SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2505. [2000 : 46] *Proposed by Hayo Ahlburg, Benidorm, Spain, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that A, B and C are the angles of a triangle. Determine the best lower and upper bounds of $\prod_{\text{cyclic}} \sin(B-C)$.

Combination of solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, and Kee-Wai Lau, Hong Kong.

We show that

$$-\frac{3\sqrt{3}}{8} \;\le\; \prod_{\text{cyclic}} \sin(B-C) \;\le\; \frac{3\sqrt{3}}{8}\,.$$

First, for the best upper bound we consider two cases.

Case 1: $\sin(B-C) \ge 0$. Using the identity

$$\sin X \sin Y \;=\; \frac{1}{2}\Big(\cos(X-Y)-\cos(X+Y)\Big)\,,$$

we have

$$\begin{aligned}\prod_{\text{cyclic}} \sin(B-C) &= \frac{1}{2}\sin(B-C)\Big(\cos(B+C-2A)-\cos(C-B)\Big) \\ &\le \frac{1}{2}\sin(B-C)\Big(1-\cos(B-C)\Big)\,.\end{aligned}$$

The maximum of $\sin(B-C)(1-\cos(B-C))$ occurs at the same value as for its square

$$\sin^2(B-C)\Big(1-\cos(B-C)\Big)^2 \;=\; (1+c)(1-c)^3\,,$$

where $c=\cos(B-C)$. By the AM–GM inequality,

$$\left[(1+c)\left(\frac{1-c}{3}\right)^3\right]^{1/4} \le \frac{1}{4}\left[(1+c)+3\left(\frac{1-c}{3}\right)\right] \;=\; \frac{1}{2}\,,$$

with equality only when all four numbers are equal; that is, when $(1-c)/3 = 1+c$ or $c=-1/2$. Therefore,

$$(1+c)(1-c)^3 \;\le\; \left(\frac{1}{2}\right)^4\cdot 27 \;=\; \frac{27}{16}\,,$$

with equality only when $\cos(B - C) = c = -1/2$. Thus, the upper bound will be

$$\prod_{\text{cyclic}} \sin(B - C) \;\le\; \frac{1}{2}\sqrt{\frac{27}{16}} \;=\; \frac{3\sqrt{3}}{8}\,.$$

The degenerate triangle with $A = \pi/3$, $B = 2\pi/3$, $C = 0$ shows that the upper bound cannot be improved.

Case 2: $\sin(B - C) < 0$. Similarly we have

$$\begin{aligned}\prod_{\text{cyclic}} \sin(B - C) &= -\frac{1}{2}\sin(B - C)\Big(\cos(C - B) - \cos(B + C - 2A)\Big) \\ &\le -\frac{1}{2}\sin(B - C)\Big(\cos(B - C) + 1\Big) \\ &= \frac{1}{2}\sqrt{(1 - c)(1 + c)^3}\,,\end{aligned}$$

with c as above. The maximum occurs when $(1 + c)/3 = 1 - c$ or when $\cos(B - C) = c = 1/2$. This gives the same upper bound as before, but here there are no corresponding angles of a triangle.

For the lower bound, we get

$$\prod_{\text{cyclic}} \sin(B - C) \;\ge\; -\frac{3\sqrt{3}}{8}\,.$$

[*Editorial note*: The easiest way to see this, as some readers noted, is because

$$\prod_{\text{cyclic}} \sin(B - C) \;=\; -\prod_{\text{cyclic}} \sin(C - B) \;\ge\; -\frac{3\sqrt{3}}{8}\,,$$

using the upper bound.] The degenerate triangle with $A = 2\pi/3$, $B = \pi/3$, $C = 0$ shows that the lower bound cannot be improved either.

Editorial note: Klamkin observes that these bounds hold for ***arbitrary*** angles A, B, C, as the above proof (which is largely Klamkin's, but uses some elements of Lau's solution as well) shows. Thus readers might like to investigate the best bounds, in terms of n, for

$$\prod_{\text{cyclic}} \sin(A_1 - A_2),$$

where $A_1, A_2, \ldots, A_n$ are arbitrary angles.

Also solved by AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HENRY LIU, student, University of Cambridge, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, *UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One other reader misread the product for a sum.*

Most readers obtained strict inequality in both bounds, which is correct if degenerate triangles are not allowed.

2512. [2000 : 46] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$, the sides satisfy $a \geq b \geq c$. Let R and r be the circumradius and the inradius respectively. Prove that

$$bc \;\leq\; 6Rr \;\leq\; a^2\,,$$

with equality if and only if $a = b = c$.

Solution by Henri Liu, graduate student, University of Cambridge, UK.

Let A be the area of $\triangle ABC$. Combining the well-known identities $4AR = abc$ and $2A = r(a+b+c)$, we have

$$6Rr \;=\; \frac{3abc}{a+b+c}\,.$$

Using $a+b+c \leq 3a$, we obtain

$$6Rr \;\geq\; \frac{3abc}{3a} \;=\; bc\,.$$

Equality holds if and only if $a = b = c$. Using $a+b+c \geq 3c$, we get

$$6Rr \;\leq\; \frac{3abc}{3c} \;=\; ab \;\leq\; a^2$$

with equality if and only if $a = b = c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JONATHAN CAMPBELL, student, Chapel Hill High School, North Carolina, USA; GORAN CONAR, student, University of Zagreb, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; ROMÁN FRESNEDA, Universidad de la Habana, Cuba; KARTHIK GOPALRATNAM, student, Angelo State University, Texas, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY PAN, student, East York C.I., Toronto, Ontario; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. Most of the submitted solutions are similar to the one given above.

