THE OLYMPIAD CORNER

No. 212

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We begin this number with the problems of the IV$^{\text{th}}$ class of the Croatian National Mathematical Competition, Novi Vinodolski, May 8–11, 1997. Thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for collecting them.

CROATIAN NATIONAL MATHEMATICAL COMPETITION

Novi Vinodolski, May 8–11, 1997

IV$^{\text{th}}$ Class

1. Find the last four digits of the number 3^{1000} and the number 3^{1997}.

2. A circle k and the point K are on the same plane. For every two distinct points P and Q on k, the circle k' contains the points P, Q, and K. Let M be the intersection of the tangent to the circle k' at the point K and the line PQ. Find the locus of the points M when P and Q move over all points on k.

3. A function f is defined on the set of positive numbers, which has the following properties

$$f(1) = 1, \quad f(2) = 2,$$

$$f(n+2) = f(n+2-f(n+1)) + f(n+1-f(n)), \quad (n \geq 1).$$

(a) Show that $f(n+1) - f(n) \in \{0,1\}$ for every $n \geq 1$.

(b) If $f(n)$ is odd, show that $f(n+1) = f(n) + 1$.

(c) For every natural number k determine all numbers n for which

$$f(n) = 2^{k-1} + 1.$$

4. Let k be a natural number. Determine the number of non-congruent triangles whose vertices are the vertices of the regular polygon with $6k$ sides.

We continue with the Additional Competition for selection of the 38th IMO team of the Croatian National Mathematical Competition, also sent to us by Richard Nowakowski.

CROATIAN NATIONAL MATHEMATICAL COMPETITION

ADDITIONAL COMPETITION FOR SELECTION OF THE 38th IMO TEAM

May 10, 1997

1. Three points A, B, C, are given on the same line, such that B is between A and C. Over the segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, as diameters, the semicircles are constructed on the same side of the line. The perpendicular from B to AC intersects the largest circle at point D. Prove that the common tangent of two smaller semicircles, different from BD, is parallel to the tangent on the largest semicircle through the point D.

2. Let a, b, c, d be real numbers such that at least one is different from zero. Prove that all roots of the polynomial

$$P(x) \;=\; x^6 + ax^3 + bx^2 + cx + d$$

cannot be real.

3. Squares $ABDE$ and $BCFG$ are constructed outside the acute triangle ABC in which $|AB| < |BC|$. Let M, N be the mid-points of the sides $\overline{BC}$, $\overline{AC}$, respectively, and S be the intersection of the lines BN and GM, $S \neq N$. Suppose that the points M, C, S and N are on the same circle. Prove that $|MD| = |MG|$.

4. Prove that every natural number can be represented in the unique form

$$a_0 + 2a_1 + 2^2a_2 + \cdots + 2^n a_n$$

where $a_k \in \{-1, 0, 1\}$ and $a_k \cdot a_{k+1} = 0$ for every $0 \leq k \leq n-1$.

Next we turn to the Selection Round of the 1997 St. Petersburg City Mathematical Olympiad. Thanks go to Richard Nowakowski for collecting the problems while at the IMO in Argentina.

1997 ST. PETERSBURG CITY MATHEMATICAL OLYMPIAD

Selection Round – 10^{th} Grade

March 10, 1997

1. Positive integers x, y, z satisfy the equation $2x^x + y^y = 3z^z$. Prove that they are equal.

2. The number N is the product of k different primes ($k \geq 3$). Two players play the following game: they in turn write on the blackboard **composite** divisors of N. One cannot write the number N. It is also not permitted that two coprime numbers or two numbers one of which divides the other are written on the blackboard. The player who cannot move loses. Which of them has a winning strategy: the player starting the game or his adversary?

3. K, L, M, N are the mid-points of sides AB, BC, CD, DA respectively of an inscribed quadrangle $ABCD$. Prove that the orthocentres of triangles AKN, BKL, CLM, DMN are vertices of a parallelogram.

4. A 100×100 checked square is folded several times along the lattice lines. Two straight cuts are made also going along the lattice lines. What is the maximum number of parts that the square can be cut into?

5. All sides of a convex polyhedron are triangles. At least 5 edges go from each of its vertices, and no two vertices of degree 5 are connected by an edge. Prove that this polyhedron has a face whose vertices have degrees 5, 6, and 6 respectively.

6. $2n + 1$ lines lie on the plane. Prove that there are not more than $n(n+1)(2n+1)/6$ different acute triangles with sides on these lines.

7. Prove that the collection of all 12-digit numbers cannot be divided into groups of four numbers in such a way that the digits of the four numbers in each group coincide in 11 positions, with the remaining position using four consecutive digits.

8. 360 points divide the circle into equal arcs. These points are connected by 180 non-intersecting chords. Consider another 180 chords obtained from these by rotation of the circle by the angle $38°$. Prove that the union of all these 360 chords cannot be a closed broken line.

11^{th} Grade

1. Can a 75×75 table be partitioned into dominoes (that is, 1×2 rectangles) and crosses (that is, five-square figures consisting of a square and its four horizontal and vertical neighbours)?

2. Prove that for $x \geq 2$, $y \geq 2$, $z \geq 2$

$$(y^3 + x)(z^3 + y)(x^3 + z) \geq 125xyz\,.$$

3. Circles S_1 and S_2 intersect at points A and B. A point Q is chosen on S_1. The lines QA and QB meet S_2 at points C and D; the tangents to S_1 at A and B meet at point P. The point Q lies outside S_2, the points C and D lie outside S_1. Prove that the line QP goes through the mid-point of CD.

4. A convex 50-gon with vertices at integral points is drawn on a checked paper. What maximum number of its diagonals can lie on the lattice lines?

