SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2505. [2000 : 46] Proposed by Hayo Ahlborg, Benidorm, Spain, and Walther Janous, Ursulengymnasium, Innsbruck, Austria.

Suppose that \( A, B \) and \( C \) are the angles of a triangle. Determine the best lower and upper bounds of \( \prod_{\text{cyclic}} \sin(B - C) \).

Combination of solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, and Kee-Wai Lau, Hong Kong.

We show that

\[
- \frac{3\sqrt{3}}{8} \leq \prod_{\text{cyclic}} \sin(B - C) \leq \frac{3\sqrt{3}}{8}.
\]

First, for the best upper bound we consider two cases.

Case 1: \( \sin(B - C) \geq 0 \). Using the identity

\[
\sin X \sin Y = \frac{1}{2} \left( \cos(X - Y) - \cos(X + Y) \right),
\]

we have

\[
\prod_{\text{cyclic}} \sin(B - C) = \frac{1}{2} \sin(B - C) \left( \cos(B + C - 2A) - \cos(C - B) \right)
\leq \frac{1}{2} \sin(B - C) \left( 1 - \cos(B - C) \right).
\]

The maximum of \( \sin(B - C)(1 - \cos(B - C)) \) occurs at the same value as for its square

\[
\sin^2(B - C)(1 - \cos(B - C))^2 = (1 + c)(1 - c)^3,
\]

where \( c = \cos(B - C) \). By the AM–GM inequality,

\[
\left[ (1 + c) \left( \frac{1 - c}{3} \right)^3 \right]^{1/4} \leq \frac{1}{4} \left[ (1 + c) + 3 \left( \frac{1 - c}{3} \right) \right] = \frac{1}{2},
\]

with equality only when all four numbers are equal; that is, when \( (1 - c)/3 = 1 + c = c = -1/2 \). Therefore,

\[
(1 + c)(1 - c)^3 \leq \left( \frac{1}{2} \right)^4 \cdot 27 = \frac{27}{16},
\]
with equality only when \( \cos(B - C) = c = -1/2 \). Thus, the upper bound will be
\[
\prod_{\text{cyclic}} \sin(B - C) \leq \frac{1}{2} \sqrt{\frac{27}{16}} = \frac{3\sqrt{3}}{8}.
\]

The degenerate triangle with \( A = \pi/3, B = 2\pi/3, C = 0 \) shows that the upper bound cannot be improved.

**Case 2:** \( \sin(B - C) < 0 \). Similarly we have
\[
\prod_{\text{cyclic}} \sin(B - C) = -\frac{1}{2} \sin(B - C) \left( \cos(C - B) - \cos(B + C - 2A) \right)
\leq -\frac{1}{2} \sin(B - C) \left( \cos(B - C) + 1 \right)
= \frac{1}{2} \sqrt{(1 - c)(1 + c)^3},
\]
with \( c \) as above. The maximum occurs when \( (1 + c)/3 = 1 - c \) or when \( \cos(B - C) = c = 1/2 \). This gives the same upper bound as before, but here there are no corresponding angles of a triangle.

For the lower bound, we get
\[
\prod_{\text{cyclic}} \sin(B - C) \geq -\frac{3\sqrt{3}}{8}.
\]

*Editorial note: The easiest way to see this, as some readers noted, is because
\[
\prod_{\text{cyclic}} \sin(B - C) = -\prod_{\text{cyclic}} \sin(C - B) \geq -\frac{3\sqrt{3}}{8},
\]
using the upper bound.] The degenerate triangle with \( A = 2\pi/3, B = \pi/3, C = 0 \) shows that the lower bound cannot be improved either.

*Editorial note: Klamkin observes that these bounds hold for arbitrary angles \( A, B, C \), as the above proof (which is largely Klamkin's, but uses some elements of Lau's solution as well) shows. Thus readers might like to investigate the best bounds, in terms of \( n \), for
\[
\prod_{\text{cyclic}} \sin(A_1 - A_2),
\]
where \( A_1, A_2, \ldots, A_n \) are arbitrary angles.

Also solved by AUSTRIAN IMO-TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I. B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HENRY LIU, student, University of Cambridge, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol,
Most readers obtained strict inequality in both bounds, which is correct if degenerate triangles are not allowed.


