suppose (1) holds. If we let $d_1 = c - b$ and $d_2 = c + b - 1$, then clearly d_1 and d_2 are divisors of $a(a-1)$, with $d_1 < d_2$, $d_1 + d_2$ being odd, and $\frac{d_2-d_1+1}{2} = b \geq a$. Therefore, the constructions described above would indeed produce all the solutions.

(b) Since $\binom{2n}{n} = \binom{2n-1}{n} + \binom{2n-1}{n-1} = \binom{2n-1}{n} + \binom{2n-1}{n}$,

we see that $(a, b, c) = (2n-1,\ 2n-1,\ 2n)$ is a solution.

(c) There are many solutions; for example, $(a, b, c) = (10, 16, 17)$, $(22, 56, 57)$, $(32, 57, 60)$, ... [Ed: Hess gave 11 solutions for this part.]

II. *Solution to* (a) *by the proposer* (*modified slightly by the editor*).

The given equation is equivalent to $a^2 - a + b^2 - b = c^2 - c$.

If we let $c = a + m = b + n$, where $0 \leq n \leq m$, then, from $a^2 - a + b^2 - b = (a+m)^2 - (b+n)$, we get

$$\begin{aligned} 2am &+ a + m^2 - n = b^2 = (a + m - n)^2 \quad \text{or} \\ a^2 &- (2n+1)a + n^2 - 2mn + n = 0 \,. \end{aligned} \tag{2}$$

The discriminant of this quadratic equation is

$$D = (2n+1)^2 - 4(n^2 - 2mn + n) = 8mn + 1 \,.$$

Since D must be a perfect square, we have $8mn + 1 = (2k-1)^2$ for some $k \in \mathbb{N}$, from which we get $mn = \dfrac{k(k-1)}{2} = \dbinom{k}{2}$ for some $k \geq 2$. [Ed: Note that if $k = 1$, then $mn = 0$ implies $n = 0$ or $m = 0$; that is, either $c = b$ or $c = a = b$. In either case we get $\dbinom{a}{2} = 0$, a contradiction.]

Then, the solutions of (2) are given by

$$a = \frac{2n + 1 \pm \sqrt{D}}{2} = \frac{2n + 1 + 2k - 1}{2} = n + k \,.$$

[Ed: If we take $-\sqrt{D}$, then $a = n - k + 1$. However, $n^2 \leq mn = \dbinom{k}{2} < k^2$ implies $n < k$ and therefore, $a = n - k + 1 \leq 0$, a contradiction.] It follows that $b = a + m - n = m + k$ and $c = b + n = m + n + k$. Conversely, it is straightforward to verify that these values of a, b, c do yield a solution. Therefore, all solutions are given by $(a, b, c) = (n + k, m + k, m + n + k)$ where $k \geq 2$, $mn = \binom{k}{2}$, $n \leq m$.

Also solved (completely) by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SHOICHI HIROSE, Mita High School, Tokyo, Japan; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, Trinity College, Cambridge, England; and the proposer.

Partial solutions were submitted by HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; PANOS E. TSAOUSSOGLOU, Athens, Greece; and KENNETH M. WILKE, Topeka, KS, USA.

There were also one incomplete and one partially incorrect solution.

Hirose gave a few references, including a paper of his own ([1], [2] below) in which this problem was studied. In particular, he provided a list of all solutions (a, b, c) to part (a) in which $2 \leq a \leq b \leq c \leq 50$. For part (c), he stated that there are five solutions in which $3 \leq a \leq b \leq c \leq 100$, 34 solutions in which $3 \leq a \leq b \leq c \leq 1000$, and 82 solutions in which $3 \leq a \leq b \leq c \leq 10000$. Parts of his claims were substantiated by a computer search by Wilke. Both of them discovered the solution $(a, b, c) = (132, 190, 200)$ to $\binom{a}{4} + \binom{b}{4} = \binom{c}{4}$ besides the "trivial" solution $(7, 7, 8)$ covered by the general solution to part (b).