2514. [2000 :] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle ABC$ and $\angle BCA$ meet CA and AB at D and E respectively. Suppose that $AE = BD$ and that $AD = CE$. Characterize $\triangle ABC$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

As is known,

$$AE^2 = \left(\frac{bc}{b+a}\right)^2 , \qquad AD^2 = \left(\frac{bc}{c+a}\right)^2 ,$$
$$BD^2 = ca\left(1 - \frac{b^2}{(c+a)^2}\right) , \qquad CE^2 = ba\left(1 - \frac{c^2}{(b+a)^2}\right) ,$$

so that $\left(\frac{AE}{AD}\right)^2 = \left(\frac{BD}{CE}\right)^2$ reduces to

$$b(c+a)^4\left((b+a)^2 - c^2\right) = c(b+a)^4\left((c+a)^2 - b^2\right) ,$$

or successively to

$$(c+a)^2(b+a)^2\left((b(c+a)^2 - c(b+a)^2\right) = bc\left(c(c+a)^4 - b(b+a)^4\right) ,$$

or

$$\begin{aligned} 0 &= (b-c)\left((c+a)^2(b+a)^2a^2\right) \\ &\quad +bc\left(b(b+a)^4 - c(c+a)^4 - (b-c)(c+a)^2(b+a)^2\right) . \end{aligned}$$

The second term also has a factor $(b - c)$. When this is factored out, there remains a polynomial all of whose coefficients are positive. [Ed. I checked — it has 19 terms!] Hence, $b = c$, and the triangle is isosceles.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

All of the other solvers, with one exception, also showed that the base angles of the triangle are 36°.

2515. [2000 : 114] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$, $\angle ABC$ and $\angle BCA$ meet BC, AC and AB at D, E and F respectively. Let p be the perimeter of $\triangle ABC$. Suppose that $AF + BD + CE = \frac{1}{2}p$. Characterize $\triangle ABC$.

I. *Solution by David Loeffler, student, Cotham School, Bristol, UK.*

We show first that the angle bisector of a triangle divides the opposite side in the ratio of the other two sides. By the Law of Sines on $\triangle ABD$, we have

$$BD \;=\; c\,\frac{\sin\left(\frac{1}{2}\angle BAC\right)}{\sin(\angle ADB)}\,.$$

Similarly, from $\triangle ADC$ we get

$$DC \;=\; b\,\frac{\sin\left(\frac{1}{2}BAC\right)}{\sin(\angle ADC)}\,.$$

Thus,

$$\frac{BD}{DC} \;=\; \frac{c}{b}\left(\frac{\sin(\angle ADC)}{\sin(180^\circ - \angle ADC)}\right) \;=\; \frac{c}{b}\,.$$

Likewise, we have $\dfrac{CE}{EA} = \dfrac{a}{c}$ and $\dfrac{AF}{FB} = \dfrac{b}{a}$. Thus, we have

$$AF \;=\; \frac{bc}{a+b}\,, \qquad BD \;=\; \frac{ca}{b+c}\,, \qquad CE \;=\; \frac{ab}{c+a}\,.$$

The condition that $AF + BD + CE = \frac{1}{2}p$ is now equivalent to

$$\frac{2bc}{a+b} + \frac{2ca}{b+c} + \frac{2ab}{c+a} \;=\; a+b+c\,,$$

or, upon multiplying by $(a+b)(b+c)(c+a)$,

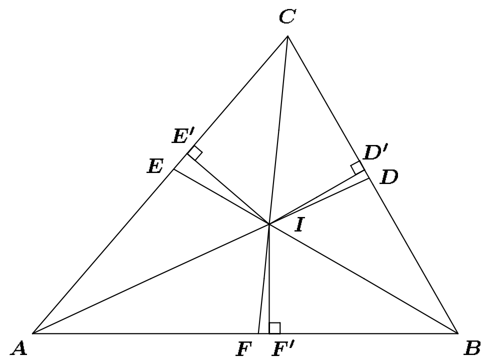
$$\begin{aligned}&2bc(b+c)(c+a) + 2ca(c+a)(a+b) + 2ab(a+b)(b+c)\\ &\quad=\; (a+b+c)(a+b)(b+c)(c+a)\,.\end{aligned}$$

After expanding, most of the terms cancel and the equation becomes

$$ab^3 + bc^3 + ca^3 \;=\; a^3b + b^3c + c^3a\,,$$

which factors as $(a+b+c)(a-b)(b-c)(c-a) = 0$. Thus $a = b$, or $b = c$, or $c = a$; that is, the triangle is isosceles.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editors.*



Without loss of generality, assume $A \le B \le C$. Let r be the inradius, and I the incentre. Let the projections of I upon BC, CA, AB be D', E', F' respectively as shown in the diagram above. Let $s = \frac{1}{2}(a+b+c)$ where $a = BC$, $b = CA$, $c = AB$. Then

$$\angle DID' \;=\; \angle ADB - \frac{\pi}{2} \;=\; \frac{A}{2} + C - \frac{A+B+C}{2} \;=\; \frac{C-B}{2}\,.$$

Similarly, $\angle EIE' = (C-A)/2$ and $\angle FIF' = (B-A)/2$.

Notice that $AF' + BD' + CE' = (s-a) + (s-b) + (s-c) = s = AF + BD + CE$. Hence $EE' = FF' + DD'$; that is,

$$r \tan\frac{1}{2}(B-A) + r\tan\frac{1}{2}(C-B) \;=\; r\tan\frac{1}{2}(C-A)\,.$$

Let $\alpha = (B-A)/2$ and $\beta = (C-B)/2$. The above equation becomes:

$$\tan\alpha + \tan\beta \;=\; \tan(\alpha+\beta)\,;$$

$$\text{that is,}\quad \tan\alpha + \tan\beta \;=\; \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\,.$$

Therefore, we have $\tan\alpha = -\tan\beta$ or $\tan\alpha\tan\beta = 0$. The first case leads to $\alpha = -\beta$ and thus $A = C$; the second case yields either $A = B$ or $B = C$. Therefore, $\triangle ABC$ is isosceles. Conversely, if $\triangle ABC$ is isosceles, it is trivial to see that $AF + BD + CE = s$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were two incomplete and two incorrect solutions.

