THE OLYMPIAD CORNER

No. 214

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Murray Klamkin has written to point out that "Quickies" did not originate with him. They first appeared in the March 1950 issue of Mathematics Magazine. Quickies originated with the late Charles W. Trigg, who was then Problems Editor. In those days, many Quickies were published by the late Leo Moser as well as by Murray. But most probably it is true that Murray has published more Quickies than anyone else.

We start this number with an additional set of five Klamkin Quickies. Thanks go to Murray Klamkin, University of Alberta. Try them before looking ahead to the solutions!

FIVE MORE KLAMKIN QUICKIES

1. Determine the maximum value of

\[ S = 4(a^4 + b^4 + c^4 + d^4) - (a^2 b c + b^2 c d + c^2 d a + d^2 a b) - (a b^2 c + b c^2 d + c d^2 a + d a^2 b), \]

where \( 1 \geq a, b, c, d \geq 0 \).

2. If \( a, b, c, d \) are \( > 0 \), prove or disprove the two inequalities:

(i) \( \frac{a b}{c} + \frac{b c}{a} + \frac{c d}{a} + \frac{d a}{b} \geq a + b + c + d \),

(ii) \( a^2 b + b^2 c + c^2 d + d^2 a \geq a b c + b c d + c d a + d a b \).

3. Determine all the points \( P(x, y, z) \), if any, such that all the points of tangency of the enveloping (tangent) cone from \( P \) to the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) (\( a > b > c \)), are coplanar.

4. Determine whether or not there exists a set of 777 distinct positive integers such that for every seven of them, their product is divisible by their sum.

5. If \( R \) is any non-negative rational approximation to \( \sqrt{5} \), determine an always better rational approximation.

Next we give the problems of the 28th Austrian Mathematics Olympiad 1997, Final Round Advanced Level. Thanks go to Walther Janous, Ursulinen-gymnasium, Innsbruck, Austria for sending them to us.
**28th AUSTRIAN MATHEMATICS OLYMPIAD 1997**

**Final Round Advanced Level**

*First Day — June 4, 1997 (Time: 4 hours)*

1. Let \(a\) be a fixed whole number.

Determine all solutions \(x, y, z\) in whole numbers to the system of equations:

\[
\begin{align*}
5x + (a + 2)y + (a + 2)z &= a, \\
(2a + 4)x + (a^2 + 3)y + (2a + 2)z &= 3a - 1, \\
(2a + 4)x + (2a + 2)y + (a^2 + 3)z &= a + 1.
\end{align*}
\]

2. Let \(K\) be a positive whole number. The sequence \(\{a_n : n \geq 1\}\) is defined by \(a_1 = 1\) and \(a_n\) is the \(n^{th}\) natural number greater than \(a_{n-1}\) which is congruent to \(n\) modulo \(K\).

(a) Determine an explicit formula for \(a_n\).

(b) What is the result if \(K = 2\)?

3. We are given a triangle \(ABC\). On side \(AC\) a point \(P\) is chosen. On the production of ray \(CB\) (beyond \(B\)) there lies the point \(Y\) which subtends equal angles with \(AP\) and \(PC\), respectively.

On side \(BC\), point \(Q\) is chosen. On the production of ray \(AC\) (beyond \(C\)) there is point \(X\), subtending equal angles with \(BQ\) and \(QC\), respectively.

Furthermore, \(R\) is the point of intersection of lines \(YP\) and \(XB\), \(S\) is the point of intersection of lines \(XQ\) and \(YA\), and \(D\) is the point of intersection of lines \(XB\) and \(YA\).

Prove: \(PQRS\) is a parallelogram if and only if \(ACBD\) is inscribable.