5. The number $99\ldots99$ (1997 nines) is written on a blackboard. Every minute one of the numbers written on the blackboard is factored into two factors, then wiped out, and these two factors, independently increased or diminished by 2, are written instead. Can it be that at last all the numbers on the blackboard will be equal to 9?

6. A device consists of $4n$ elements. Every pair of them are connected by a red or a blue wire. The numbers of red and blue wires are equal. The device is totally disabled when two wires of the same colour connecting four elements are removed. An agent of a supposed enemy found the number of ways to disable the device by removing two blue wires. Prove that there are as many ways to disable the device by removing two red wires.

7. See problem 7, 10^{th} grade.

8. An Aztek diamond of rank n is a figure consisting of those cells of a checked coordinate plane that are wholly contained in the square $\{(x, y) : |x| + |y| \leq n + 1\}$. For any covering of an Aztek diamond by dominoes (1×2 rectangles) a "switch" operation is permitted: one can choose any 2×2 square covered by exactly two dominoes and rotate it by $90°$. Prove that not more than $n(n+1)(2n+1)/6$ such operations are required to transform an arbitrary covering into the covering consisting only of horizontal dominoes.

Next we give the problems of the 33rd Spanish Mathematical Olympiad, 2nd Round, March 1997, First Day. Thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for collecting and forwarding this material to us.

33rd SPANISH MATHEMATICAL OLYMPIAD
Second Round — March 1997
First Day (Time: 4.5 hours)

1. Calculate the sum of the squares of the first 100 terms of an arithmetic progression, given that the sum of the first 100 terms of the progression equals -1, and that the sum of the even numbered terms equals $+1$.

2. Let A be the set of the 16 lattice points forming a square of side 4. Find, with reasons, the largest number of points of A such that any THREE of them do NOT form an isosceles right triangle.

3. Consider the parabolas

$$y = x^2 + px + q,$$

that intersect the coordinate axes at three distinct points. For these three points, a circle is drawn. Show that all the circles drawn when p and q vary over $\mathbb{R}$ pass through a fixed point, and determine the point.

Second Day (Time: 4.5 hours)

4. Let p be a prime number. Find all integers $k \in \mathbb{Z}$ such that $\sqrt{k^2 - pk}$ is a positive integer.

5. Show that for any convex quadrilateral with unit area, the sum of the sides and diagonals is not less than $2(2 + \sqrt{2})$.

6. The exact quantity of gasoline necessary for ONE complete round of a circular circuit is distributed in n reservoirs, situated at any n fixed points of the circuit. At the beginning, the car has no gasoline.

Prove that, for any distribution of the gasoline in the reservoirs, there exists an initial point such that it is possible to make a complete circuit.

Notes: The consumption of gasoline is uniform and proportional to the distance. The car can contain all the gasoline.

As a final problem set this number we give the Final Round of the 7th Japan Mathematical Olympiad (1997). Again, my thanks go to Richard Nowakowski who collected the problem set while in Argentina.

7th JAPAN MATHEMATICAL OLYMPIAD
Final Round — February 1997
Time: 4.5 hours

1. Prove that, whenever we put ten points in any way on a circle whose diameter is 5, we can find two points whose distance is less than 2.

2. Let a, b, c be positive integers. Prove that the inequality

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}$$

holds. Determine also when the equality holds.

3. Let G be a graph with 9 vertices (with no self-looping edges). Assume that "given any five points of G, there are at least edges e_1, ..., e_n, ($n \geq 2$), forming a path between two of these five points".

What is the minimum number of edges such a graph G must have?

4. Let A, B, C, D be points in space in a general position. Assume that $AX + BX + CX + DX$ is a minimum at $X = X_0$ ($X_0 \neq A, B, C, D$). Prove that $\angle AX_0B = \angle CX_0D$.

5. Let n be a positive integer. To each vertex of a regular 2^n–gon, we assign one of letters "A" or "B". Prove that we can do this in such a way that all possible sequences of n letters which appear in this 2^n–gon as an arc directed clockwise from some vertex are mutually distinct.

Next we turn to solutions by readers to the problems of the Dutch Mathematical Olympiad [1999 : 134–135].

1. A kangaroo jumps from lattice-point to lattice-point in the (x, y)–plane. She can make only two kinds of jumps:

Jump A: 1 to the right (in the positive x-direction) and 3 up (in the positive y-direction).
Jump B: 2 to the left and 4 down.

(a) The start position of the kangaroo is the origin $(0, 0)$. Show that the kangaroo can jump to the point $(19, 95)$ and determine the number of jumps she needs to reach that point.

(b) Take the start position to be the point $(1, 0)$. Show that it is impossible for her to reach the point $(19, 95)$.

(c) The start position of the kangaroo is once more the origin $(0, 0)$. Which points (m, n) with m, $n \geq 0$ can she reach, and which points can she not reach?

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

(a) and (c). Clearly, the point (m, n) is reachable if and only if the system

$$\begin{cases} x - 2y = m \\ 3x - 4y = n \end{cases}$$

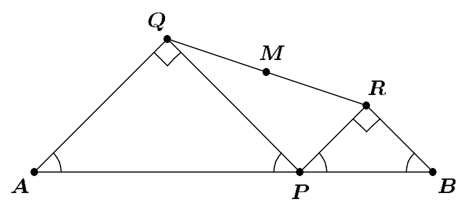
has solutions in non-negative integers x and y, where x and y denote the number of jumps of type A and type B, respectively. Since the solution of the system is $x = n - 2m$ and $y = \frac{n-3m}{2}$, we conclude that (m, n) is reachable if and only if m and n have the same parity such that $3m \leq n$. In particular, when $m = 19$, $n = 95$ we get $x = 57$ and $y = 19$; that is, she can reach $(19, 95)$ by making 57 jumps of type A and 19 jumps of type B.