2516. [2000 : 115] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In isosceles $\triangle ABC$ (with $AB = AC$), let D and E be points on sides AB and AC respectively such that $AD < AE$. Suppose that BE and CD meet at P. Prove that $AE + EP < AD + DP$.

I. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let Ax and Ay be the half-lines starting at A and containing B and C respectively, and let Az be the bisector of $\angle xAy$. The bisector of $\angle DCy$ meets Az at K, which is therefore an excentre of triangle ADC; the bisector of $\angle EBx$ meets Az at L, an excentre of triangle ABE. We have

$$\angle LBx \;=\; \frac{1}{2}\angle EBx \;<\; \frac{1}{2}\angle DCy \;=\; \angle KCy\,,$$

and hence,

$$AL \;>\; AK\,.$$

If F on AB is the symmetric point of P with respect to the angle bisector DK, and G on AC is the symmetric point of P with respect to the angle bisector EL, then we have

$$DP \;=\; DF\,, \quad \text{or} \quad AF \;=\; AD + DP\,, \quad \text{and}$$

$$EP \;=\; EG\,, \quad \text{or} \quad AG \;=\; AE + EP\,.$$

Since K is on the same side of EL as P, then $KG > KP = KF$. Finally, from the fact that K and A lie on opposite sides of the line EF, we have that $AF > AG$, or $AD + DP > AE + EP$.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

We shall generalize the problem using Green's Theorem with some basic calculus. I hope that the solution has a sufficient degree of elegance to befit the celebration of the 90th birthday of Prof. Seimiya.

Theorem. Let CD, BE be cevians of a triangle ABC where $BD \geq CE$. Let CD intersect BE at P. Then $AD + DP > AE + EP$.

Remark. The given problem is a corollary: Let A' be on AC such that $AA' = AD$. Let BA' intersect CD at D'. Then

$$AD + DD' \;=\; AA' + A'D'$$

by symmetry, and

$$A'D' + D'P \;>\; A'E + EP$$

by the theorem. Adding the two expressions then gives the result.

Proof of the theorem. Let $x_1 = \angle BCD$, $x_2 = \angle BCA$, $y_1 = \angle CBE$, $y_2 = \angle CBA$. Assume $BC = 1$ unit. For any point X in the plane of $\triangle ABC$ let $\angle XCB = x$ and $\angle CBX = y$. The point is inside quadrangle $ADPE$ when both $x_1 < x < x_2$ and $y_1 < y < y_2$. When X is on AD, if x is increased by dx, X would move along AD from D to A through ds. From the Law of Sines,

$$BX = \frac{BC \sin x}{\sin(x+y)}.$$

Thus $\dfrac{\partial s}{\partial x} = \dfrac{\sin(x+y)\cos x - \sin x \cos(x+y)}{\sin^2(x+y)} = \dfrac{\sin y}{\sin^2(x+y)}$.
Consequently,

$$\begin{aligned} DA &= \int_{x_1}^{x_2} \frac{\sin y_2}{\sin^2(x+y_2)} dx \\ &= \text{line integral} \int_{DA} \frac{\sin y dx + \sin x dy}{\sin^2(x+y)}. \end{aligned}$$

Similar results hold for DP, PE, EA, so that

$$AE + EP - PD - DA = \text{line integral} \int_{ADPE} \frac{\sin y dx + \sin x dy}{\sin^2(x+y)}.$$

By Green's Theorem, this equals

$$\int\int \left[\frac{\partial}{\partial x} \frac{\sin x}{\sin^2(x+y)} - \frac{\partial}{\partial y} \frac{\sin y}{\sin^2(x+y)} \right] dxdy$$

over the interior of $ADPE$.

Let $\alpha = y + x$ and $\beta = y - x$. Then the integrand simplifies to

$$\begin{aligned} & \sin^{-3}\alpha[\sin\alpha\cos x - 2\sin x\cos\alpha - \sin\alpha\cos y + 2\sin y\cos\alpha] \\ = & \sin^{-3}\alpha[\sin\alpha(\cos x - \cos y) + 2\cos\alpha(\sin y - \sin x)] \\ = & \sin^{-3}\alpha\left[\sin\alpha\left(2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\right) + 4\cos\alpha\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\right] \\ = & \sin^{-3}\alpha\sin\left(\frac{\beta}{2}\right)\left[\cos\left(\frac{\alpha}{2}\right) - \cos\left(\frac{3\alpha}{2}\right) + 2\cos\left(\frac{3\alpha}{2}\right) + 2\cos\left(\frac{\alpha}{2}\right)\right] \\ = & \sin^{-3}\alpha\sin\left(\frac{\beta}{2}\right)\left[3\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{3\alpha}{2}\right)\right]. \end{aligned}$$

With a bit of calculus, it can been seen that $3\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{3\alpha}{2}\right)$ is a decreasing function for $0 \le \alpha \le \pi$, reaching 0 only when $\alpha = \pi$. This proves that the integrand is positive or negative according as β (that is, $y - x$) is positive or negative; so too with the double integral and therefore also the line integral. The theorem specifies that $y_2 \le x_1$, so that $y \le x$ and the line integral is negative. Thus $AE + EP - PD - DA < 0$. ■

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain (two solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HENRY LIU, student, *Trinity College Cambridge; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA (a second solution); and the proposer.*

Both Yiu and the proposer showed that the result follows quickly from an 1846 result of Steiner:

> *Given a convex quadrangle $ABCD$ whose sides AB and CD meet at E while AD and BC meet at F, labeled so that B is between A and E, C is between E and D and also between F and B, while D is between F and A, then $AB + BC = AD + DC$ if and only if $AE + EC = AF + FC$, if and only if the productions of the four sides of $ABCD$ are tangent to a circle.*