*Second Day — June 5, 1997 (Time: 4 hours)*

4. Determine all quadruples \((a, b, c, d)\) of real numbers satisfying the equation:

\[
256a^3b^3c^3d^3 = (a^6 + b^6 + c^6 + d^6)(a^2 + b^2 + c^2 + d^2)
\]

\[
\times (a^2 + b^2 + c^2 + d^2)(a^2 + b^2 + c^2 + d^2).
\]

5. We define the following operation which will be applied to a row of bars being situated side-by-side on positions 1, \ldots, \(N\):

Each bar situated at an odd numbered position is left as is, while each bar at an even numbered position is replaced by two bars. After that, all bars will be put side-by-side in such a way that all bars form a new row and are situated (side-by-side) on positions 1, \ldots, \(M\).

From an initial number \(a_0 > 0\) of bars there originates (by successive application of the above-defined operation) a sequence, \(\{a_n : n \geq 0\}\) of
natural numbers, where $a_n$ is the number of bars after having applied the operation $n$ times.

(a) Prove that for all $n > 0$ we have $a_n \neq 1997$.
(b) Determine the natural numbers that can only occur as $a_0$ or $a_1$.

6. Let $n$ be a fixed natural number. Determine all polynomials $x^2 + ax + b$, where $a^2 \geq 4b$, such that $x^2 + ax + b$ divides $x^{2n} + ax^n + b$.

Next we turn to the Icelandic Olympiad of 1995–1996. Thanks go to Mohammed Aassila, Strasbourg, France for sending this to us.

ÍSLENZKA STAERÖFRÆÖIKEPPNI
FRAMHÄLDSSKÓLANEMMA 1995–1996
Úrslitakeppni
Laugardagur 23. mars 1996 kl. 10–14

1. Calculate the area of the region in the plane determined by the inequality

$$|x| + |y| + |x + y| \leq 2.$$ 

2. Suppose that $a$, $b$ and $c$ are the three roots of the polynomial $p(x) = x^3 - 19x^2 + 26x - 2$. Calculate

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$ 

3. A collection of 52 integers is given. Show that amongst these numbers it is possible to find two such that 100 divides either their sum or their difference.

4. (i) Show that the sum of the digits of every integer multiple of 99, from $1 \cdot 99$ up to and including $100 \cdot 99$, is 18.

   (ii) Show that the sum of the digits of every integer multiple of the number $10^n - 1$, from $1 \cdot (10^n - 1)$ up to and including $10^n \cdot (10^n - 1)$, is $n \cdot 9$.

5. The sequence $\{a_n\}$ is defined by $a_1 = 1$ and, for $n \geq 1$, 

$$a_{n+1} = \frac{a_n}{1 + na_n}.$$ 

Find $a_{1996}$. 
6. In a square bookcase two identical books are placed as shown in the figure. Suppose the height of the bookcase is 1. How thick are the books?

\begin{center}
\includegraphics[width=0.2\textwidth]{bookcase.png}
\end{center}

As a third set of Olympiad problems we give the Second Round of the 1997 Iranian Mathematical Olympiad. Thanks go to Mohammed Aassila, Strasbourg, France.

**1997 IRANIAN MATHEMATICAL OLYMPIAD**

**Second Round**

**Time:** 2 × 4 hours

1. Suppose that $S$ is a finite set of real numbers with the property that any two distinct elements of $S$ will form an arithmetic progression with another element of $S$. Give an example of such a set with 5 elements and prove that no such set exists with at least 6 elements.

2. Suppose that ten points are given in the plane such that any five of them contain four points which are concyclic. What is the largest number $N$ for which we can correctly say: "At least $N$ of the ten points lie on a circle"? (4 ≤ $N$ ≤ 10.)

3. Suppose that $\Gamma$ is a semi-circle with centre $O$ and diameter $AB$. Assume that $M$ is a point on the extension of $AB$ such that $MA > MB$. A line through $M$ intersects $\Gamma$ at $C$ and $D$ such that $MC > MD$. Circum-circles of the triangles $AOC$ and $BOD$ will intersect at points $O$ and $K$. Prove that $OK \perp MK$.

4. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}\setminus\{1\}$ such that for all $n \in \mathbb{N}\setminus\{0\}$ we have,
$$f(n + 1) + f(n + 3) = f(n + 5)f(n + 7) - 1375.$$ 

5. Suppose that $ABC$ is an acute triangle with $AC < AB$ and the points $O$, $H$, and $P$ are circumcentre, orthocentre, and the foot of the altitude drawn, from $C$ on $AB$, respectively. The line perpendicular to $OP$ at $P$ intersects the line $AC$ at $Q$. Prove that $\angle PHQ = \angle BAC$. 

6. Suppose that \( A \) is a symmetric \((0,1)\)-matrix such that all of its diagonal entries are 1. Prove that there exist \( 0 \leq i_1 < i_2 < \cdots < i_k \leq n \) such that \( A_{i_1} + A_{i_2} + \cdots + A_{i_k} \equiv (1,1,\ldots,1) \pmod{2} \), where \( A_i \) is the \( i \)th row of \( A \).

As a final set for this number we give the problems of the Final Round of the 1997 Iranian Mathematical Olympiad. Again, thanks go to Mohammed Aassila, Strasbourg, France.

1997 IRANIAN MATHEMATICAL OLYMPIAD
Final Round
Time: 2 \( \times \) 4 hours

1. Let \( n \) be a positive integer. Prove that there exist polynomials \( f(x) \) and \( g(x) \) with integer coefficients such that,

\[
f(x)(x+1)^{2n} + g(x)(x^{2n}+1) = 2.
\]

2. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) has the following properties:
   (a) \( \forall x \in \mathbb{R}, \ f(x) \leq 1 \)
   (b) \( \forall x \in \mathbb{R}, \ f \left( x + \frac{13}{42} \right) + f(x) = f \left( x + \frac{1}{6} \right) + f \left( x + \frac{1}{7} \right) \).

Prove that \( f \) is periodic; that is, there exists a non-zero real number \( T \) such that for every real number \( x \), we have \( f(x+T) = f(x) \).

3. Suppose that \( w_1, w_2, \ldots, w_k \) are distinct real numbers with a non-zero sum. Prove that there exist integer numbers \( n_1, n_2, \ldots, n_k \) such that \( \sum_{i=1}^{k} n_i w_i > 0 \) and for any non-identity permutation \( \pi \) on \( \{1,2,\ldots,k\} \) we have \( \sum_{i=1}^{k} n_i w_{\pi(i)} < 0 \).

4. Suppose that \( P \) is a variable point on arc \( BC \) of the circumcircle of triangle \( ABC \), and let \( I_1, I_2 \) be the incentre of the triangles \( PAB \) and \( PAC \), respectively. Prove that,
   (a) The circumcircle of \( PI_1I_2 \) passes through a fixed point.
   (b) The circle with diameter \( I_1I_2 \) passes through a fixed point.
   (c) The mid-point of \( I_1I_2 \) lies on a fixed circle.

5. Suppose that \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is a decreasing continuous function that fulfils the following condition for all \( x, y \in \mathbb{R}^+ \):

\[
f(x+y) + f(y) = f(x + f(y)) + f(y + f(x)).
\]

Prove that \( f(x) = f^{-1}(x) \).
6. A one story building consists of a finite number of rooms which have been separated by walls. There are one or more doors on some of these walls which can be used to travel in this building. Two of the rooms are marked by $S$ and $E$. An individual begins walking from $S$ and wants to reach to $E$.

By a program $P = (P_i)_{i \in I}$, we mean a sequence of $R$'s and $L$'s. The individual uses the program as follows: after passing through the $n^{th}$ door, he chooses the rightmost or the leftmost door, meaning that $P_n$ is $R$ or $L$. For the rooms with one door, any symbol means selecting the door that he had just passed. Notice that the person stops as soon as he reaches $E$.

Prove that there exists a program $P$ (possibly infinite) with the property that no matter how the structure of the building is, one can reach from $S$ to $E$ by following it. [Editor's comment: one has to assume that there is a way of getting from any room to any other room.]

Now we give Klamkin's solutions to the five Quickies given at the beginning of this Corner.

