MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z5 (NEW!)**. The electronic address is

NEW! mayhem-editors@cms.math.ca NEW!

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University).

Mayhem Problems

The Mayhem Problems editors are:

Chris Cappadocia	*Mayhem Problems Editor,*
Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 5 of 2002.

Mayhem Problems — NEW!

More changes! We have decided to make the problem section of MAYHEM more like that in CRUX. So, starting with this issue, **each** issue will contain problems for you to solve and, eventually, solutions. For the next few issues we will print solutions to outstanding problems from past High School, Advanced and Challenge sections. Sometime in the new year we should be totally in our new format.

To help with the changes to MAYHEM, the editors have secured a grant from the CMS endowment fund. We will be using this money to promote MAYHEM, as well as making 2002 a year of prizes for MAYHEM! Send in problems and solutions to proposed problems and you may win a prize.

There will be random prizes drawn from names of solvers and proposers as well as prizes for "problem solvers of the year". Watch for details in future problem sections.

Mayhem assistant editor Chris Cappadocia will be taking over the new Mayhem Problems section. We are still in the process of setting up a permanent address that problems and solutions can be sent to, so in the interim correspondence can be sent to the MAYHEM editor.

I would like to take a moment to thank Adrian Chan, Donny Cheung and David Savitt for their hard work on the MAYHEM problem sections over the years. We will still see their work for a bit as we round out the material from their columns, and we may see them pop up in some other capacity in the future. Thanks again guys, your effort is greatly appreciated!

To start the new problem section, we will begin with seven questions from various contests put on by the Australian Mathematics Trust. The questions were, originally, multiple choice but the choices have been removed (and possibly a word changed to get you to find, rather than choose) so that you have to do all the work yourself!

My thanks go to Peter Taylor, Executive Director of the Australian Mathematics Trust for permitting us to use the questions. For more information on the Trust, its contests and publications, you can visit its website `http://www.amt.canberra.edu.au` or email the Executive Director at

`pjt@amt.canberra.edu.au`

To facilitate their consideration, please send your proposals and solutions on signed and separate sheets of paper. These may be typewritten or neatly handwritten and should be received no later than 1 January 2002. They may also be sent by email (it would be appreciated if it was in $\LaTeX$). Solutions received after this date will also be considered if there is sufficient time before the date of publication.

Australian Mathematics Trust Questions

M1. Four singers take part in a musical round of 4 equal lines, each finishing after singing the round through three times. The second singer begins when the first singer begins the second line,the third singer begins when the first singer begins the third line,the fourth singer begins when the first singer begins the fourth line. Find the fraction of the total singing time that all four are singing at the same time.

M2. When 5 new classrooms were built for Wingecarribee School the average class size was reduced by 6. When another 5 classrooms were built, the average class size reduced by another 4. If the number of students remained the same throughout the changes, how many students were there at the school?

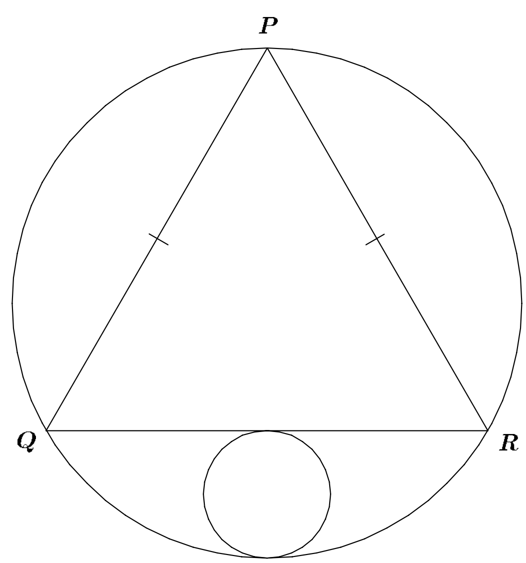
M3. How many years in the 21^{st} century will have the property that, dividing their year number by each of 2, 3, 5, and 7 always leaves a remainder of 1?

M4. We write down all the numbers $2, 3, \ldots, 100$, together with all their products taken two at a time, their products taken three at a time, and so on up to and including the product of all 99 of them. Find the sum of the reciprocals of all the numbers written down.