(b) In this case, the system to be solved becomes

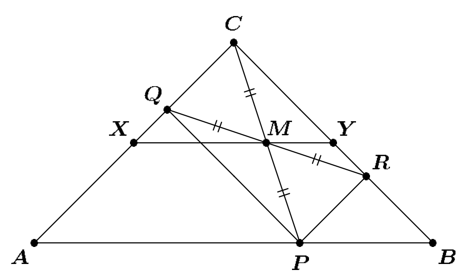
$$\begin{cases} x - 2y &= 18 \\ 3x - 4y &= 95\,. \end{cases}$$

Since $y = \frac{41}{2}$ is not an integer, $(19, 95)$ is no longer reachable.

2. On a segment AB a point P is chosen. On AP and PB, isosceles right-angled triangles AQP and PRB are constructed with Q and R on the same side of AB. M is the mid-point of QR. Determine the set of all points M for all points P on the segment AB.



Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution of Seimiya.



We consider the problem on one side of AB. Let C be the intersection of AQ and BR. Since $\angle CAB = \angle QAP = 45^\circ$ and $\angle CBA = \angle RBP = 45^\circ$, we have that $\triangle CAB$ is isosceles and a right-angled triangle.

Since $\angle QPA = \angle RBP$ $(= 45^\circ)$, we have $PQ \| BR$. Similarly we have $PR \| AQ$, so that $CQPR$ is a parallelogram.

Since M is the mid-point of QR, M is also the mid-point of CP.

Let X and Y be the mid-points of CA and CB respectively.

Since X, M are mid-points of CA, CP respectively, we have $XM \| AP$.

Similarly we have $MY \| PB$. Hence, X, M, Y are collinear.

If P varies on the segment AB from A to B, then M varies on the segment XY from X to Y.

Thus, the set of all points M is the segment XY.

3. 101 marbles are numbered from 1 to 101. The marbles are divided over two baskets A and B. The marble numbered 40 is in basket A. This marble is removed from basket A and put in basket B. The average of all the numbers on the marbles in A increases by $\frac{1}{4}$. The average of all the numbers of the marbles in B increases by $\frac{1}{4}$ too. How many marbles were there originally in basket A?

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

Let $x_0, x_1, \ldots, x_n$ (respectively $x_{n+1}, \ldots, x_{100}$) denote the numbers on the marbles originally in basket A (respectively in basket B) with $x_0 = 40$. The original averages are

$$m_A = \frac{40 + x_1 + \cdots + x_n}{n+1} \quad \text{and} \quad m_B = \frac{x_{n+1} + \cdots + x_{100}}{100 - n}\,.$$

After the marble numbered 40 has left basket A for basket B, the averages become

$$m'_A = \frac{x_1 + \cdots + x_n}{n} \quad \text{and} \quad m'_B = \frac{40 + x_{n+1} + \cdots + x_{100}}{101 - n}\,.$$

Expressing $m'_A = m_A + \frac{1}{4}$ and $m'_B = m_B + \frac{1}{4}$, we easily get:

$$\frac{x_1 + \cdots + x_n}{n(n+1)} = \frac{40}{n+1} + \frac{1}{4}$$

and

$$\frac{x_{n+1} + \cdots + x_{100}}{(100-n)(101-n)} = \frac{40}{101-n} - \frac{1}{4}$$

which yield

$$x_1 + \cdots + x_n = \frac{n^2 + 161n}{4} \qquad (1)$$

and

$$x_{n+1} + \cdots + x_{100} = \frac{5900 + 41n - n^2}{4}\,. \qquad (2)$$

Since $x_1 + \cdots + x_n + x_{n+1} + \cdots + x_{100} = (1 + 2 + \cdots + 100 + 101) - 40 = \frac{101 \times 102}{2} - 40$, we obtain by addition of (1) and (2):

$$5900 + 202n = 204 \times 101 - 160\,.$$

Hence, $n = 72$ and there were initially 73 marbles in basket A.

4. A number of spheres, all with radius 1, are being placed in the form of a square pyramid. First, there is a layer in the form of a square with $n \times n$ spheres. On top of that layer comes the next layer with $(n-1) \times (n-1)$ spheres, and so on. The top layer consists of only one sphere. Determine the height of the pyramid.

Solutions by Mohammed Aassila, Strasbourg, France; and by Michel Bataille, Rouen, France. We give Bataille's solution.

We observe that there is a plane parallel to the base containing all the centres of the spheres of a layer, and we first determine the distance d between the planes so associated to two successive layers. Any sphere (S) of the $(k+1)^{\text{th}}$ layer rests on four spheres (S_1), (S_2), (S_3), (S_4) of the k^{th} layer so that (S) is tangent to these four spheres and (S_1), (S_2) [and (S_2), (S_3), and (S_3), (S_4) and (S_4), (S_1)] are tangent to each other

Let Ω, A, B, C, D be the centres of (S), (S_1), (S_2), (S_3), (S_4) respectively. Then d is the height of the regular pyramid $\Omega ABCD$ and we have

$$\Omega A = \Omega B = \Omega C = \Omega D = AB = BC = CD = DA = 2 .$$

Let H be the projection of Ω on the plane $(ABCD)$ and K be the projection of H on, say, AB. Then K is also the projection of Ω on AB and Pythagoras' theorem gives:

$$d^2 = \Omega H^2 = \Omega K^2 - HK^2 = \left(2\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{2}{2}\right)^2 = 2 .$$

Hence, $d = \sqrt{2}$.

Now, there are n planes associated to the different layers, so that the distance between the lowest (P_1) and the highest (P_n) of these planes is $(n-1)\sqrt{2}$.