In \(\triangle ABC\), the sides satisfy \(a \geq b \geq c\). Let \(R\) and \(r\) be the circumradius and the inradius respectively. Prove that

\[
bc \leq 6Rr \leq a^2,
\]

with equality if and only if \(a = b = c\).

Solution by Henri Liu, graduate student, University of Cambridge, UK.

Let \(A\) be the area of \(\triangle ABC\). Combining the well-known identities \(4AR = abc\) and \(2A = r(a + b + c)\), we have

\[
6Rr = \frac{3abc}{a + b + c}.
\]

Using \(a + b + c \leq 3a\), we obtain

\[
6Rr \geq \frac{3abc}{3a} = bc.
\]

Equality holds if and only if \(a = b = c\). Using \(a + b + c \geq 3c\), we get

\[
6Rr \leq \frac{3abc}{3c} = ab \leq a^2
\]

with equality if and only if \(a = b = c\).

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO-TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNANDEZ, i. B. Praedexes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JONATHAN CAMPBELL, student, Chapel Hill High School, North Carolina, USA; CORAN CONAR, student, University of Zagreb, Croatia; NIKOLAOS DERGIADIES, Thessaloniki, Greece; ROMAN FRESNEDA, Universidad de la Habana, Cuba; KARTHIK GOPALRATNAM, student, Angelo State University, Texas, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; HOJOON LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY PAN, student, East York C.I., Toronto, Ontario; COTTIFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIRO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; M. JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. Most of the submitted solutions are similar to the one given above.
2514. [2000:] Proposed by Toshio Seimiya, Kawasaki, Japan.

In \( \triangle ABC \), the internal bisectors of \( \angle ABC \) and \( \angle BCA \) meet \( CA \) and \( AB \) at \( D \) and \( E \) respectively. Suppose that \( AE = BD \) and that \( AD = CE \). Characterize \( \triangle ABC \).

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

As is known,

\[
AE^2 = \left( \frac{bc}{b + a} \right)^2, \quad AD^2 = \left( \frac{bc}{c + a} \right)^2,
\]

\[
BD^2 = ca \left( 1 - \frac{b^2}{(c + a)^2} \right), \quad CE^2 = ba \left( 1 - \frac{c^2}{(b + a)^2} \right),
\]

so that \( \left( \frac{AE}{AD} \right)^2 = \left( \frac{BD}{CE} \right)^2 \) reduces to

\[
b(c + a)^4 ((b + a)^2 - c^2) = c(b + a)^4 ((c + a)^2 - b^2),
\]

or successively to

\[
(c + a)^2(b + a)^2 ((b + a)^2 - c(b + a)^2) = bc(c(c + a)^4 - b(b + a)^4),
\]

or

\[
0 = (b - c) ((c + a)^2(b + a)^2a^2)
\]

\[
+bc(b(b + a)^4 - c(c + a)^4 - (b - c)(c + a)^2(b + a)^2).
\]

The second term also has a factor \( (b - c) \). When this is factored out, there remains a polynomial all of whose coefficients are positive. [Ed. I checked — it has 19 terms!] Hence, \( b = c \), and the triangle is isosceles.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DEGIADES, Thessaloniki, Greece; WALTER JANOUS, Ursulinen Gymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

All of the other solvers, with one exception, also showed that the base angles of the triangle are \( 36^\circ \).

In \( \triangle ABC \), the internal bisectors of \( \angle BAC \), \( \angle ABC \) and \( \angle BCA \) meet \( BC \), \( AC \) and \( AB \) at \( D \), \( E \) and \( F \) respectively. Let \( p \) be the perimeter of \( \triangle ABC \). Suppose that \( AF + BD + CE = \frac{1}{2}p \). Characterize \( \triangle ABC \).