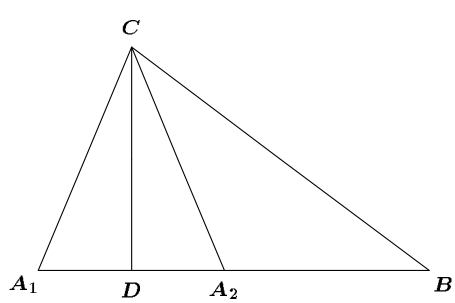
[1]. *A.S. Fraenkel. "Diophantine Equations Involving Generalized Triangular and Tetrahedral Numbers".* Computers in Number Theory. *New York: Academic Press, 1971, pp. 99–114.*

[2]. *S. Hirose. "On Some Polygonal Number which are, at the Same Time the Sums, Differences, and Products of Two Other Polygonal Numbers."* The Fibonacci Quarterly *24 (1986) pp. 99–106.*

2565. [2000 : 373] *Proposed by K.R.S. Sastry, Dodballapur, India.*

A *Heron* triangle has integer sides and integer area. Show that there are exactly three pairs of Heron Triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that $B_1C_1 = B_2C_2$, $A_1C_1 = A_2C_2$, $\angle A_1B_1C_1 = \angle A_2B_2C_2$ and $A_2B_2 - A_1B_1 = 10$.

Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany.



$AD = 5$ implies that $CD = 12$ and $A_nC = 13$ $(n = 1, 2)$. Let $BC = a$ and $BD = d$.

From $CD^2 = BC^2 - BD^2$, we have $144 = a^2 - d^2 = (a+d)(a-d)$. Note that $a+d$ and $a-d$ must have the same parity. Thus,

$$144 = 2 \cdot 72 = 4 \cdot 36 = 6 \cdot 24 = 8 \cdot 18$$

implies that

$$(a,d) = (37,35) \quad \text{or} \quad (20,16) \quad \text{or} \quad (15,9) \quad \text{or} \quad (13,5)\,.$$

Therefore, we have

$$\begin{array}{lrlcl} \text{First pair} & :(a,b,c) = & (37,13,40) & \text{and} & (37,13,30), \\ \text{Second pair} & : & (20,13,21) & \text{and} & (20,13,11), \\ \text{Third pair} & : & (15,13,14) & \text{and} & (15,13,4), \end{array}$$

and these are all the solutions.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHAI CIPU, IMAR, Bucharest, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Cambridge, England; KENNETH M. WILKE, Topeka, KS, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; CALVIN LIN ZHIWEI, Singapore; and the proposer. There was one incorrect solution.

Janous commented that it was a neat problem, and that he intended to pose it to his gifted students.

2567. [2000 : 373] *Proposed by K.R.S. Sastry, Dodballapur, India.*

In triangle ABC, points P and Q are on the line segment BC such that AP and AQ are trisectors of $\angle BAC$ and $BQ = QC$. If $AC = \sqrt{2}\,AQ$, find the measure of $\angle BAC$.

Solution by Panos E. Tsaoussoglou, Athens, Greece.

If the sides of $\triangle ABC$ are a, b, and c (in the usual notation), then $AC = b$, $AQ = \dfrac{b\sqrt{2}}{2}$, and $BQ = QC = \dfrac{a}{2}$. Let $\theta = \dfrac{A}{3}$. The Sine Law applied to $\triangle ABQ$ gives

$$\frac{a}{2\sin 2\theta} = \frac{b\sqrt{2}}{2\sin B}\,,$$

and to $\triangle ABC$ gives

$$\frac{a}{\sin 3\theta} = \frac{b}{\sin B}\,.$$

Consequently, θ satisfies

$$\frac{\sin 3\theta}{\sin 2\theta} = \sqrt{2}\,.$$

Using the identities

$$\sin 3\theta = \sin\theta(3 - 4\sin^2\theta) = \sin\theta(4\cos^2\theta - 1)\,,$$

and

$$\sin 2\theta = 2\sin\theta\cos\theta\,,$$

this becomes

$$4\cos^2\theta - 2\sqrt{2}\cos\theta - 1 = 0\,.$$

Since the negative root is extraneous, $\cos\theta = \dfrac{(\sqrt{2}+\sqrt{6})}{4}$; hence, $\theta = 15^\circ$, so that $A = 3\theta = 45^\circ$.