Taking into account that the base is lower than (P_1) by a radius of the spheres and that the apex is higher than (P_n) by a radius as well, we finally obtain the height of the pyramid: $2 + (n-1)\sqrt{2}$.

5. We consider arrays $(a_1, a_2, \ldots, a_{13})$ containing 13 integers. An array is called "tame" when for each $i \in \{1, 2, \ldots, 13\}$ holds: if you leave a_i out, the remaining twelve integers can be divided in two groups in such a way that the sum of the numbers in one group is equal to the sum of the numbers in the other group. A "tame" array is called "turbo tame" if you can always divide the remaining twelve numbers in two groups of six numbers having the same sum.

(a) Give an example of an array of 13 integers (not all equal!) that is "tame". Show that your array is "tame".

(b) Prove that in a "tame" array all numbers are even or all numbers are odd.

(c) Prove that in a "turbo tame" array all numbers are equal.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

(a) For $i \in \{1, \ldots, 12\}$, let $a_i = 1$ and $a_{13} = 11$.

Leaving out a_{13}, then $a_1 + \cdots + a_6 = 6 = a_7 + \cdots + a_{12}$.

If a_{i_0}, for some $i_0 \in \{1, \ldots, 12\}$, then

$$a_{13} \;=\; 11 \;=\; \sum_{\substack{i \neq i_0 \\ i \leq 12}} a_i \,.$$

Thus, $(1, 1, \ldots, 1, 11)$ is "tame".

(b) Let $(a_1, \ldots, a_{13})$ be a "tame" array. Denote $S = a_1 + a_2 + \cdots + a_{13}$. Then there exist A_i and B_i, subsets of $\{1, \ldots, 13\} - \{i\}$ such that $A_i \cap B_i = \emptyset$, $A_i \cup B_i = \{1, \ldots, 13\} - \{i\}$ and

$$\sum_{a_j \in A_i} a_j \;=\; \sum_{a_j \in B_i} a_j \,.$$

Denote $S_i = \sum_{a_j \in A_i} a_j$. Thus,

$$S \;=\; a_i + 2S_i \,.$$

It follows, for every $i \in \{1, \ldots, 13\}$, that $a_i \equiv S \pmod 2$. Thus, all the a_i have the same parity, the parity of S.

(c) Let $(a_1, \ldots, a_{13})$ be "turbotame". Since $(a_1, \ldots, a_{13})$ is "tame", all the a_i have the same parity (from (b)). It follows that $a_j - a_1$ is even for $j = 2, \ldots, 13$. And it is easy to see that $(0, a_2 - a_1, \ldots, a_{13} - a_1)$ is also "turbotame". Suppose, for a contradiction, that there exists $i_0 \in \{2, \ldots, 13\}$ such that $a_{i_0} \neq a_1$. Then $a_{i_0} - a_1 \in \mathbb{Z}^*$ and $a_{i_0} - a_1$ is even.

Let p be the greatest power of 2 that divides $a_{i_0} - a_1$ (that is equivalent to, $a_{i_0} - a_1 \equiv 0 \pmod{2^p}$ and $a_{i_0} - a_1 \not\equiv 0 \pmod{2^{p+1}}$).

Denote, for $i \in \{1, \ldots, 13\}$,

$$\begin{aligned} U_0(i) &= a_i \,, \\ U_1(i) &= a_i - a_1 \,, \\ U_{n+1}(i) &= \frac{1}{2} U_n(i) \quad \text{for } n \geq 1 \,. \end{aligned}$$

Then, for all $n \geq 1$, $U_n(1) = 0$.

We know that $(U_1(1), U_1(2), \ldots, U_1(13))$ is "turbotame", and

$$U_1(i) \equiv 0 \pmod 2 \quad \text{for all } i \in \{1, \ldots, 13\} \,.$$

By an easy induction, for $n \geq 1$, $(U_n(1), U_n(2), \ldots, U_n(13))$ is "turbotame" and $U_n(i) \equiv 0 \pmod 2$ for all $i \in \{1, \ldots, 13\}$. (All the $U_n(i)$ are even since $(U_n(1), \ldots U_n(13))$ is "tame", and from (b), $U_n(i) \equiv U_n(1) \pmod 2$).

It follows that $U_{p+1}(i_0)$ is an even integer.

But $U_{p+1}(i_0) = \frac{1}{2}U_p(i_0) = \cdots = \frac{a_{i_0}-a_1}{2^p}$, an odd integer (from the maximality of p). This is a contradiction.

Then, for $i \in \{1, \ldots, 13\}$, $a_i = a_1$. It follows that all numbers are equal.

Remark. With the same reasoning, 13 can be replaced with any odd integer $k \geq 5$. I think that this general statement is a problem from the William Lowell Putnam Competition (but I have lost the reference ...).

Next we turn to the May 1999 number of the *Corner* and readers' solutions to problems of the XXXIX Republic Competition of Mathematics in Macedonia Class I [1999 : 196].

1. The sum of three integers a, b and c is 0. Prove that $2a^4+2b^4+2c^4$ is the square of an integer.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; by Andrei Simion, student, Brooklyn Technical High School, New York; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Simion, welcoming a new contributor to the Corner.

Let $p(x) = x^3 + sx^2 + qx + r$ be a third degree polynomial having a, b, and c as roots.