**SOLUTIONS TO FIVE MORE KLAMKIN QUICKIES**

1. $S = 4(a^2 + b^2 + c^2 + d^2) - (a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2) - (a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2)$.

Since the expression on the right hand side is linear in $a^2$, $b^2$, $c^2$, and $d^2$, it takes on its maximum at the endpoints 0, 1 for each variable. By inspection, $S_{\text{max}} = 9$ and is taken on for $a = b = c = 1$ and $d = 0$.

2. Neither inequality is valid.

(i) Just consider the case: $b = 2$, $c = 5$, $d = 1$ and $a$ is very large.

(ii) Just consider the case: $a = 2$, $b = 1$, $c = 8$ and $d$ is very small.

3. Consider the affine transformation $x' = \frac{x}{a}$, $y' = \frac{y}{b}$, $z' = \frac{z}{c}$ which takes the ellipsoid into a sphere. Under this transformation, lines go to lines, planes go to planes, and tangency is preserved. Consequently, any enveloping cone of the ellipsoid goes into an enveloping cone of the sphere and which by symmetry is a right circular one and its points of tangency are a circle (coplanar) of the sphere. Thus, $P$ can be any exterior point of the ellipsoid.

4. Just take any 777 distinct positive integers and multiply each one by the product of the sums of every 7 of them.

5. Assuming the better approximation has the form $\frac{aR + b}{cR + d}$ where $a$, $b$, $c$, $d$ are rational, we must satisfy

$$\left| \frac{aR + b}{cR + d} - \sqrt{5} \right| < |R - \sqrt{5}|.$$ (1)
If $R \to \sqrt{5}$, the left hand side must $\to 0$. Thus, we must have $\frac{a\sqrt{5}+b}{c\sqrt{5}+d} = \sqrt{5}$, so that $a = d = a$ and $b = 5c$. Then substituting these values in (1) and dividing both sides by the common factor $|R - \sqrt{5}|$, we get

$$|cR + a| > |c\sqrt{5} - a|$$

and which can easily be satisfied by letting $a = 2$ and $c = 1$. Finally, our better approximation is $\frac{2R + 5}{R + 2}$.


Next we turn to readers' solutions to problems from the September 1999 Corner, and the Georg Mohr Konkurrencen I Matematik 1996 [1999: 261–262].

**GEORG MOHR KONKURRENCEN I**
**MATEMATIK 1996**
January 11, 9–13

*Only writing and drawing materials are allowed.*

1. $\angle C$ in $\triangle ABC$ is a right angle and the legs $BC$ and $AC$ are both of length 1. For an arbitrary point $P$ on the leg $BC$ construct points $Q$, respectively, $R$, on the hypotenuse, respectively, on the other leg, such that $PQ$ is parallel to $AC$ and $QR$ is parallel to $BC$. This divides the triangle into three parts.

Determine positions of the point $P$ on $BC$ such that the rectangular part has greater area than each of the other two parts.
Denote $x = CP$, $x \in (0, 1)$. Then $PB = 1 - x$, and, from Thales' Theorem,

$$QB = (1 - x)\sqrt{2}, \quad QA = x\sqrt{2}, \quad RC = 1 - x, \quad AR = x.$$  

Thus,

$$[RQPC] = x(1 - x), \quad [PBQ] = \frac{1}{2}(1 - x)^2, \quad [AQR] = \frac{1}{2}x^2.$$  

It remains to solve

$$\begin{cases} x(1 - x) > \frac{1}{2}(1 - x)^2, \\ x(1 - x) > \frac{1}{2}x^2, \end{cases}$$

which is equivalent to

$$\begin{cases} x > \frac{1}{3}, \\ x < \frac{2}{3}. \end{cases}$$

Thus, we will have the desired result if and only if $PC = x$ with $x \in \left(\frac{1}{3}, \frac{2}{3}\right)$.

2. Determine all triples $(x, y, z)$, satisfying

$$xy = z, \quad xz = y, \quad yz = x.$$  

Solution by Pierre Bornszttein, Pontoise, France.