Seimiya provides the reference F. G.-M., Exercices de géométrie, *p. 318, Théorème 157. Liu calls the result "Urquhart's Theorem," but provides no reference, presumably because it was easier to give a proof than to find a reference. The story of how M.L. Urquhart (1902-1966) got his name on the result is told by Dan Pedoe in "The Most 'Elementary' Theorem of Euclidean Geometry," Math. Magazine* **49** *: 1 (January 1976), 40–42.*

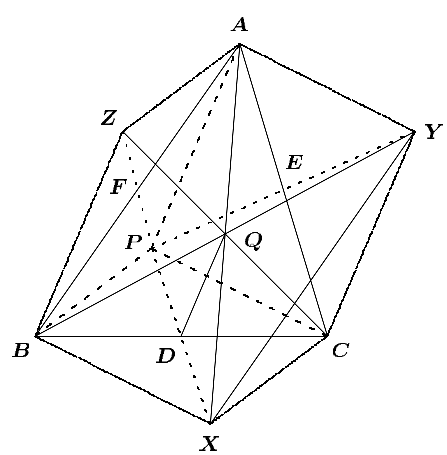
2517. [2000 : 115] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that D, E, F are the mid-points of the sides BC, CA, AB, respectively, of $\triangle ABC$. Let P be any point in the plane of the triangle, distinct from A, B and C.

1. Show that the lines parallel to AP, BP, CP, through D, E, F, respectively, are concurrent (at Q, say).
2. If X, Y, Z are the symmetrics of P with respect to D, E, F, respectively, show that AX, BY, CZ are concurrent at Q.

I. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

[Ed.: The most elementary solution was given by Nikolaos Dergiades.]



Since $PBXC$ and $PCYA$ are parallelograms, it follows that $ABXY$ is a parallelogram, and hence, BY passes through the mid-point Q of AX. Similarly, CZ passes through Q. Hence, AX, BY, CZ are concurrent at Q. In $\triangle APX$, note that D is the mid-point of PX, and that Q is the mid-point of AX. Therefore, DQ is parallel to AP. Likewise, EQ is parallel to BP, and FQ is parallel to CP.

II. *Solution by Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland.*

[Ed.: A number of solvers used a transformational argument based on properties of homotheties. However, no-one else saw Gunther's short solution presented below.]

Denote by $H(R, \lambda)$ the homothety (central dilatation) with centre R and ratio λ. Note that the composition $H(R, \lambda) \circ H(S, \mu) = H(T, \lambda\mu)$, where the point T lies on the line RS (in the case where $\lambda\mu = 1$, this is a translation when T lies at infinity and the identity when $R = S = T$).

In $\triangle ABC$, let G be the centroid. Observe first that $H(G, -\frac{1}{2})$ takes $\triangle ABC$ to $\triangle DEF$. Since a homothety takes a line to a parallel line, it follows that AP, BP, CP are taken to the respectively parallel lines DQ, EQ, FQ, where Q is the image of P under $H(G, -\frac{1}{2})$. Next note that $H(P, 2)$ takes $\triangle DEF$ to $\triangle XYZ$. Hence, the composition $H(P, 2) \circ H(G, -\frac{1}{2})$ is a homothety which takes $\triangle ABC$ to $\triangle XYZ$, and whose centre lies on the line PG. It is easy to verify that this composition has Q as a fixed point, and hence $H(P, 2) \circ H(G, -\frac{1}{2}) = H(Q, -1)$. Thus AX, BY, CZ are all concurrent with Q.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL COVAS, Mallorca, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

A number of solutions employed vectors; others used coordinates (both affine and areal). Klamkin noted that a generalization of part (1) *is given in Problem 39, Math. Horizons, April 1996, by T.S. Bolis and M. Klamkin:* "Let $n + 1$ points be given on a sphere. From the centroid of any n of these points a line is drawn normal to the tangent plane to the sphere at the remaining point. Prove that all of these $n + 1$ lines are concurrent." *In the solution to this problem, it was noted that* (1) *had appeared on a Cambridge Scholarship Examination. As well, Benito and Fernández noted that a problem similar to* (2) *is exercise #196 of the Leçons de Géométrie Élémentaire, vol. 1 by J. Hadamard.*

2518. [2000 : 115] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

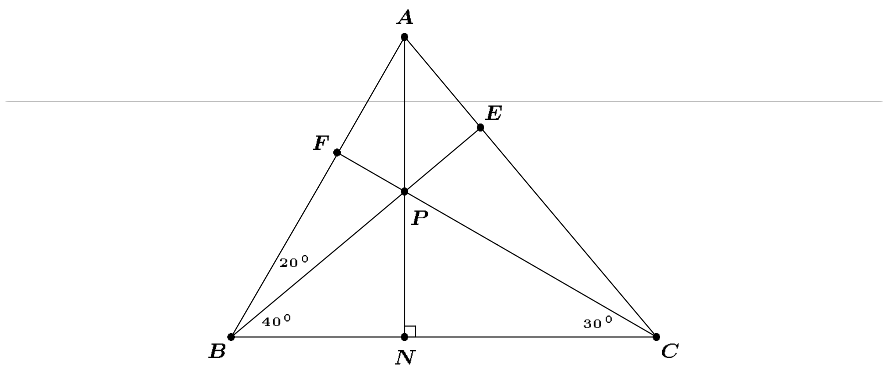
If P is a point on the altitude AN of $\triangle ABC$, if $\angle PBA = 20°$, if $\angle PBC = 40°$ and if $\angle PCB = 30°$, without using trigonometry, find $\angle PCA$.

Solution by Henri Liu, student, Trinity College Cambridge, England.

See figure on page 150

Let BP and CP meet AC and AB at the points E and F, respectively. Then $\angle BFC = 180° - 20° - 40° - 30° = 90°$, so CF is an altitude of $\triangle ABC$. Hence P is the orthocentre of $\triangle ABC$. Then $BCEF$ is a cyclic quadrilateral and we obtain $\angle PCA = \angle FCE = \angle FBE = 20°$.