According to Viète's Theorem $s = -(a+b+c) = 0$. Then

$$\begin{aligned} a^3 + qa + r &= 0 \\ b^3 + qb + r &= 0 \\ c^3 + qc + r &= 0\,. \end{aligned}$$

Upon multiplying each equation by $2a$, $2b$ and $2c$, respectively, and adding we have

$$2a^4 + 2b^4 + 2c^4 + 2q\left(a^2+b^2+c^2\right) = 0$$

(the term with r vanishing since $a+b+c=0$). However,

$$\begin{aligned} a^2+b^2+c^2 &= (a+b+c)^2 - 2(ab+bc+ac) \\ &= -2q\,. \end{aligned}$$

Thus,

$$2a^4 + 2b^4 + 2c^4 - 4q^2 = 0$$

and

$$2a^4 + 2b^4 + 2c^4 = (2q)^2$$

and we are done.

2. Prove that if

$$a_0^{a_1} = a_1^{a_2} = \cdots = a_{1995}^{a_{1996}} = a_{1996}^{a_0}, \quad a_1 \in \mathbb{R}^*,$$

then

$$a_0 = a_1 = \cdots = a_{1996}.$$

Solution by Pierre Bornsztein, Courdimanche, France.

I think that the a_i are supposed to be non-negative real numbers, and give that solution.

If $a_0, \ldots, a_{1996}$ are non-negative real numbers such that

$$a_0^{a_1} = \cdots = a_{1996}^{a_0}, \quad \text{then } a_0 = a_1 = \cdots = a_{1996}.$$

We first prove that $a_i \neq 0$ for all i.

Indeed, we are given $a_1 \neq 0$. Then, $a_1^{a_2} = a_0^{a_1} \neq 0$. Thus, $a_0 \neq 0$. An easy induction yields $a_i \neq 0$ for all i.

Next we show that either $a_i = 1$ for all i, or $0 < a_i < 1$ for all i or $1 < a_i$ for all i.

First suppose $a_i = 1$. Then $a_i^{a_{i+1}} = 1 = a_{i+2}^{a_{i+3}}$ (indices read modulo 1997). Now $a_{i+2} \neq 1$ entails $a_{i+3} = 0$, contrary to the last claim. It follows then that $a_i = 1$ for some i implies that $a_j = 1$ for all $j = 0, 1, \ldots, 1996$. Thus, we may suppose that $a_i \neq 0, 1$ for all $i = 0, 1, \ldots, 1996$. Suppose, for a contradiction, that $0 < a_i < 1$ for some i and $1 < a_j$ for some value of j. Then, we may suppose, without loss of generality, that $0 < a_i < 1$ and $1 < a_{i+1}$. But then $1 > a_i^{a_{i+1}} = a_{i+1}^{a_{i+2}}$ gives $a_{i+2} < 0$, contrary to the hypothesis. The claim now follows.

To complete the proof we distinguish two cases:

Case 1. $0 < a_i < 1$ for all $i = 0, 1, \ldots, 1996$.

Now first suppose $a_0 < a_1$. From $a_1^{a_2} = a_0^{a_1} < a_1^{a_1}$ and the fact that a_1^x is monotone decreasing, we obtain $a_1 < a_2$.

From this we get $a_0 < a_1 < a_i < \cdots < a_{1996} < a_{1997} = a_0$, a contradiction. The assumption that $a_1 < a_0$ similarly leads to a contradiction, completing this case.

Case 2. $1 < a_i$ for all i.

Suppose for a contradiction, that $a_0 < a_1$. Then $a_1^{a_2} = a_0^{a_1} < a_1^{a_1}$. It follows that $a_2 < a_1$. Thus, $a_1^{a_2} = a_2^{a_3} < a_1^{a_3}$, and we obtain $a_2 < a_3$.

By an easy induction we get $a_{2p} < a_{2p+1}$ for all p (subscripts are read modulo 1997).

But $a_{1998} = a_1$ and $a_{1999} = a_2$, giving $a_2 > a_1$, a contradiction. Thus, we have $a_0 \geq a_1$.

Next suppose $a_0 > a_1$. By similar reasoning we obtain $a_{2p} > a_{2p+1}$ for all p, leading to a similar contradiction.

Thus, $a_0 = a_1$.

From the cyclic symmetry of the assumptions we get $a_i = a_{i+1}$ for all i; that is, $a_0 = a_1 = a_2 = \cdots = a_{1996}$.

3. Let h_a, h_b and h_c be the altitudes of the triangle with edges a, b and c, and r be the radius of the inscribed circle in the triangle. Prove that the triangle is equilateral if and only if $h_a + h_b + h_c = 9r$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Luyun-Zhong-Qiao, Columbia International College, Hamilton, Ontario; by Toshio Seimiya, Kawasaki, Japan; and by Andrei Simion, student, Brooklyn Technical High School, New York. We give Bataille's write-up.

Clearly $h_a + h_b + h_c = 9r$ when the triangle is equilateral since in that case, we have

$$h_a = h_b = h_c = a\frac{\sqrt{3}}{2} \quad \text{and} \quad r = \frac{1}{3} \cdot a\frac{\sqrt{3}}{2}.$$

Conversely, let S denote the area of a $\triangle ABC$ in which $h_a + h_b + h_c = 9r$. Then, on the one hand,

$$2S = ah_a = bh_b = ch_c \quad \text{so that } h_a + h_b + h_c = 2S\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

and on the other hand

$$2S = ar + br + cr, \quad \text{so that} \quad r = \frac{2S}{a+b+c}.$$

From the hypothesis, we get

$$\frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{3}{a+b+c}.$$

In other words, the harmonic mean of a, b, c equals their arithmetic mean. This implies $a = b = c$ and the triangle is equilateral.