M5. The ratio of the speeds of two trains is equal to the ratio of the time they take to pass each other going in the same direction to the time they take to pass each other going in the opposite directions. Find the ratio of the speeds of the two trains.

M6. A city railway network has for sale one-way tickets for travel from one station to another station. Each ticket specifies the origin and destination. Several new stations were added to the network, and an additional 76 different ticket types had to be printed. How many new stations were added to the network?

M7. A circle of radius 6 has an isosceles triangle PQR inscribed in it, where $PQ = PR$. A second circle touches the first circle and the mid-point of the base QR of the triangle as shown. The side PQ has length $4\sqrt{5}$. Find the radius of the smaller circle.



Problem of the Month

Jimmy Chui, student, University of Toronto

Problem.

Problem. The roots of the equation $x^2 + 4x - 5 = 0$ are also the roots of the equation $2x^3 + 9x^2 - 6x - 5 = 0$. What is the third root of the second equation?

(1996 COMC, Problem A1)

Solution 1. The first equation factors as $(x + 5)(x - 1) = 0$. Hence the roots of the first equation are -5 and 1. These are also two roots of the second equation.

Since -5 is a root, we can factor the second equation to get $(x + 5)\left(2x^2 - x - 1\right) = 0$. Another root is 1, so we can factor again to make $(x + 5)(x - 1)(2x + 1) = 0$.

Thus, the three roots of the second equation are -5, 1, and $-1/2$.

Hence, the third root of the second equation is $-1/2$.

Solution 2. From the given information, we can deduce that the second equation can be factored into the form $\left(x^2 + 4x - 5\right)(ax + b) = 0$ for some numbers a and b.

If we expand this last equation and compare coefficients with the given numbers, we would get $a = 2$ and $-5b = -5$. (This is done by comparing the x^3 terms and the constant terms, respectively.)

The third factor would then be $(2x + 1)$. Hence the third root would be $-1/2$.

Note: For the paranoid, we could check to make sure that $-1/2$ is indeed the third root of the equation. But I will let you do that yourself, because it is late, and I have to work tomorrow :(.

Writer's Guide For Mayhem

Shawn Godin

MATHEMATICAL MAYHEM was conceived as a mathematics journal by students, for students. The original audience of the journal was high school and university students. Since MAYHEM is now contained within CRUX, we, the editors, thought it was time to refine our scope so that we can cut down the overlap between CRUX and MAYHEM. We thought this would be best achieved by having the MAYHEM section focus on a high school audience.

The change in focus to a high school audience does not mean that there will be nothing here for anyone beyond that level; quite the contrary. The only restrictions that we see this change of focus putting on material is the prerequisite knowledge needed by the reader to be able to read it.

Many great mathematics and science writers like Martin Gardner and Ivars Peterson have taken quite complex material and made it accessible to the layperson . It is our hope that items to be published in MAYHEM do not presuppose knowledge beyond the high school level. In cases where some other knowledge is needed, it should be self contained and included in the item.

We seek articles that would be of interest to a wide audience that does not presuppose mathematical background beyond the high school level. The topics do not have to be new, but it is hoped that well known topics will be presented in some new light.

The editors of MAYHEM will be working closely with the editors of CRUX to permit the passing of material from one board to the other as we see an item fitting into the journal. Thus, a piece submitted to CRUX may end up in the MAYHEM section (with the author's approval) or vice versa.

To help to promote MAYHEM, the editors have secured a grant from the CMS endowment fund. Starting in 2002, there will be a wide selection of prizes for solutions to problems and articles submitted and used in MAYHEM. Watch for specific details in future issues.

As we slowly reshape MAYHEM, we want to make sure we are on the right track. Please feel free to contact us at either of the addresses above and let us know how we are doing.