1. Solution by David Loeffler, student, Cotham School, Bristol, UK.

We show first that the angle bisector of a triangle divides the opposite side in the ratio of the other two sides. By the Law of Sines on \( \triangle ABD \), we have

\[
BD = c \frac{\sin \left( \frac{1}{2} \angle BAC \right)}{\sin(\angle ADB)}.
\]

Similarly, from \( \triangle ADC \) we get

\[
DC = b \frac{\sin \left( \frac{1}{2} \angle BAC \right)}{\sin(\angle ADC)}.
\]

Thus,

\[
\frac{BD}{DC} = c \frac{\sin(\angle ADC)}{b \sin(180^\circ - \angle ADC)} = \frac{c}{b}.
\]

Likewise, we have \( \frac{CE}{EA} = \frac{a}{b} \) and \( \frac{AF}{FB} = \frac{b}{a} \). Thus, we have

\[
AF = \frac{bc}{a+b}, \quad BD = \frac{ca}{b+c}, \quad CE = \frac{ab}{c+a}.
\]

The condition that \( AF + BD + CE = \frac{1}{2}p \) is now equivalent to

\[
\frac{2bc}{a+b} + \frac{2ca}{b+c} + \frac{2ab}{c+a} = a + b + c,
\]

or, upon multiplying by \((a+b)(b+c)(c+a)\),

\[
2bc(b+c)(c+a) + 2ca(c+a)(a+b) + 2ab(a+b)(b+c) = (a+b+c)(a+b)(b+c)(c+a).
\]

After expanding, most of the terms cancel and the equation becomes

\[
ab^3 + bc^3 + ca^3 = a^3b + b^3c + c^3a,
\]

which factors as \((a+b+c)(a-b)(b-c)(c-a) = 0\). Thus \(a = b\), or \(b = c\), or \(c = a\); that is, the triangle is isosceles.
II. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editors.

Without loss of generality, assume \( A \leq B \leq C \). Let \( r \) be the inradius, and \( I \) the incentre. Let the projections of \( I \) upon \( BC, CA, AB \) be \( D', E', F' \) respectively as shown in the diagram above. Let \( s = \frac{1}{2}(a + b + c) \) where \( a = BC, b = CA, c = AB \). Then

\[
\angle DID' = \angle ADB - \frac{\pi}{2} = \left( A\frac{1}{2} + C - \frac{A + B + C}{2} \right) = \frac{C - B}{2}.
\]

Similarly, \( \angle EIE' = (C - A)/2 \) and \( \angle FIF' = (B - A)/2 \).

Notice that \( AF' + BD' + CE' = (s - a) + (s - b) + (s - c) = s = AF + BD + CE \). Hence \( EE' = FF' + DD' \); that is,

\[
r \tan \frac{1}{2}(B - A) + r \tan \frac{1}{2}(C - B) = r \tan \frac{1}{2}(C - A).
\]

Let \( \alpha = (B - A)/2 \) and \( \beta = (C - B)/2 \). The above equation becomes:

\[
\tan \alpha + \tan \beta = \frac{\tan(\alpha + \beta)}{1 - \tan \alpha \tan \beta}.
\]

Therefore, we have \( \tan \alpha = -\tan \beta \) or \( \tan \alpha \tan \beta = 0 \). The first case leads to \( \alpha = -\beta \) and thus \( A = C \); the second case yields either \( A = B \) or \( B = C \). Therefore, \( \triangle ABC \) is isosceles. Conversely, if \( \triangle ABC \) is isosceles, it is trivial to see that \( AF + BD + CE = s \).

Also solved by MIGUEL AMENGUAL COVAS, Cala Figueras, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinegymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HOJOO LEE, student, Kwangwoon University, Kwangwon-Do, South Korea; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer. There were two incomplete and two incorrect solutions.

In isosceles \( \triangle ABC \) (with \( AB = AC \)), let \( D \) and \( E \) be points on sides \( AB \) and \( AC \) respectively such that \( AD < AE \). Suppose that \( BE \) and \( CD \) meet at \( P \). Prove that \( AE + EP < AD + DP \).

1. Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let \( Ax \) and \( Ay \) be the half-lines starting at \( A \) and containing \( B \) and \( C \) respectively, and let \( Az \) be the bisector of \( \angle Ax Ay \). The bisector of \( \angle DCy \) meets \( Az \) at \( K \), which is therefore an excentre of \( \triangle ADC \); the bisector of \( \angle EBx \) meets \( Az \) at \( L \), an excentre of \( \triangle ABE \). We have

\[
\frac{1}{2} \angle EBx < \frac{1}{2} \angle DCy = \angle KCy,
\]

and hence,

\[
AL > AK.
\]

If \( F \) on \( AB \) is the symmetric point of \( P \) with respect to the angle bisector \( DK \), and \( G \) on \( AC \) is the symmetric point of \( P \) with respect to the angle bisector \( EL \), then we have

\[
DP = DF, \quad \text{or} \quad AF = AD + DP, \quad \text{and} \quad EP = EG, \quad \text{or} \quad AG = AE + EP.
\]

Since \( K \) is on the same side of \( EL \) as \( P \), then \( KG > KP = KF \). Finally, from the fact that \( K \) and \( A \) lie on opposite sides of the line \( EF \), we have that \( AF > AG \), or \( AD + DP > AE + EP \).

II. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

We shall generalize the problem using Green's Theorem with some basic calculus. I hope that the solution has a sufficient degree of elegance to befit the celebration of the 90th birthday of Prof. Seimiya.

Theorem. Let \( CD, BE \) be cevians of a triangle \( ABC \) where \( BD \geq CE \). Let \( CD \) intersect \( BE \) at \( P \). Then \( AD + DP > AE + EP \).

Remark. The given problem is a corollary: Let \( A' \) be on \( AC \) such that \( AA' = AD \). Let \( BA' \) intersect \( CD \) at \( D' \). Then

\[
AD + DD' = AA' + A'D'
\]

by symmetry, and

\[
A'D' + D'P > A'E + EP
\]

by the theorem. Adding the two expressions then gives the result.

Proof of the theorem. Let \( x_1 = \angle BCD, x_2 = \angle BCA, y_1 = \angle CBE, y_2 = \angle CBA \). Assume \( BC = 1 \) unit. For any point \( X \) in the plane of \( \triangle ABC \) let \( \angle XCB = x \) and \( \angle CBX = y \). The point is inside quadrangle \( ADPE \) when both \( x_1 < x < x_2 \) and \( y_1 < y < y_2 \). When \( X \) is on \( AD \), if \( x \) is
increased by $dx$, $X$ would move along $AD$ from $D$ to $A$ through $ds$. From the Law of Sines,

$$BX = \frac{BC \sin x}{\sin(x + y)}.$$

Thus

$$\frac{\partial s}{\partial x} = \frac{\sin(x + y) \cos x - \sin x \cos(x + y)}{\sin^2(x + y)} = \frac{\sin y}{\sin^2(x + y)}.$$

Consequently,

$$DA = \int_{x_1}^{x_2} \frac{\sin y_2}{\sin^2(x + y_2)} dx = \text{line integral } \int_{DA} \frac{\sin y dx + \sin x dy}{\sin^2(x + y)}.$$

Similar results hold for $DP, PE, EA$, so that

$$AE + EP - PD - DA = \text{line integral } \int_{ADPE} \frac{\sin y dx + \sin x dy}{\sin^2(x + y)}.$$

By Green’s Theorem, this equals

$$\int \int \left[ \frac{\partial}{\partial x} \frac{\sin x}{\sin^2(x + y)} - \frac{\partial}{\partial y} \frac{\sin y}{\sin^2(x + y)} \right] dx dy$$

over the interior of $ADPE$.

Let $\alpha = y + x$ and $\beta = y - x$. Then the integrand simplifies to

$$\sin^{-3} \alpha \left[ \sin \alpha \cos x - 2 \sin x \cos \alpha - \sin \alpha \cos y + 2 \sin y \cos \alpha \right]$$

$$= \sin^{-3} \alpha \left[ \sin \alpha (\cos x - \cos y) + 2 \cos \alpha (\sin y - \sin x) \right]$$

$$= \sin^{-3} \alpha \left[ \sin \alpha \left( 2 \sin \left( \frac{\alpha}{2} \right) \sin \left( \frac{\beta}{2} \right) \right) + 4 \cos \alpha \cos \left( \frac{\alpha}{2} \right) \sin \left( \frac{\beta}{2} \right) \right]$$

$$= \sin^{-3} \alpha \sin \left( \frac{\beta}{2} \right) \left[ \cos \left( \frac{\alpha}{2} \right) - \cos \left( \frac{3\alpha}{2} \right) + 2 \cos \left( \frac{3\alpha}{2} \right) + 2 \cos \left( \frac{\alpha}{2} \right) \right]$$

$$= \sin^{-3} \alpha \sin \left( \frac{\beta}{2} \right) \left[ 3 \cos \left( \frac{\alpha}{2} \right) + \cos \left( \frac{3\alpha}{2} \right) \right] .$$