4. Prove that each square can be cut into n ($n \geq 6$) squares.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Let S denote a given square. For any integer $k \geq 2$, we can divide S into k^2 equal (smaller) squares (as on a $k \times k$ chessboard). If we erase all the internal line segments forming all the $(k-1)^2$ smaller squares situated at the southeast corner of S, then we obtain a square T of dimension $(k-1) \times (k-1)$ which, together with the $2k - 1$ smaller squares in the first row and first column of S, constitute a decomposition of S into $2k$ smaller squares. (Figures 1 and 2 below depict the cases when $k = 3$ and 4, respectively). If we further divide T into four equal squares as shown in Figure 3, then we have a decomposition of S into $2k + 3$ squares. Since $k \geq 2$ is arbitrary we can always divide S into n squares for any $n \geq 6$ (as well as $n = 4$, of course).

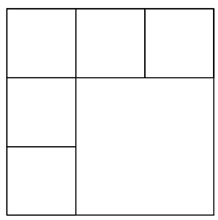


Figure 1

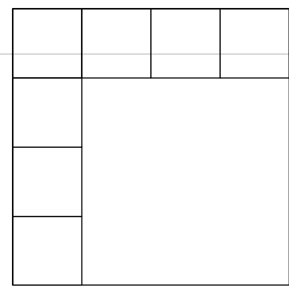


Figure 2

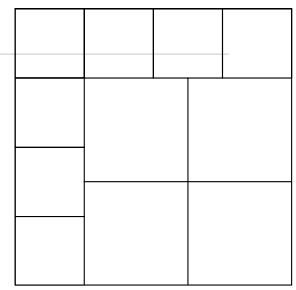


Figure 3

Remarks. (1) If we require that all the smaller squares must have different dimensions, then we have the famous "Squaring the Square" problem first studied by Brooks, Smith, Stone, and Tutte in 1936. (See, for example, *Ingenuity in Mathematics* by Ross Honsberger, New Mathematical Library, pp. 46–60).

(2) A similar problem about dissecting an equilateral triangle into n smaller equilateral triangles was proposed jointly by (the late) Helen Sturtevant and E.T.H. Wang in [1986 : 27; Solution 1987: 189]. It was shown that such dissection is possible for all n except when $n = 2, 3, 5$. It seems that the same conclusion is also true for dissecting a square. It would be interesting to see a proof (or a counterexample) of this.

Next are solutions to Class II problems of the XXXIX Republic Competition of Mathematics in Macedonia [1999 : 196–197]

1. Prove that for positive real numbers a and b

$$2 \cdot \sqrt{a} + 3 \cdot \sqrt[3]{b} \geq 5 \cdot \sqrt[5]{ab}\,.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's solution.

Set $a_1 = \sqrt{a}$, $a_2 = \sqrt{a}$, $a_3 = \sqrt[3]{b}$, $a_4 = \sqrt[3]{b}$, $a_5 = \sqrt[3]{b}$. Then, by the A.M.–G.M. inequality, we have

$$\sqrt[5]{ab} = \sqrt[5]{a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5} \leq \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = \frac{2}{5}\sqrt{a} + \frac{3}{5}\sqrt[3]{b}$$

and the result follows.

Remarks. The generalization is immediate: if m and n are integers ≥ 2, then we have

$$m\sqrt[m]{a}+n\sqrt[n]{b} \;\geq\; (m+n)\sqrt[m+n]{ab} \quad \text{for all positive real numbers } a \text{ and } b\,.$$

This also results from Hölder's Inequality:

$$uv \;\leq\; \frac{u^p}{p}+\frac{v^q}{q} \quad \text{when } u,v>0 \text{ and } \frac{1}{p}+\frac{1}{q} \;=\; 1\,,$$

by taking

$$u \;=\; a^{1/(m+n)}\,, \quad v \;=\; b^{1/(m+n)}\,, \quad p \;=\; \frac{m+n}{m}\,, \quad q \;=\; \frac{m+n}{n}\,.$$

3. Let $A = \{z_1, z_2, \ldots, z_{1996}\}$ be a set of complex numbers and for each $i \in \{1, 2, \ldots, 1996\}$ suppose $\{z_iz_1, z_iz_2, \ldots, z_iz_{1996}\} = A$.

(a) Prove that $|z_i| = 1$ for each i.

(b) Prove that $z \in A$ implies $\overline{z} \in A$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

We note the obvious misprint in the original statement [1999 : 197] and solve the general problem by assuming that $A = \{z_1, z_2, \ldots, z_n\}$ is a set of n (distinct) complex numbers, $n \geq 2$, such that for each $i \in \{1, 2, \ldots, n\}$ we have

$$\{z_iz_1, z_iz_2, \ldots, z_iz_n\} \;=\; A\,, \tag{1}$$

(a) Let i be fixed, $1 \leq i \leq n$. Then by (1) we have

$$\prod_{j=1}^{n} z_iz_j \;=\; \prod_{j=1}^{n} z_j\,, \quad \text{or} \quad z_i^n\left(\prod_{j=1}^{n} z_j\right) \;=\; \prod_{j=1}^{n} z_j\,.$$

If $z_j = 0$ for any j, then $\{z_jz_1, z_jz_2, \ldots, z_jz_n\} = \{0\} \neq A$, since A has at least two elements. Thus, $\prod_{j=1}^{n} z_j \neq 0$, and we get $z_i^n = 1$. Hence, $|z_i|^n = |z_i^n| = 1$ from which $|z_i| = 1$ follows.

(b) Let $z \in A$ be fixed. Then by (1), $\{zz_1, zz_2, \ldots, zz_n\} = A$, and thus, $z = zz_i$ or $z_i = 1$ for some i, $1 \leq i \leq n$. It follows that $zz_k = 1$ for some k, $1 \leq k \leq n$. Since $z\overline{z} = |z|^2 = 1$, we conclude that $\overline{z} = z_k \in A$.

Remark. An example of a set A which satisfies the assumption (and hence, the conclusion) of the problem would be the set of all n of the n^{th} roots of unity.