An Extension of Ptolemy's Theorem

David Loeffler

Ptolemy's Theorem is a well known result that states that if $ABCD$ is a convex cyclic quadrilateral (with vertices in this order), then $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

This may be generalized in the following way: let $ABCD$ be any quadrilateral, not necessarily cyclic, not even necessarily convex. Let $A = \angle BAD$ and $C = \angle DCB$.[1] Then we have

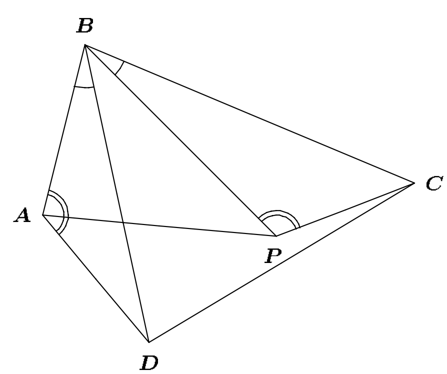
$$AC^2 \cdot BD^2 \;=\; AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2AB \cdot BC \cdot CD \cdot DA \cos(A+C)\,.$$

This immediately implies Ptolemy's Theorem, since if $ABCD$ is cyclic, $A + C = \pi$, so that $\cos(A + C) = -1$.

I do not know if this formula is known; I certainly have not seen it before, and I have not met anybody who has.[2]

Proof.

Consider this diagram.



1. If $ABCD$ is not convex, care should be taken that angles are measured in the right sense; that is, so that $\angle BAD$ and $\angle DCB$ have the same orientation.

2. Ed. – See Crux problem 1015 [1985 : 128]: "the following extension of Ptolemy's Theorem has recently appeared (E. Kraemer, Zobecneni Vety Ptolemaivoy, *Rozhledy Mathematickofyzikalni* (Czechoslovakia) 63, no. 8, 345-349):

$$a_{13}^2 a_{24}^2 \;=\; a_{12}^2 a_{34}^2 + a_{23}^2 a_{41}^2 - 2a_{12}a_{23}a_{34}a_{41}\cos(A_1 + A_3)."$$

Here the point P is constructed in such a way that $\triangle PBC \sim \triangle ABD$. Thus, we have $\frac{PC}{AD} = \frac{BC}{BD}$, or $BD \cdot PC = AD \cdot BC$.

However, we also have $\angle ABP = \angle CBD$, and by the similarity mentioned above $\frac{AB}{BP} = \frac{BD}{BC}$. Hence, $\triangle ABP$ and $\triangle DBC$ have the same angle at B and the same ratio of the sides adjacent to that angle, so that they are similar. Hence, $\frac{AP}{CD} = \frac{AB}{BD}$, or $BD \cdot AP = AB \cdot CD$.

Furthermore, we have $\angle BPC = \angle BAD = A$, and $\angle APB = \angle DCB = C$, so that $\angle APC = A + C$.

We may now apply the Cosine Rule to $\triangle APC$, obtaining

$$AC^2 \;=\; AP^2 + PC^2 - 2AP \cdot PC \cos(A + C)\,.$$

Multiplying this by BD^2, we have

$$AC^2 \cdot BD^2 \;=\; (BD \cdot AP)^2 + (BD \cdot PC)^2 - 2(BD \cdot AP)(BD \cdot PC)\cos(A + C)\,.$$

Substituting the expressions found above for $BD \cdot AP$ and $BD \cdot PC$, this becomes

$$AC^2 \cdot BD^2 \;=\; AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2AB \cdot BC \cdot CD \cdot DA \cos(A + C)\,,$$

as required.

I originally discovered this result while working on a problem from the 1998 IMO shortlist:

Let $ABCDEF$ be a convex hexagon in the plane, with

$$\angle ABC + \angle CDE + \angle EFA \;=\; 2\pi\,.$$

Prove that, if

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA\,,$$

then

$$BC \cdot DF \cdot AE = EF \cdot AC \cdot DB\,.$$

Of course, there are many other solutions to this problem, but one of the simplest purely geometric methods uses the above result.

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Convergent and Divergent Infinite Series

Sandra Pulver

Every irrational number, such as π, e, and $\sqrt{2}$, is the limit of an infinite series. The definite integral, one of the fundamental tools of calculus, is the limit of infinite series.