With a bit of calculus, it can be seen that $3 \cos \left( \frac{\alpha}{2} \right) + \cos \left( \frac{3\alpha}{2} \right)$ is a decreasing function for $0 \leq \alpha \leq \pi$, reaching 0 only when $\alpha = \pi$. This proves that the integrand is positive or negative according as $\beta$ (that is, $y - x$) is positive or negative; so too with the double integral and therefore also the line integral. The theorem specifies that $y_2 \leq x_1$, so that $y \leq x$ and the line integral is negative. Thus $AE + EP - PD - DA < 0$.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain (two solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HENRY LIU, student.
Given a convex quadrangle $ABCD$ whose sides $AB$ and $CD$ meet at $E$ while $AD$ and $BC$ meet at $F$, labeled so that $B$ is between $A$ and $E$, $C$ is between $E$ and $D$ and also between $F$ and $B$, while $D$ is between $F$ and $A$, then $AB + BC = AD + DC$ if and only if $AE + EC = AF + FC$, if and only if the productions of the four sides of $ABCD$ are tangent to a circle.


2517. [2000:115] Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

Suppose that $D$, $E$, $F$ are the mid-points of the sides $BC$, $CA$, $AB$, respectively, of $\triangle ABC$. Let $P$ be any point in the plane of the triangle, distinct from $A$, $B$ and $C$.

1. Show that the lines parallel to $AP$, $BP$, $CP$, through $D$, $E$, $F$, respectively, are concurrent (at $Q$, say).

2. If $X$, $Y$, $Z$ are the symmetries of $P$ with respect to $D$, $E$, $F$, respectively, show that $AX$, $BY$, $CZ$ are concurrent at $Q$.

I. Solution by Nikolaos Dergiades, Thessaloniki, Greece.

[Ed.: The most elementary solution was given by Nikolaos Dergiades.]

Since $PBXC$ and $PCYA$ are parallelograms, it follows that $ABXY$ is a parallelogram, and hence, $BY$ passes through the mid-point $Q$ of $AX$. Similarly, $CZ$ passes through $Q$. Hence, $AX$, $BY$, $CZ$ are concurrent at $Q$. In $\triangle APX$, note that $D$ is the mid-point of $PX$, and that $Q$ is the mid-point of $AX$. Therefore, $DQ$ is parallel to $AP$. Likewise, $EQ$ is parallel to $BP$, and $FQ$ is parallel to $CP$.

II. Solution by Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland.

[Ed.: A number of solvers used a transformational argument based on properties of homotheties. However, no-one else saw Gunther’s short solution presented below.]
Denote by $H(R, \lambda)$ the homothety (central dilation) with centre $R$ and ratio $\lambda$. Note that the composition $H(R, \lambda) \circ H(S, \mu) = H(T, \lambda\mu)$, where the point $T$ lies on the line $RS$ (in the case where $\lambda\mu = 1$, this is a translation when $T$ lies at infinity and the identity when $R = S = T$).

In $\triangle ABC$, let $G$ be the centroid. Observe first that $H(G, -\frac{1}{2})$ takes $\triangle ABC$ to $\triangle DEF$. Since a homothety takes a line to a parallel line, it follows that $AP, BP, CP$ are taken to the respectively parallel lines $DQ, EQ, FQ$, where $Q$ is the image of $P$ under $H(G, -\frac{1}{2})$. Next note that $H(P, 2)$ takes $\triangle DEF$ to $\triangle XYZ$. Hence, the composition $H(P, 2) \circ H(G, -\frac{1}{2})$ is a homothety which takes $\triangle ABC$ to $\triangle XYZ$, and whose centre lies on the line $PG$. It is easy to verify that this composition has $Q$ as a fixed point, and hence $H(P, 2) \circ H(G, -\frac{1}{2}) = H(Q, -1)$. Thus $AX, BY, CZ$ are all concurrent with $Q$.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Práxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL COVAS, Mallorca, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKEN, University of Alberta, Edmonton, Alberta; HOJOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; TOSHIO SEIIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