4. Find the biggest value of the difference $x - y$ if $2 \cdot (x^2 + y^2) = x + y$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Heinz-Jürgen Seiffert, Berlin, Germany; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Seiffert's generalization and solution.

More generally: Let $p > 0$ and $q > 1$. If the real numbers x and y satisfy the equation

$$p(|x-y|^q + |x+y|^q) \;=\; x+y\,, \qquad (1)$$

then there holds the sharp inequality

$$x - y \;\le\; (pq)^{1/(1-q)}(q-1)^{1/q}\,. \qquad (2)$$

Proof. The function $f(t) = t/p - t^q$, $t \ge 0$, has first derivative $f'(t) = \frac{1}{p} - qt^{q-1}$, $t > 0$. Let $t_0 = (pq)^{1/(1-q)}$. From $f'(t) > 0$ if $0 < t < t_0$, and $f'(t) < 0$ if $t > t_0$, it follows that $f(t) \le f(t_0)$ for all $t \ge 0$; that is,

$$\frac{t}{p} - t^q \;\le\; (pq)^{q/(1-q)}(q-1), \quad t \;\ge\; 0\,.$$

Suppose that the real numbers x and y satisfy (1). Then

$$\begin{aligned} |x-y|^q &= \frac{x+y}{p} - |x+y|^q \;\le\; \frac{|x+y|}{p} - |x+y|^q \\ &\le\; (pq)^{q/(1-q)}(q-1)\,. \end{aligned}$$

Hence,

$$x - y \;\le\; |x-y| \;\le\; (pq)^{1/(1-q)}(q-1)^{1/q}\,,$$

proving (2). It is easily verified that

$$x \;=\; \frac{1}{2}(pq)^{1/(1-q)}(1 + (q-1)^{1/q})$$

and

$$y \;=\; \frac{1}{2}(pq)^{1/(1-q)}(1 - (q-1)^{1/q})$$

satisfy (1), and that there is equality in (2) for these values. Thus, (2) is sharp.

With $p = 1$ and $q = 2$, (1) becomes $2(x^2 + y^2) = x + y$. Then, by (2), there holds the sharp inequality $x - y \le \frac{1}{2}$. Hence, $\frac{1}{2}$ is the biggest value asked for.

Now we look at readers' solutions to Class III of the XXXIX Republic Competition of Mathematics in Macedonia [1999 : 197].

1. Solve the equation $x^{1996} - 1996x^{1995} + \cdots + 1 = 0$ (the coefficients in front of $x, \ldots, x^{1994}$ are unknown), if it is known that its roots are positive real numbers.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Andrei Simion, student, Brooklyn Technical High School, New York. We give Bataille's solution.

Let $x_1, x_2, \ldots, x_{1996}$ be the roots of the given polynomial. By hypothesis, $x_k > 0$ for all k in $\{1, 2, \ldots, 1996\}$ and from the known coefficients: $x_1 + x_2 + \cdots + x_{1996} = 1996$ and $x_1 \cdot x_2 \cdots x_{1996} = 1$. It follows that:

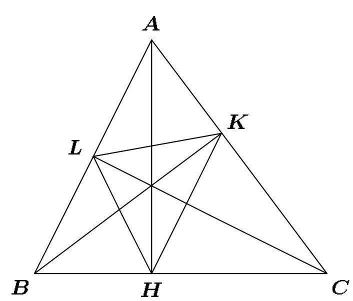
$$\frac{x_1 + x_2 + \cdots + x_{1996}}{1996} = 1 = \sqrt[1996]{x_1 \cdot x_2 \cdots x_{1996}}\,.$$

Therefore, we are in the case where the A.M.–G.M. inequality is actually an equality. As it is well known, this means that $x_1 = x_2 = \cdots = x_{1996}$ from which $x_1 = x_2 = \cdots = x_{1996} = 1$ immediately follows. Thus, the solution of the given equation is 1 with multiplicity 1996.

2. Let AH, BK and CL be the altitudes of arbitrary triangle ABC. Prove that

$$\overline{AK} \cdot \overline{BL} \cdot \overline{CH} = \overline{AL} \cdot \overline{BH} \cdot \overline{CK} = \overline{HK} \cdot \overline{KL} \cdot \overline{LH}\,.$$

Solutions by Toshio Seimiya, Kawasaki, Japan; and by Andrei Simion, student, Brooklyn Technical High School, New York. We give Seimiya's answer.



Since $\angle BKC = \angle BLC = 90^\circ$, B, C, K, L are concyclic. Hence, $\angle AKL = \angle ABC$, so that we have

$$\triangle AKL \sim \triangle ABC\,.$$

Thus,

$$\frac{\overline{AK}}{\overline{KL}} = \frac{\overline{AB}}{\overline{BC}}\,. \qquad (1)$$

Similarly we have

$$\frac{\overline{BL}}{\overline{LH}} = \frac{\overline{BC}}{\overline{CA}}, \tag{2}$$

and

$$\frac{\overline{CH}}{\overline{HK}} = \frac{\overline{CA}}{\overline{AB}}. \tag{3}$$

From (1), (2), (3) we get

$$\frac{\overline{AK}}{\overline{KL}} \cdot \frac{\overline{BL}}{\overline{LH}} \cdot \frac{\overline{CH}}{\overline{HK}} = \frac{\overline{AB}}{\overline{BC}} \cdot \frac{\overline{BC}}{\overline{CA}} \cdot \frac{\overline{CA}}{\overline{AB}} = 1.$$

It follows that

$$\overline{AK} \cdot \overline{BL} \cdot \overline{CH} = \overline{HK} \cdot \overline{KL} \cdot \overline{LH}.$$

Similarly we have

$$\overline{AL} \cdot \overline{BH} \cdot \overline{CK} = \overline{HK} \cdot \overline{KL} \cdot \overline{LH}.$$

3. An initial triple of numbers 2, $\sqrt{2}$, $\frac{1}{\sqrt{2}}$ is given. It is admitted to obtain a new triple from an old one as follows: two numbers a and b of the triple are changed to $\frac{(a+b)}{\sqrt{2}}$ and $\frac{(a-b)}{\sqrt{2}}$ and the third number is unchanged. Is it possible after a finite number of such steps to obtain the triple $(1, \sqrt{2}, 1+\sqrt{2})$?