If $(x, y, z)$ is a solution, then, multiplying, we have $(xyz)^2 = xyz$. Thus, $xyz = 0$ or $xyz = 1$.

Case 1. If $xyz = 0$ then, for example, $z = 0$.

From (2) we get $y = 0$, and from (3) we obtain $x = 0$. Conversely, it is easy to see that $(0, 0, 0)$ is a solution.
Case 2. If \( xyz = 1 \) then \( z = \frac{1}{xy} \).

From (1), we deduce \( z^2 = 1 \). Thus, \( z \in \{-1, 1\} \).

In the same way, \( x \in \{-1, 1\} \) and \( y \in \{-1, 1\} \). Moreover, since \( xyz = 1 \), the number of \(-1\)'s in \((x, y, z)\) is even. This leads to

\[(−1, −1, 1), (−1, 1, −1), (1, −1, −1), (1, 1, 1).\]

Conversely, it is easy to see that these triples are solutions. Then,

\[S = \{(0, 0, 0), (1, 1, 1), (−1, −1, 1), (−1, 1, −1), (1, −1, −1)\}.\]

3. This year's idea for a gift is from "BabyMath", namely a series of 9 coloured plastic figures of decreasing sizes, alternating cube, sphere, cube, sphere, etc. Each figure may be opened and the succeeding one may be placed inside, fitting exactly. The largest and the smallest figures are both cubes. Determine the ratio between their side-lengths.

Solution by Pierre Bornsztein, Pontoise, France.

If a sphere with radius \( R \) is circumscribed to a cube with edge \( a \) then the sphere and the cube have the same centre, and the vertices of the cube are points of the sphere.

From Pythagoras' Theorem:

\[R^2 = d^2 + \frac{a^2}{4} = \frac{3a^2}{4}.\]

Thus,

\[R = \frac{\sqrt{3}}{2} a.\] (1)
If a sphere with radius $R$ is inscribed in a cube with edge $b$, then the sphere and the cube have the same centre, and the centres of the sides of the cube are points of the sphere. Then

$$R = \frac{b}{2}. \quad (2)$$

From (1) and (2), it follows that the ratio between the side-lengths of the "outside cube" and the "inside cube" is

$$\frac{b}{a} = \sqrt{3}.$$  

Since there are 5 cubes, the ratios between the side-lengths of the largest and the smallest figures is $(\sqrt{3})^4 = 9$.

4. $n$ is a positive integer. It is known that the last but one digit in the decimal expression of $n^2$ is 7. What is the last digit?

_Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's solution._

We prove that this last digit is 6.

Write $n$ as $10a + b$ where $a$ is a non-negative integer and $b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then $n^2 = 100a^2 + 20ab + b^2$ and the last two digits of $n^2$ are also those of $20ab + b^2$.

But $20ab = 10 \times 2ab$ ends with the digits 00 or 20 or 40 or 60 or 80. Since it is given that the penultimate digit of $20ab + b^2$ is 7, we see that the penultimate digit of $b^2$ must be odd; this can occur only when $b = 4$ or $b = 6$, and then $b^2 = 16$ or $b^2 = 36$. Adding these two values to any of the integers 00, 20, 40, 60, 80, we obtain only one result whose last but one digit is 7, and it is 76. This completes the proof.

5. In a ballroom 7 gentlemen, $A, B, C, D, E, F$ and $G$ are sitting opposite 7 ladies $a, b, c, d, e, f$ and $g$ in arbitrary order. When the gentlemen walk across the dance floor to ask each of their ladies for a dance, one observes that at least two gentlemen walk distances of equal length. Is that always the case?

The figure shows an example. In this example $Bb = Ee$ and $Dd = Cc.
Solutions by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Note first that the observation is only correct if we assume that the 7 gentlemen are "evenly" spaced.

We show that in general, if there are n gentlemen and n ladies, then the same conclusion holds when $n \equiv 2, 3 \pmod{4}$.