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JONATHAN CAMPBELL, student, Chapel Hill High School, North Carolina, USA; NIKOLAOS DERGIADES,



Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; MICHAEL SHIAW-TIAN, Biola University, La Mirada, CA, USA; SKIDMORE COLLEGE PROBLEM GROUP, Skidmore College, New York, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; JENS WINDELBAND, Hegel-Gymnasium, Magdeburg, Germany; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Most of the submitted solutions are similar to the one given above.

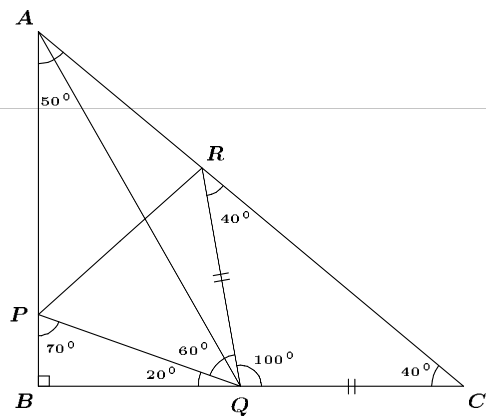
2519. [2000 : 115] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

In $\triangle ABC$, $\angle ACB = 40^\circ$, $AB \perp BC$, P and Q are points on AB and BC respectively with $\angle PQB = 20^\circ$. Without using trigonometry, prove that $AQ = 2BQ$ if and only if $PQ = CQ$.

Solution by Toshio Seimiya, Kawasaki, Japan. Since $\angle ABC = 90^\circ$ and $\angle ACB = 40^\circ$, we have $\angle BAC = 50^\circ$. Let R be the point on the ray CA for which $\angle QRC = \angle QCA = 40^\circ$. Then $QR = QC$ and $\angle RQC = 100^\circ$. Hence $\angle PQR = 60^\circ$. See figure on page 151.

(1) If $PQ = CQ$, then $PQ = CQ = QR$. Since $\angle PQR = 60^\circ$, $\triangle PQR$ is equilateral, so that $\angle RPQ = 60^\circ$. Since $\angle BPQ = 70^\circ$, we have $\angle APR = 180^\circ - \angle BPQ - \angle RPQ = 180^\circ - 70^\circ - 60^\circ = 50^\circ = \angle PAR$. Thus, we have $RA = RP = RQ$. Consequently, $\angle RAQ = \angle RQA = \frac{1}{2}\angle QRC = 20^\circ$. Then, $\angle BAQ = \angle BAC - \angle QAC = 50^\circ - 20^\circ = 30^\circ$. Therefore, $AQ = 2BQ$. Thus,

$$PQ = QC \Longrightarrow AQ = 2BQ\,.$$



(2) If $AQ = 2BQ$, then $\angle BAQ = 30°$. As a consequence, $\angle QAC = \angle BAC - \angle BAQ = 50° - 30° = 20°$. Also, $\angle RQA = \angle QRC - \angle QAR = 40° - 20° = 20°$. Hence, $RA = RQ$.

Let O be the circumcentre of $\triangle APQ$. Since $\angle APQ = 180° - \angle BPQ = 110°$, the major arc AQ is $220°$. The minor arc AQ is then $140°$, so that $\angle AOQ = 140°$. However, points P and R are in different half-planes with respect to the line AQ. Also, $\angle ARQ = 180° - \angle QRC = 140°$ and $RA = RQ$. Therefore, point R coincides with O. Thus, R is the circumcentre of $\triangle APQ$ and $RP = RQ$. Since $\angle PQR = 60°$, $\triangle PQR$ is equilateral. Hence, $PQ = QR$, and since $QR = QC$, we obtain $PQ = QC$. Thus,

$$AQ = 2BQ \Longrightarrow PQ = QC .$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRI LIU, student, Trinity College Cambridge, England; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

2521. [2000 : 116] *Proposed by Eric Postpschil, Nashua, New Hampshire, USA.*

Given a permutation τ, determine all pairs of permutations α and β, such that $\tau = \beta \circ \alpha$ and $\alpha^2 = \beta^2 = \imath$ (the identity permutation). That is, determine all factorizations of τ into two permutations, each composed of **disjoint** transpositions.

Solution by the proposer.

Define the **order** of an element i in τ to be the least positive integer m such that $\tau^m(i) = i$; that is, the length of the cycle containing i.

Construct α by selecting any two elements of the permutation, i and j, of the same order in τ and not necessarily distinct, and make these assignments to α:

$$\text{For each integer } t \text{, let } \alpha \text{ transpose } \tau^t(i) \text{ and } \tau^{-t}(j). \tag{1}$$

In particular, for $t = 0$, this gives $\alpha(i) = j$. A number of other elements are assigned transpositions with i and j, but many elements may remain unassigned. Continue the construction of α by repeating the above selection (from unassigned elements) and assignment until α is completely defined. Because the i and j of each selection are of the same order, the transpositions specified in (1) are consistent even though they are redundant when t exceeds the order of i and j. It is also clear that the transpositions defined for each selection use elements distinct from those used for other selections, so, when the construction of α is complete, its defintion is consistent and composed solely of disjoint transpositions.

When α is complete, β is determined:

$$\text{For each element } i,\ \beta(i) = \tau(\alpha(i)). \tag{2}$$

We will see that each such construction of an α and a β satisfies the conditions of the problem and that each solution takes the form of such a construction.

In (2), substitute $\alpha(i)$ for i to get

$$\beta(\alpha(i)) \ = \ \tau(\alpha(\alpha(i))) \ = \ \tau(i)\,,$$

showing that $\tau = \beta \circ \alpha$, so it remains only to show that β is composed solely of transpositions.