Solution by Pierre Bornsztein, Courdimanche, France.

The answer is no.

Let T be the transformation described in the statement of the problem. If

$$T(a,b,c) = (x,y,z),$$

it is clear that

$$a^2+b^2+c^2 = x^2+y^2+z^2.$$

Then, from an initial triple (a,b,c), the number $S = x^2+y^2+z^2$ is invariant by the use of T. Starting from $\left(2, \sqrt{2}, \frac{1}{\sqrt{2}}\right)$ with $2^2 + \sqrt{2}^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 6.5$ we cannot obtain $\left(1, \sqrt{2}, 1+\sqrt{2}\right)$ after any number of steps because

$$1^2 + \left(\sqrt{2}\right)^2 + \left(1+\sqrt{2}\right)^2 = 6+2\sqrt{2} \neq 6.5.$$

4. A finite number of points in the plane are given such that not all of them are collinear. A real number is assigned to each point. The sum of the numbers for each line containing at least two of the given points is zero. Prove that all numbers are zeros.

Solution by Pierre Bornsztein, Courdimanche, France.

Let M_1, M_2, $\ldots$, M_n be the points. Then $n \geq 3$. Denote by a_i, the real number assigned to M_i, and let $S = a_1 + a_2 + \cdots + a_n$.

For a fixed $i_0 \in \{1, \ldots, n\}$, denote by n_{i_0} the number of lines containing M_{i_0} and at least one of the M_j for $j \neq i_0$.

Since not all the M_i are collinear, we have $n_i \geq 2$ for each i. Let Δ be one of the lines containing M_{i_0} and another of the M_j.

We then have

$$S(\Delta) \;=\; \sum_{M_j \in \Delta} a_j \;=\; 0\,.$$

Adding all these equalities, for all such Δ, we obtain

$$\sum_{M_{i_0} \in \Delta} S(\Delta) \;=\; 0 \;=\; S + (n_{i_0} - 1)a_{i_0}\,.$$

Since M_{i_0} is arbitrary, we then have

$$S + (n_i - 1)a_i \quad \text{for each } i\,. \tag{1}$$

Suppose that $S \neq 0$. Then there exists $i_0 \in \{1, \ldots, n\}$ such that a_{i_0} has the same sign as S.

For this choice of i_0, we cannot have $S + (n_{i_0} - 1)a_{i_0} = 0$. This is a contradiction.

Then $S = 0$, and, from (1), we have $a_i = 0$ for each i.

Next we look at solutions to Class IV, XXXIX Republic Competition of Mathematics in Macedonia, [1999 : 197].

1. Let a_1, a_2, $\ldots$, a_n be real numbers which satisfy:

There exists a real number M such that $|a_i| \leq M$ for each $i \in \{1, \ldots, n\}$.

Prove that $a_1 + 2a_2 + \cdots + na_n \leq \frac{Mn^2}{4}$.

Comments and solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Aassila's remarks.

This problem is similar to the one proposed by Morocco but not used by the jury at the 28th IMO in Cuba. A solution appeared in [1998 : 38]. The problem as stated is false: If $a_i = M$ for $i = 1, 2, \ldots, n$, then $\frac{Mn(n+1)}{2} \not\leq \frac{Mn^2}{4}$. One has to assume that $a_1 + a_2 + \cdots + a_n = 0$ as in [1998 : 38].

2. Two circles with radii R and r touch from inside. Find the side of an equilateral triangle having one vertex at the common point of the circles and the other two vertices lying on the two circles.

Solution by Michel Bataille, Rouen, France.

Let Γ, γ denote the two given circles (with centres Ω, ω and radii R, r respectively) and let A be their point of tangency.

An equilateral triangle ABC having B on γ and C on Γ is easily obtained by drawing the circle Γ' image of Γ under a rotation ρ with centre A and angle 60°. B is the point other than A common to γ and Γ', and C is the image of B under ρ^{-1}. [There are two such triangles — because we can use either a direct or an indirect rotation; they are symmetrical about the line $A\Omega$ and, consequently, have the same length of sides].

Let Ω' be the centre of Γ' (so that $A\Omega = A\Omega' = R$). Then $AB = 2h$ where h is the length of the altitude from A in $\Delta A\omega\Omega'$. Since $A\omega = r$, $A\Omega' = R$ and $\angle\omega A\Omega' = 60^\circ$, the Cosine Law gives: $\omega\Omega'^2 = r^2 + R^2 - rR$.

Now, the area of $\triangle A\omega\Omega'$ is $\frac{1}{2}h \times \omega\Omega'$ as well as $\frac{1}{2} \times rR\sin(\angle\omega A\Omega') = \frac{\sqrt{3}}{4}rR$. This provides immediately:

$$h = \frac{\sqrt{3}}{2}\frac{rR}{\sqrt{r^2+R^2-rR}} \quad \text{and} \quad AB = \sqrt{3}\frac{rR}{\sqrt{r^2+R^2-rR}}.$$

That completes the *Corner* for this issue. Olympiad Season is coming up! Send me your nice contests and your nice solutions to problems from the *Corner* for future use.