Suppose the n ladies are situated at $(k, 0)$ and the n gentlemen, at $(k, 1); k = 1, 2, \ldots, n$. Suppose that the gentleman at $(k, 1)$ walks a distance of $d_k$ to the lady at $(a_k, 0), a_k \in \{1, 2, \ldots, n\}$. Then $(a_1, a_2, \ldots, a_n)$ is a permutation of $(1, 2, \ldots, n)$ and thus, $\sum_{k=1}^{n} (a_k - k) = 0$. Since $d_k = (1 + (a_k - k)^2)^{1/2}$, we have $d_k^2 = 1 + (a_k - k)^2$. We show that the values of the $d_k$'s cannot all be distinct.

Note that $a_k - k \in \{0, \pm 1, \pm 2, \ldots, \pm (n-1)\}$. Suppose to the contrary that $(a_k - k)^2 \neq (a_j - j)^2$ for all $j \neq k, j = 1, 2, \ldots, n$. Then we have $\{|a_k - k|: k = 1, 2, \ldots, n\} = \{0, 1, 2, \ldots, n-1\}$ and thus,

$$\sum_{k=1}^{n} |a_k - k| = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}.$$ (1)

On the other hand, since $|t| - t$ must be even for all integers $t$, we have

$$\sum_{k=1}^{n} |a_k - k| = 2d + \sum_{k=1}^{n} (a_k - k) = 2d, \text{ for some integer } d.$$ (2)

Comparing (1) and (2) we get $n(n-1) = 4d$ which implies that $n \equiv 0, 1 \pmod{4}$. Therefore, if $n \equiv 2, 3 \pmod{4}$ then we must have $d_j = d_k$ for some $j \neq k$.

Editor's question. If $n$ is congruent to either 0 or 1, is it always possible to arrange the n gentlemen and the n ladies in a way such that the distance are all different?

Next we turn to solutions to problems of the St. Petersburg City Mathematical Olympiad, Third Round, 1996 [1999: 262].

ST. PETERSBURG CITY MATHEMATICAL OLYMPIAD
Third Round – February 25, 1996
11th Grade (Time: 4 hours)

1. Serge was solving the equation $f(19x - 96/x) = 0$ and found 11 different solutions. Prove that if he tried hard he would be able to find at least one more solution.
Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang’s write-up.

Note first that \(x = 0\) is not a solution. If \(r \neq 0\) is a solution, then so is \(t = -\frac{96}{19r}\), since

\[
f \left( 19t - \frac{96}{t} \right) = f \left( -\frac{96}{r} + 19r \right) = 0.
\]

Since \(r = \frac{96}{19t}\) is impossible, different \(r\)'s will correspond to different \(t\)'s. Therefore, the number of solutions must be even (if it is finite) and the conclusion easily follows.

2. The numbers 1, 2, \ldots, \(2n\) are divided into two groups of \(n\) numbers. Prove that pairwise sums of numbers in each group (sums of the form \(a + a\) included) have identical sets of remainders on division by \(2n\).

Solution by Pierre Bornsztein, Pontoise, France.

Let \(G_1, G_2\) be the two groups, and let \(S_1 = \{p \in \{1, 2, \ldots, 2n\} : \) there exist \(a, b \in G_1\) for which \(a + b \equiv p \pmod{2n}\}. \) \(S_2\) is defined in the same way. We must prove that \(S_1 = S_2\).

Let \(p \in \{1, \ldots, 2n\}\).

We will have \(a + b \equiv p \pmod{2n}\), with \(a, b \in \{1, \ldots, 2n\}\), if and only if \(a + b = p\) or \(a + b = 2n + p\).

Case 1. If \(p\) is odd, then \(p = 2k + 1\) for some integer \(k\) such that \(0 \leq k < n\).

Let \(a, b \in \{1, \ldots, 2n\}\), with \(a \leq b\).