Given a sequence of numbers

$\{a_n\} : a_1,\, a_2,\, a_3,\, \ldots,\, a_n,\, \ldots$

suppose that you form a new sequence $\{s_n\}$ of "partial sums" as follows:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n \\ &\vdots \end{aligned}$$

The sequence $\{s_n\}$ is derived from the sequence $\{a_n\}$. If s_n is an infinite series, it is denoted by the symbol

$s_\infty = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Zeno of Elea, a Greek philosopher of the fifth century B.C., demonstrated with a famous series of paradoxes how easily one falls into logical traps when talking about infinite series. How, Zeno asked, can a runner get from A to B? First he must go half the distance. Then he must go half the remaining distance, which brings him to the $\frac{3}{4}$ point. But before completing the last quarter he must again go halfway, to the $\frac{7}{8}$ point. In other words, he goes a distance equal to the sum of the following series:

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

How can a runner traverse an infinite series of lengths in a finite period of time? If you keep adding the terms of this series, you will never reach the goal of 1; you are always short by the distance equal to the last fraction added.

The series in the previous problem, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$, the halving series, has a "sum" of 1. In this series, or any infinite series, there is no way to arrive at a "sum" in the usual sense of the word because there is no end to the terms that must be added. The sum — more precisely the limit — of an infinite series, means a number that the value of the series approaches without bound. By "approach" we mean that the difference between the value of the series and its limit can be made as small as one pleases. In every case of an infinite series that has a sum or "converges" one can always find a partial sum that differs from the limit by an amount smaller than any fraction one cares to name. Besides this, in order for an infinite series to have a limit its terms must approach zero.

Consider the halving series again. As the number of terms increases, the last term gets closer to zero. (For example, $\frac{1}{16}$ is closer to zero than $\frac{1}{8}$ is.)

A series is said to be convergent if the sequence of partial sums $\{s_n\}$ has a limit. If $\lim_{n\to\infty} s_n = S$, we say that the series converges to S. If the sequence of partial sums $\{s_n\}$ does not have a limit, the series is said to be divergent. In the case that the series is convergent, the limit S of the series of partial sums $\{s_n\}$ is written

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

This means that the series indicated by the right side converges to the number on the left side.

Consider the following sequence: $10, 50, 250, 1250, \ldots$. This is called a geometric sequence. This means that every term after the first is obtained by multiplying the term preceding it by a constant, called the common ratio of the sequence. The common ratio of this sequence is 5. The common ratio of the halving series is $\frac{1}{2}$.

Finding the limit of a converging series is often extremely difficult. But when the terms decrease in a geometric progression, as in the halving series, there is a simple method to use. First, let X equal the sum of the entire series. Because each term is twice as large as the next, then multiply each side of the equation by 2:

$$\begin{aligned} 2X &= 2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) \\ 2X &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots . \end{aligned}$$

The new series, beyond 1, is the same as the original series X. Thus,

$$2X = 1 + X$$

so that $X = 1$. Therefore, the limit of the halving series is 1.

This method for finding the limit of a converging series where the terms decrease in geometric progression can be applied to another of Zeno's paradoxes: the race of Achilles and the tortoise. Assume that Achilles runs ten times as fast as the tortoise, and that the animal has a lead of 100 yards. After Achilles has gone 100 yards the tortoise has moved 10. After Achilles has run 10 yards the tortoise has moved 1. If Achilles takes the same length of time to run each segment of this series, he will never catch the tortoise, but if they move at uniform speed, he will. How far has Achilles gone by the time he overtakes the tortoise? The answer is the limit of the series

$$100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

Each term is ten times the next term. Let X equal the sum of the series, and then multiply each side by 10:

$$10X = 1000 + 100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots$$

This series, after 1000, is the original series. Therefore $10X = 1000 + X$, or $9X = 1000$, and $X = 111\frac{1}{9}$ yards, the number of yards Achilles travels.