In (1) we may substitute 1 for t and $\alpha(i)$ for j to obtain the property $\alpha(\tau(i)) = \tau^{-1}(\alpha(i))$. We can apply $\alpha \circ \tau$ to both sides and use $\alpha^2(i) = i$ to obtain:

$$\alpha(\tau(\alpha(\tau(i)))) \ = \ i\,. \tag{3}$$

To see that β is composed of disjoint transpositions, we consider whether $\beta(\beta(i)) = i$:

$$\begin{aligned} \beta(\beta(i)) &= \tau(\alpha(\tau(\alpha(i)))), \quad \text{by two applications of (2)} \\ \alpha(\beta(\beta(i))) &= \alpha(\tau(\alpha(\tau(\alpha(i))))) \\ \alpha(\beta(\beta(i))) &= \alpha(i)\,, \quad \text{by (3)} \\ \beta(\beta(i)) &= i\,. \end{aligned}$$

Thus any α and β constructed as described give a desired factorization of τ. To see that all such factorizations are the results of such constructions, we show that $\alpha(i) = j$ requires i and j to be of the same order and determines the factorization.

Suppose $\alpha(i) = j$. This is $\alpha(\tau^t(i)) = \tau^{-1}(j)$ for $t = 0$. We assume this holds for some t and prove it holds for $t+1$. Since $\tau = \beta \circ \alpha$, $\tau(x) = \beta(\alpha(x))$. Applying β to both sides yields:

$$\beta(\tau(x)) \;=\; \alpha(x)\,. \tag{4}$$

Alternately, substituting $\alpha(\tau(\tau^t(i)))$ for x yields:

$$\begin{aligned}
\tau(\alpha(\tau(\tau^t(i)))) &= \beta(\alpha(\alpha(\tau(\tau^t(i))))) \\
\tau(\alpha(\tau(\tau^t(i)))) &= \beta(\tau(\tau^t(i))) \\
\tau(\alpha(\tau(\tau^t(i)))) &= \alpha(\tau^t(i)), \quad \text{by (4)} \\
\alpha(\tau(\tau^t(i))) &= \tau^{-1}(\alpha(\tau^t(i))), \quad \text{applying } \tau^{-1} \text{ to both sides} \\
\alpha(\tau(\tau^t(i))) &= \tau^{-1}(\tau^{-t}(j)), \quad \text{by induction hypothesis} \\
\alpha(\tau^{t+1}(i)) &= \tau^{-(t+1)}(j).
\end{aligned}$$

So induction holds, and $\alpha(\tau^t(i)) = \tau^{-t}(j)$ for all non-negative integers t. The negative integers follow similarly from $\alpha(j) = i$, itself a consequence of α's composition of transpositions. This property also demonstrates i and j have the same order: since $i = \tau^m(i)$ implies

$$j \;=\; \alpha(i) \;=\; \alpha(\tau^m(i)) \;=\; \tau^{-m}(j),$$

which implies $\tau^m(j) = j$, and the converse holds similarly.

Also solved by MICHEL BATAILLE, Rouen, France; and HENRY LIU, student, Trinity College Cambridge, UK. There was one incomplete solution.

2522*. [2000 : 116] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that a, b and c are positive real numbers. Prove that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) \;\geq\; \frac{9}{1+abc}\,.$$

Solution by Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.

Applying the Arithmetic-Geometric-Mean Inequality twice, we have

$$\begin{aligned}\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} &= \frac{\frac{1}{a}}{1+\frac{1}{a}}+\frac{\frac{1}{b}}{1+\frac{1}{b}}+\frac{\frac{1}{c}}{1+\frac{1}{c}}\\ &\geq 3\cdot\sqrt[3]{\frac{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}{(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})}}\\ &= \frac{3}{\sqrt[3]{abc}}\cdot\frac{1}{\sqrt[3]{(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})}}\\ &\geq \frac{3}{\sqrt[3]{abc}}\cdot\frac{3}{3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\,,\end{aligned}$$

and so by the Geometric-Harmonic-Mean Inequality we have

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) &\geq \frac{9}{\sqrt[3]{abc}\left(\dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}+1\right)}\\ &\geq \frac{9}{\sqrt[3]{abc}\;\left(\sqrt[3]{abc}+1\right)}\,. \qquad (1)\end{aligned}$$

We also have, for $x \in \mathbb{R}^+$, that

$$(1+x^3)-x(x+1) \;=\; (x-1)(x^2-1) \;=\; (x-1)^2(x+1) \;\geq\; 0\,,$$

whence $x(x+1) \leq 1+x^3$ with equality if and only if $x=1$. (2)

The result now follows immediately from (1) and (2) with $x=\sqrt[3]{abc}$. It is easily seen by the conditions on the preceding inequalities that equality holds if and only if $a=b=c=1$.

Also solved by the AUSTRIAN IMO–TEAM 2000, GEORGE BALOGLOU, SUNY at Oswego, NY, USA; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong; and PETER Y. WOO, Biola University, La Mirada, CA, USA. Partial or incomplete solution were submited by MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; and VEDULA N. MURTY, Visakhapatnam, India. There was also one incorrect solution.

The solution given by Woo was quite similar to the one above and was based on the following interesting lemma which can be proved easily by the AM-GM Inequality:

Lemma: For any finite set S of positive real numbers, let $\mathrm{AM}(S)$ and $\mathrm{GM}(S)$ denote the arithmetic mean and the geometric mean of the elements in S, respectively. Then for any p, q, $r>0$, we have

$$\mathrm{GM}(\{p,q,r\}) \;\leq\; \mathrm{GM}(\{1+p,1+q,1+r\})-1 \;\leq\; \mathrm{AM}(\{p,q,r\})\,.$$

2523. [2000 : 116] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove that, if $t \geq 1$, then

$$\ln t \;\leq\; \frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right)\;.$$

Also, prove that, if $0 < t \leq 1$, then

$$\ln t \;\geq\; \frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right)\;.$$

I. *Solution by Michel Bataille, Rouen, France.*

First we remark that

$$\frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right)$$

is changed into its negative when t is replaced by $1/t$ (just as $\ln t$ is). The second inequality of the problem is thus an immediate consequence of the first one which we will only consider. Thus we suppose $t > 1$ (there is equality for $t = 1$).

