THE ACADEMY CORNER

No. 42

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In this issue, we present problems of the Undergraduate Mathematics Competition held at Memorial University of Newfoundland on 27 March 2001. This competition was designed to be a little easier than usual, and some local high school students were invited to participate. Accordingly, we especially invite high school students everywhere to send us their solutions to these problems. A copy of *Inequalities* by E.J. Barbeau and B.L.R. Shawyer (in the **ATOM** series) will be awarded to the best solution set sent in by a high school student. The deadline to receive such solutions is 14 December 2001. High school students should enclose a note from a teacher verifying that they are indeed enrolled in a high school on the date on which the solutions are mailed. (No FAX or email entries accepted.)

Memorial's Local Undergraduate Competition Winter 2001

1. When $x^3 + px + 5$ is divided by $x - 1$, the remainder is the same as when it is divided by $x + 1$. Find p.

2. Two astronauts, Pat and Chris, are orbiting the earth (in circular orbits) in separate capsules. They are orbiting in the same direction along the equator. Pat orbits in 3 hours and Chris in $7\frac{1}{2}$ hours. At 12 noon Chris sees Pat directly below. How long will it be before they are one above the other again?

3. Given $\triangle ABC$ with right angle C and leg lengths a, b. From a point P on AB, perpendiculars are drawn to meet AC at S and BC at T. Find the minimum possible length of ST.

4. A unit square and a unit equilateral triangle share an edge. There is a unique circle that passes through the vertex of the triangle and two vertices of the square that are not on the shared edge. Determine the radius of the circle.

   [*This problem is due to E.J. Barbeau, University of Toronto, Toronto, Ontario.*]

5. Show that
$$\frac{\sqrt{y^2+1}+y+x}{x\sqrt{y^2+1}-xy+1}$$
   is independent of x, assuming that the denominator is not equal to zero.

6. Let P_n be the number of permutations $(a_1, a_2, \ldots, a_n)$ of the numbers $1, \ldots, n$, with the following property: there exists exactly one index $i \in \{1, \ldots, n-1\}$ such that, $a_i > a_{i+1}$.

   (a) Find P_2, P_3 and P_4.

   (b) Show that $P_n = 2^n - n - 1$, $(n > 1)$.

   [*This is a modified version from the national Bulgarian Olympiad of 1995.*]

Next, we present some more solutions to problems of the 2000 Atlantic Provinces Council on the Sciences Mathematics Competition [2000 : 450, 2001 : 86]. These solutions were sent in by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

2000 Atlantic Provinces Council on the Sciences Mathematics Competition

5. The three-term geometric progression $(2, 10, 50)$ is such that

$$(2 + 10 + 50) * (2 - 10 + 50) \; = \; 2^2 + 10^2 + 50^2 \,.$$

   (a) Generalize this (with proof) to other three-term geometric progressions.
   (b) Generalize (with proof) to geometric progressions of length n.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

(a) In general, for three-term geometric progressions, we have

$$\begin{aligned} & \left(a + ar + ar^2\right)\left(a - ar + ar^2\right) \\ &= \left(a + ar^2\right)^2 - a^2r^2 \\ &= a^2 + a^2r^2 + a^2r^4 \; = \; a^2 + (ar)^2 + (ar^2)^2 \,. \end{aligned}$$

(b) For **odd** n, consider a geometric progression of length n, $(a, ar, ar^2, \ldots, ar^{n-1})$. If $r \neq \pm 1$, we have

$$\begin{aligned} & \left(a + ar + ar^2 + \cdots + ar^{n-1}\right)\left(a - ar + ar^2 - \cdots + ar^{n-1}\right) \\ &= \frac{a\,(1 - r^n)}{1 - r} \times \frac{a\,(1 + r^n)}{1 + r} \\ &= a^2\left(1 + r^2 + r^2 \cdots + r^{2n-2}\right) \\ &= a^2 + (ar)^2 + (ar^2)^2 + \cdots + (ar^{n-1})^2 \,. \qquad (1) \end{aligned}$$

It can also readily be verified that when $r = \pm 1$, both sides of (1) equal na^2, and, hence, (1) holds for all n.

We note that the conclusion does not hold for even n; for example, for $n = 2$, we have

$$(a + ar)(a - ar) \;=\; a^2 - a^2r^2 \;\neq\; a^2 + (ar)^2$$

in general.

7. Without calculator or elaborate computation, show that

$$3^{2701} \equiv 3 \pmod{2701}\,.$$

NOTE: $2701 = 37 \times 73$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Note first that $2701 = 37 \times 73$ and that both 37 and 73 are primes.

By Fermat–Euler's Theorem, we have $3^{37} \equiv 3 \pmod{37}$ and $3^{73} \equiv 3 \pmod{73}$. Hence,

$$3^{2701} \;=\; \left(3^{37}\right)^{73} \;\equiv\; 3^{73} \pmod{37}\,, \tag{1}$$

$$3^{2701} \;=\; \left(3^{73}\right)^{37} \;\equiv\; 3^{37} \pmod{73}\,. \tag{2}$$

By Fermat's Little Theorem, we have $3^{36} \equiv 1 \pmod{37}$, and further, $3^{72} \equiv 1 \pmod{37}$. Hence,

$$3^{73} \;\equiv\; 3 \pmod{37}\,. \tag{3}$$

Also, from $3^4 = 81 \equiv 8 \pmod{73}$, we have $3^6 \equiv 72 \equiv -1 \pmod{73}$, so that $3^{36} \equiv 1 \pmod{73}$. Hence,

$$3^{37} \;\equiv\; 3 \pmod{73}\,. \tag{4}$$

From (1) and (3), we have $3^{2701} \equiv 3 \pmod{37}$.

From (2) and (4), we have $3^{2701} \equiv 3 \pmod{73}$.

Finally, since $\gcd(37, 73) = 1$, we have $3^{2701} \equiv 3 \pmod{2701}$.

Wang also sent in a solution to problem 2 — we have already published a solution in [2001 : 86].