This problem can also be solved using another method. Let x equal the distance the tortoise has run. Achilles must run ten times as fast as the tortoise, so Achilles runs a distance of $10x$. You also must consider the tortoise's lead of 100 yards. From this information you get the equation:

$$\begin{aligned} 100 + x &= 10x \\ 100 &= 9x \\ 11\tfrac{1}{9} &= x\,. \end{aligned}$$

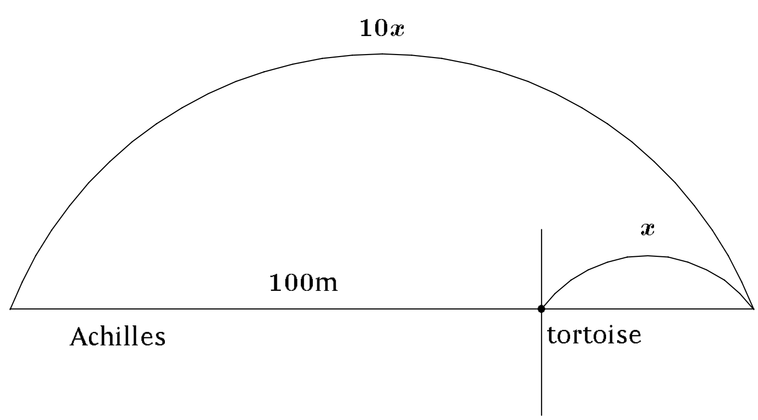

Figure 1.

By solving for x you see that the distance the tortoise has run is $11\frac{1}{9}$ yards. Therefore, since Achilles runs 100 yards more than the tortoise, he overtakes the tortoise after running $111\frac{1}{9}$ yards.

Suppose Achilles runs seven times as fast as the tortoise, which has the same head start of 100 yards. The total distance Achilles travels before overtaking the tortoise is the limit of the series

$$100 + \frac{100}{7} + \frac{100}{7\times 7} + \frac{100}{7\times 7\times 7} + \cdots$$

Each term is seven times the next term. Let x equal the sum of the series; then multiply each side by 7:

$$7x = 700 + 100 + \frac{100}{7} + \frac{100}{7\times 7} + \cdots$$

This series, after 700, is the original series. Therefore $7x = 700 + x$, or $6x = 700$, and $x = 116\frac{2}{3}$, the number of yards Achilles travels.

When no limit, S, exits, a series is called divergent. It is easy to see that $1 + 2 + 3 + 4 + 5 + \cdots$ does not converge. Suppose, however, that each new term in a series joined by a plus sign is smaller than the preceding one. Must such a series converge? It may be hard to believe at first, but the answer is no.

Consider the series known as the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

The terms get smaller and smaller; in fact, they approach zero as a limit. Nevertheless, the sum increases without bound. To prove this consider the terms in groups of two, four, eight, and so on, beginning with $\frac{1}{3}$. The first group, $\frac{1}{3} + \frac{1}{4}$, sums to more than $\frac{1}{2}$ because $\frac{1}{3}$ is greater than $\frac{1}{4}$, and a pair of fourths sum to $\frac{1}{2}$. Similarly, the second group, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$, is more than $\frac{1}{2}$ because each term except the last exceeds $\frac{1}{8}$, and a quadruple of eighths sums to $\frac{1}{2}$. In the same way the third group, of eight terms, exceeds $\frac{1}{2}$ because every term except the last $\frac{1}{16}$ is greater than $\frac{1}{16}$, and $\frac{8}{16}$ is $\frac{1}{2}$. Each succeeding group can thus be shown to exceed $\frac{1}{2}$, and since the number of such groups is unlimited the series must diverge.

This can be shown as follows:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)$$

$$+ \left(\frac{1}{17} + \frac{1}{18} + \cdots + \frac{1}{32}\right) + \cdots.$$

Now the harmonic series itself has the sequence of partial sums $\{s_n\}$:

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&\vdots \\
s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\
&\vdots
\end{aligned}
$$

Note that the new series, obtained by the insertion of the parentheses, has for its partial sums s_1, s_2, s_4, s_8, s_{16}, s_{32}, ..., $s_{2^{n-1}}$, ..., of the harmonic series. As said before, the sum in each set of parentheses is larger than $\frac{1}{2}$. By taking enough of these numbers one can make the partial sum of the new series bigger than any positive number, however large. Therefore the series diverges.

The harmonic series diverges very slowly. The first 100 terms, for instance, total only a bit more than 5. To reach 100 requires more than 2^{143} terms, but less than 2^{144} terms. The harmonic series is involved in the following problem. If one brick is placed on another, the greatest offset is obtained by having the centre of gravity of the top brick fall directly above the end of the lower brick, as shown by A in figure 2.