Editor's remarks. The majority of the submitted solutions ordered the points B, P, Q, C along BC as in the featured solution; however, there is no reason why the order could not be B, Q, P, C. In the latter case, the above method leads to the equation

$$\frac{\sin 3\theta}{\sin\theta} \;=\; \sqrt{2}\,,$$

in which case $4\sin^2\theta = 3 - \sqrt{2}$, and

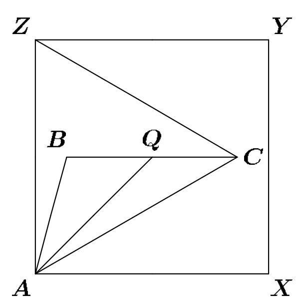
$$A \;=\; 3\sin^{-1}\frac{\sqrt{3-\sqrt{2}}}{2} \;\approx\; 117.07^\circ\,.$$

Only Loeffler included both interpretations.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, Bucharest, Romania; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Manila, Philippines; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, Cambridge, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; CHALLA K.S.N.M. SANKAR, Andhra Pradesk, India; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Bataille and Yiu both showed how to construct $\triangle ABC$ in the case where Q lies between P and C. Here is Yiu's construction:

> *Inside square $AXYZ$ construct the equilateral triangle AZC on side AZ. Then Q is the centre of the square (because $\angle CAQ = 60^\circ - 45^\circ = 15^\circ$, while $AQ = \frac{1}{\sqrt{2}}AZ = \frac{1}{\sqrt{2}}AC$). Finally, B is the point of CQ for which $\angle CAB = 3\angle CAQ$ [and the half-turn about Q interchanges B with C so that Q is the mid-point as desired]. It is clear from Yiu's figure that $\angle ACB = 30^\circ$ and $\angle ABC = 105^\circ$.*



Several of the submitted solutions exploited a variant of this figure to avoid the use of any trigonometry. Konečný's solution in particular was based on such a figure; the idea was suggested to him by his published solution to Mathematics Magazine *problem* #825 [46 pp. 45–46].

2568. [2000 : 373] *Proposed by K.R.S. Sastry, Bangalore, India.*

The sides a, b and c of a non-degenerate triangle ABC satisfy the relations $b^2 = ca + a^2$ and $c^2 = ab + b^2$. Find the measures of the angles of triangle ABC.

Solution by Václav Konečný, Ferris State University, Big Rapids, MI, USA.

Let a, b and c be the sides and A, B and C be the angles of $\triangle ABC$. Clearly, $a < b < c$, so that $A < B < C$. Consider the first condition, $b^2 = ca + a^2$. By the Law of Cosines, $b^2 = c^2 + a^2 - 2ca\cos B$.

Hence $ca = c^2 - 2ca\cos B$, so that $\frac{c}{a} - 2\cos B = 1$. By the Law of Sines, $\frac{c}{a} = \frac{\sin C}{\sin A}$. Then $\frac{\sin C}{\sin A} - 2\cos B = 1$, which gives $\sin A\cos B + \cos A\sin B - 2\cos B\sin A = \sin A$, or, $\sin(B - A) = \sin A$. Consequently, $B = 2A$. Similarly, the condition $c^2 = ab + b^2$ implies $C = 2B$. Therefore, $C = 2B = 4A$, so that $A = \pi/7$, $B = 2\pi/7$ and $C = 4\pi/7$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIHAI CIPU, IMAR, Bucharest, Romania; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD EDEN, Ateneo de Manila University, Philippines; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; HENRY LIU, Trinity College, Cambridge, England (2 solutions); DAVID LOEFFLER, student, Cotham School, Bristol, UK; GERRY LEVERSHA, St. Paul's School, London, England; VEDULA N. MURTY, Visakhapatnam, India; CHALLA SANKAR, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.