Letting $x = \ln t > 0$, we readily see that our inequality is successively equivalent to

$$x \;\leq\; \frac{1}{2}\sqrt{\frac{\cosh x - 1}{\cosh x + 1}}\left(1+\sqrt{4\cosh x + 5}\right)\;,$$

$$x\left(\sqrt{4\cosh x + 5} - 1\right) \;\leq\; 2\sinh x\,,$$

$$\sqrt{4\cosh x + 5} \;\leq\; 1 + \frac{2\sinh x}{x}\,,$$

and finally,

$$\cosh x + 1 \leq \frac{\sinh x}{x} + \frac{\sinh^2 x}{x^2}\,. \qquad (1)$$

Now, from the usual expressions of $\cosh x$ and $\sinh x$ as power series, we get

$$\cosh x + 1 \;=\; 2 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$$

and (using $\sinh^2 x = (\cosh 2x - 1)/2$)

$$\frac{\sinh x}{x} + \frac{\sinh^2 x}{x^2} = 2 + \sum_{n=1}^{\infty} \left(\frac{2^{2n+1}}{(2n+2)!} + \frac{1}{(2n+1)!} \right) x^{2n}.$$

Hence, to obtain (1), it suffices to prove that, for all integers $n \geq 1$:

$$\frac{1}{(2n)!} \leq \frac{2^{2n+1}}{(2n+2)!} + \frac{1}{(2n+1)!}.$$

But this is equivalent to $n(n+1)/2 \leq 2^{2n-2}$, which is clearly true for $n = 1$ and, for $n \geq 2$, results from

$$1 + 2 + 3 + \cdots + n \leq 1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1 < 2^n \leq 2^{2n-2}.$$

The proof is now complete.

II. *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*
Consider

$$f(t) = \frac{t-1}{2(t+1)} \left(1 + \sqrt{\frac{2t^2 + 5t + 2}{t}} \right) - \ln t.$$

Clearly $f(1) = 0$. Thus, to prove the proposition, it is sufficient to prove that $f'(t) \geq 0$ for $t > 0$. Now,

$$f'(t) = \frac{(t^4 + 4t^3 + 8t^2 + 4t + 1) - 2(t^2 + t + 1)\sqrt{t(2t^2 + 5t + 2)}}{2t(t+1)^2\sqrt{t(2t^2 + 5t + 2)}},$$

and

$$(t^4 + 4t^3 + 8t^2 + 4t + 1)^2 - 4(t^2 + t + 1)^2 t(2t^2 + 5t + 2) = (t^2 - 1)^4 \geq 0$$

for $t > 0$, so that the result follows.

A curious and rather magnificent result, that I could not have contemplated solving without **DERIVE**.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Bracken's proof also used power series of hyperbolic functions. Most other solutions were similar to Solution II.

2524. [2000 : 116] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

What conditions must the real numbers x, y and z satisfy so that

$$\cot x \cot y \cot z \;=\; \cot x + \cot y + \cot z\,,$$

where x, y, $z \neq n\pi$, with n being an integer?

Solution by Eckard Specht, Otto-von-Guericke University Magdeburg, Germany.

By the angle sum relationships for trigonometric functions we obtain

$$\cot(x+y+z) \;=\; \frac{\cos(x+y+z)}{\sin(x+y+z)} \;=\; \frac{\cot x\cot y\cot z - \cot x - \cot y - \cot z}{\cot y\cot z + \cot z\cot x + \cot x\cot y - 1}\,. \qquad (1)$$

Hence the equation $\cot x \cot y \cot z = \cot x + \cot y + \cot z$ is satisfied if and only if the numerator of (1) vanishes. This leads to the zeros of the cotangent function (or cosine function):

$$x+y+z \;=\; (2k+1)\frac{\pi}{2}\,, \qquad k \in \mathbb{Z}\,. \qquad (2)$$

Clearly the denominator in (1) is non-zero in this case, so that (2) is the condition we are looking for.

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOSÉ LUIS DIAZ, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most of the solutions were similar to the one given above.

2525 March [2000 : 116] . *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Consider the recursions: $x_{n+1} = 2x_n + 3y_n$, $y_{n+1} = x_n + 2y_n$, with $x_1 = 2$, $y_1 = 1$. Show that, for each integer $n \geq 1$, there is a positive integer K_n such that

$$x_{2n+1} \;=\; 2\left(K_n^2 + (K_n+1)^2\right)\,.$$

I. *Solution by Michel Bataille, Rouen, France.*
First, for all n we have:

$$\begin{aligned} x_{n+2} &= 2x_{n+1} + 3y_{n+1} \\ &= 2x_{n+1} + 3\left(x_n + 2y_n\right) \\ &= 2x_{n+1} + 3x_n + 2\left(x_{n+1} - 2x_n\right) . \end{aligned}$$

Hence

$$x_{n+2} = 4x_{n+1} - x_n . \tag{1}$$

With the help of $x_1 = 2$ and $x_2 = 7$, we classically obtain $x_n = \frac{1}{2}\left(u^n + v^n\right)$, where $u = 2 + \sqrt{3}$ and $v = 2 - \sqrt{3}$. Noticing $uv = 1$, we deduce that, for all n

$$2x_n^2 = x_{2n} + 1 \qquad \text{and} \qquad x_n x_{n+1} = \tfrac{1}{2}x_{2n+1} + 1 . \tag{2}$$