These two bricks, resting on a third, have maximum offset when their combined centre of gravity is above the third brick's edge, as shown by B. By continuing this procedure downward one obtains a column that curves in the manner shown. How large an offset can be obtained? Can it be the full length of a brick?

The answer is that the offset can be as large as one pleases. The top brick projects half a brick's length. The second projects $\frac{1}{4}$, the third $\frac{1}{6}$ and so on down. With an unlimited supply of bricks the offset is the limit of

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \cdots$$

This is simply the harmonic series with each term cut in half. Since the sum of the harmonic series can be made larger than any number one cares to name, so can half its sum. In short, the series diverges, and therefore the offset can be increased without limit. Such a series diverges so slowly that it would take a great many bricks to achieve even a small offset.

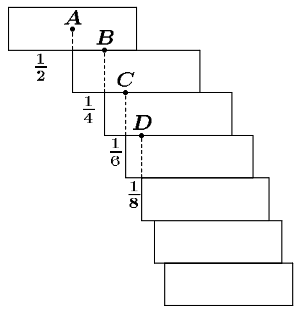

Figure 2.

With 52 playing cards, the first can be placed so that its end is flush with the table edge, and the maximum overhang is a little more than $2\frac{1}{4}$ card lengths.

The harmonic series has many curious properties. If the denominator of each term is raised to the same power n, and n is greater than 1, the series converges. If every other sign, starting with the first, is changed to minus, the resulting series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots.$$

converges to a number slightly smaller than $\frac{7}{10}$.

The following problem introduces the famous series

$$1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots,$$

which is convergent. The Swiss mathematician Leonhard Euler (1707-1783) discovered in 1763 that the sum of this series is $\frac{\pi^2}{6}$.

If all the terms of an infinite series are positive, it does not matter how the terms are grouped or rearranged; the limit remains the same. But if there are negative terms, it sometimes makes a big difference. From the seventeenth century to the middle of the nineteenth, before laws of limits were carefully formulated, all sorts of disturbing paradoxes were produced by juggling the plus and minus terms of various infinite series. Luigi Guido Grandi, a mathematician at the University of Pisa, considered the simple oscillating series $1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$. If you group the terms $(1-1)+(1-1)+(1-1)+\cdots$, the limit is 0. If you group them $1-(1-1)-(1-1)-(1-1)-\cdots$, and change the signs within the parentheses as required, the sum is 1. This shows, Grandi said, how God could take a universe with parts that added up to nothing and then, by suitable rearrangement, create something. Today, the series is recognized as divergent, so that no meaningful limit or sum can be assigned to it.

Now, consider the series $1 - 2 + 4 - 8 + 16 - \cdots$. Group it $1+(-2+4)+(-8+16)+\cdots$ and you obtain the series $1+2+8+32+\cdots$, which diverges to positive infinity. Group it $(1-2)+(4-8)+(16-32)+\cdots$ and you get the series $-1 - 4 - 16 - 64 - \cdots$ which diverges to infinity in the negative direction.

The climax to all this came in 1854 when Georg Fredrich Bernhard Riemann, the German mathematician now well known for his non-Euclidean geometry, proved a truly remarkable theorem. Whenever the limit of an infinite series can be changed by regrouping or rearranging the order of its terms, it is called conditionally convergent, in contrast to an absolutely convergent series, which is unaffected by such scrambling. Conditionally convergent series always have negative terms, and they always diverge when all their terms have been made positive. Riemann showed that any conditionally convergent series can be suitably rearranged to give a limit that is any desired number whatever, rational or irrational, or even made to diverge to infinity in either direction.

Even an infinite series without negative terms, if it diverges, can cause troubles if you try to handle it with rules that apply only to finite and converging series. For example, let x be the infinite positive sum $1 + 2 + 4 + 8 + 16 + \cdots$. Then $2x$ must equal $2 + 4 + 8 + 16 + \cdots$. This new series is merely the old series minus 1. Therefore $2x = x - 1$, which reduces to $x = -1$. This seems to prove that -1 is infinite and positive, which is not the case. This happens when one uses for divergent series, rules which apply only to infinite converging series with finite sums.

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