We will have \(a + b = p\) if and only if \((a, b)\) is one of the \(k\) pairs

\[
(1, 2k), \ (2, 2k - 1), \ \ldots, \ (k, k + 1).
\]

We will have \(a + b = 2n + p\) if and only if \((a, b)\) is one of the \(n - k\) pairs

\[
(2k + 1, 2n), \ (2k + 2, 2n - 1), \ \ldots, \ (n + k, n + k + 1).
\]

Then the numbers 1, 2, \ldots, \(2n\) are divided into \(n\) pairs to give the remainder \(p\).

If \(p \in S_1\), then \(G_1\) contains at least one of these pairs. We have to choose at most \(n - 2\) numbers to complete the group \(G_1\). It cannot be done if we want \(G_1\) to contain at least one of the members of each pair. Then at least one pair is included in \(G_2\). Thus, \(p \in S_2\).
Case 2. If \( p \) is even, then \( p = 2k \) for some integer \( k \) such that \( 1 \leq k \leq n \).

As above the numbers 1, 2, \ldots, 2n are divided into \( n + 1 \) pairs to give the remainder \( p \). The pairs are

\[
(1, 2k - 1), \ (2, 2k - 2), \ \ldots, \ (k - 1, k + 1), \ (k, k)
\]

and

\[
(2k, 2n), \ (2k + 1, 2n - 1), \ \ldots, \ (n + k - 1, n + k + 1), \ (n + k, n + k).
\]

Then we have \( n - 1 \) pairs and two "isolated" numbers \( k \) and \( n + k \).

If \( p \in S_1 \) and if at least one of the two "isolated" numbers is not in \( G_1 \), then \( k \in G_2 \) or \( n + k \in G_2 \). Thus, \( p \in S_2 \).

If \( p \in S_1 \) and if both of the two "isolated" numbers are in \( G_1 \), then \( G_1 \) contains \( n - 2 \) other numbers. Thus, \( G_1 \) cannot contain at least one of the numbers from each of the \( n - 1 \) pairs.

It follows that at least one of the pairs is included in \( G_2 \). So, we deduce that \( p \in S_2 \).

Then, in each case, if \( p \in S_1 \) then \( p \in S_2 \). That is,

\[
S_1 \subseteq S_2.
\]

By symmetry, we have \( S_2 \subseteq S_1 \). Thus,

\[
S_1 = S_2.
\]

3. No three diagonals of a convex 1996-gon meet in one point. Prove that the number of the triangles lying in the interior of the 1996-gon and having sides on its diagonals is divisible by 11.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France. We give the solution of Aassila.

If \( ABCDE \) is a 6-gon such that no three diagonals meet in one point, then the triangle formed by \( AD, BE \) and \( CF \) is the only one having sides on its diagonals. Hence, the number of such triangles for a 1996-gon is \( \binom{1996}{6} \), but since \( 1991 = 11 \times 181 \), we deduce the desired result.

4. Points \( A' \) and \( C' \) are taken on the diagonal \( BD \) of a parallelogram \( ABCD \) so that \( AA' \parallel CC' \). Point \( K \) lies on the segment \( A'C \), the line \( AK \) meets the line \( C'C \) at the point \( L \). A line parallel to \( BC \) is drawn through \( K \), and a line parallel to \( BD \) is drawn through \( C \). These two lines meet at point \( M \). Prove that the points \( D, M, L \) are collinear.
Let $O$ be the intersection of $AC$ and $BD$. Since $ABCD$ is a parallelogram we have $AO = OC$.

Since $AA'||CC'$, we obtain $A'O : OC' = AO : OC = 1 : 1$, so that $A'O = OC'$. Thus, $AA'CC'$ is a parallelogram, and further, $AC' || A'C$. Since $AD || KM$, $C'D || CM$, $AC' || KC$, and $AC' \neq KC$, we have $AK$, $C'C$, and $DM$ are concurrent at $L$.

Therefore, $D$, $M$, $L$ are collinear.

That completes this issue of the Corner. We will continue with these problems in the next number. Please send me Olympiad Contests and your nice solutions and generalizations.