Now $x_{n+1} + x_n = 3\left(x_n + y_n\right) = 3\left(y_{n+1} - y_n\right)$, so that (by addition)

$$\begin{aligned} x_1 + 2\left(x_2 + x_3 + \cdots + x_n\right) + x_{n+1} &= 3y_{n+1} - 3y_1 \\ &= 3y_{n+1} - 3 \\ &= x_{n+2} - 2x_{n+1} - 3 , \end{aligned}$$

which, using (1), easily yields

$$x_1 + x_2 + \cdots + x_n = \tfrac{1}{2}\left(x_{n+1} - x_n - 1\right) .$$

Thus, denoting by K_n the positive integer $x_1 + x_2 + \cdots + x_n$, we get

$$\begin{aligned} K_n^2 + \left(K_n + 1\right)^2 &= \tfrac{1}{4}\left[\left(x_{n+1} - x_n - 1\right)^2 + \left(x_{n+1} - x_n + 1\right)^2\right] \\ &= \tfrac{1}{4}\left(2x_{n+1}^2 + 2x_n^2 + 2 - 4x_n x_{n+1}\right) \\ &= \tfrac{1}{4}\left(x_{2n+2} + x_{2n} - 2x_{2n+1}\right) \qquad \text{using (2)} \\ &= \tfrac{1}{2}x_{2n+1} \qquad \text{using (1).} \end{aligned}$$

The result follows.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
[*Editor's remark:* As above the solver first established that $x_n = \frac{1}{2}\left(u^n + v^n\right)$ where $u = 2 + \sqrt{3}$ and $v = 2 - \sqrt{3}$.]
Then

$$x_n = \frac{1}{2}\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{2n} + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{2n}\right]$$

for $n \in \mathbb{N}^*$. Thus, we have to verify the existence of $K_n \in \mathbb{N}^*$ such that $x_{2n+1} = 2\left(2K_n^2 + 2K_n + 1\right)$; that is

$$K_n = \frac{1}{2}\left(-1 + \sqrt{x_{2n+1} - 1}\right) .$$

Clearly, $K_n \neq 0$ for $n \in \mathbb{N}^*$; otherwise, we have $x_{2n+1} = 2$, in contradiction to the fact that x_n, $n = 1, 2, 3, \ldots$, is strictly increasing (as an easy induction will show). Now,

$$\begin{aligned} x_{2n+1} - 1 &= \frac{1}{2}\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{4n+2} - 2 + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{4n+2}\right] \\ &= \left(\frac{1}{\sqrt{2}}\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{2n+1} + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{2n+1}\right]\right)^2 \\ &= \left[\frac{1+\sqrt{3}}{2}\left(2+\sqrt{3}\right)^n + \frac{1-\sqrt{3}}{2}\left(2-\sqrt{3}\right)^n\right]^2 \\ &= w_n^2\,, \end{aligned}$$

where we have used the last line to define the number w_n. Thus, $K_n = \frac{1}{2}(-1 + w_n)$. Thus, it remains to show only that w_n is an odd integer for all $n \in \mathbb{N}^*$. Looking at the structure of w_n we obtain (note $2 \pm \sqrt{3}$ are the characteristic roots!) its recursion $w_{n+2} = 4w_{n+1} - w_n$ where $w_1 = 5$ and $w_2 = 19$. Because $w_{n+2} \equiv w_n \pmod 2$, the proof is complete.

III. *Solution by David R. Stone, Georgia Southern University, Statesboro, GA, USA.*

The defining conditions and initial values characterize (x_n, y_n) as the solutions to the Pell equation $x^2 - 3y^2 = 1$. If we set

$$X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},$$

the given recursions can be written as $X_{n+1} = PX_n$. Then the odd-subscripted terms can be obtained by left multiplication by

$$P^2 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}.$$

That is,

$$\begin{aligned} x_{2n+1} &= 7x_{2n-1} + 12y_{2n-1} \\ y_{2n+1} &= 4x_{2n-1} + 7y_{2n-1}\,. \end{aligned} \qquad (1)$$

The first few examples are:

$$\begin{aligned} x_1 &= 2 \\ x_3 &= 26 = 2(2^2 + 3^2) \\ x_5 &= 362 = 2(9^2 + 10^2) \\ x_7 &= 5042 = 2(35^2 + 36^2) \\ x_9 &= 70226 = 2(132^2 + 133^2) \\ x_{11} &= 978122 = 2(494^2 + 495^2)\,. \end{aligned}$$

Finding K_n so that $x_{2n+1} = 2\left(K_n^2 + (K_n+1)^2\right) = 2\left(2K_n^2 + 2K_n + 1\right)$ is equivalent to solving:

$$4K_n^2 + 4K_n + (2 - x_{2n+1}) \;=\; 0\,,$$

which has the solution $K_n = \frac{1}{2}\left(-1 \pm \sqrt{x_{2n+1}-1}\right)$. Thus, if we let $K_n = \frac{1}{2}\left(\sqrt{x_{2n+1}-1}\right)$, we have $x_{2n+1} = 2\left(K_n^2 + (K_n+1)^2\right)$. It remains to show that a thus-defined K_n is an integer. From (1) it is clear by induction that x_{2n+1} is even and $x_{2n+1}+1$ is a multiple of 3. Thus, the consecutive integers $x_{2n+1}-1$ and $x_{2n+1}+1$ are relatively prime. Therefore, the square y_{2n+1}^2 factors into relatively prime factors:

$$y_{2n+1}^2 \;=\; \frac{x_{2n+1}^2 - 1}{3} \;=\; (x_{2n+1}-1)\,\frac{x_{2n+1}+1}{3}\,.$$

Hence $x_{2n+1} - 1$ is an (odd) perfect square, so that K_n as defined is an integer.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; the AUSTRIAN IMO–TEAM 2000; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, KS, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Most of the solvers used an approach similar to II *above. Diminnie actually solved the recurrence relation for y_n also, and then showed that*

$$K_n \;=\; \frac{y_n + y_{n+1} - 1}{2}\,.$$

He also pointed out other interesting properties of x_n and y_n, namely:

1. $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \sqrt{3}$.
2. $x_{2n} = 6y_n^2 + 1$.
3. $y_{n+1}^2 - y_n^2 = y_{2n+1}$.