A number of solutions employed vectors; others used coordinates (both affine and areal). Klamkin noted that a generalization of part (1) is given in Problem 39, Math. Horizons, April 1996, by T.S. Bolis and M. Klamkin: "Let $n + 1$ points be given on a sphere. From the centroid of any $n$ of these points a line is drawn normal to the tangent plane to the sphere at the remaining point. Prove that all of these $n + 1$ lines are concurrent." In the solution to this problem, it was noted that (1) had appeared on a Cambridge Scholarship Examination. As well, Benito and Fernández noted that a problem similar to (2) is exercise #196 of the Lecons de Géométrie Élémentaire, vol. 1 by J. Hadamard.

2518. [2000 : 115] Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.

If $P$ is a point on the altitude $AN$ of $\triangle ABC$, if $\angle PBA = 20^\circ$, if $\angle PBC = 40^\circ$ and if $\angle PCB = 30^\circ$, without using trigonometry, find $\angle PCA$.

Solution by Henri Liu, student, Trinity College Cambridge, England.

See figure on page 150

Let $BP$ and $CP$ meet $AC$ and $AB$ at the points $E$ and $F$, respectively. Then $\angle BFC = 180^\circ - 20^\circ - 40^\circ - 30^\circ = 90^\circ$, so $CF$ is an altitude of $\triangle ABC$. Hence $P$ is the orthocentre of $\triangle ABC$. Then $BCEF$ is a cyclic quadrilateral and we obtain $\angle PCA = \angle FCE = \angle FBE = 20^\circ$.

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO-TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Práxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JONATHAN CAMPBELL, student, Chapel Hill High School, North Carolina, USA; NIKOLAOS DERGIADIS,
Most of the submitted solutions are similar to the one given above.

2519. [2000 : 115] Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.

In \( \triangle ABC \), \( \angle ACB = 40^\circ \), \( AB \perp BC \), \( P \) and \( Q \) are points on \( AB \) and \( BC \) respectively with \( \angle PQB = 20^\circ \). Without using trigonometry, prove that \( AQ = 2BQ \) if and only if \( PQ = CQ \).

Solution by Toshio Seimiyaa, Kawasaki, Japan. Since \( \angle ABC = 90^\circ \) and \( \angle ACB = 40^\circ \), we have \( \angle BAC = 50^\circ \). Let \( R \) be the point on the ray \( CA \) for which \( \angle QRC = \angle QCA = 40^\circ \). Then \( QR = QC \) and \( \angle RQC = 100^\circ \). Hence \( \angle PQR = 60^\circ \). See figure on page 151.

(1) If \( PQ = CQ \), then \( PQ = CQ = QR \). Since \( \angle PQR = 60^\circ \), \( \triangle PQR \) is equilateral, so that \( \angle RPQ = 60^\circ \). Since \( \angle BPQ = 70^\circ \), we have \( \angle APR = 180^\circ - \angle BPQ - \angle RPQ = 180^\circ - 70^\circ - 60^\circ = 50^\circ = \angle PAR \).

Thus, we have \( RA = RP = RQ \). Consequently, \( \angle RAQ = \frac{1}{2} \angle QRC = 20^\circ \). Then, \( \angle BAQ = \angle BAC - \angle QAC = 50^\circ - 20^\circ = 30^\circ \). Therefore, \( AQ = 2BQ \). Thus,

\[
PQ = QC \implies AQ = 2BQ.
\]
(2) If $AQ = 2BQ$, then $\angle BAQ = 30^\circ$. As a consequence, $\angle QAC = \angle BAC - \angle BAQ = 50^\circ - 30^\circ = 20^\circ$. Also, $\angle RQA = \angle QRC - \angle QAR = 40^\circ - 20^\circ = 20^\circ$. Hence, $RA = RQ$.

Let $O$ be the circumcentre of $\triangle APQ$. Since $\angle APQ = 180^\circ - \angle BPQ = 110^\circ$, the major arc $AQ$ is $220^\circ$. The minor arc $AQ$ is then $140^\circ$, so that $\angle AOQ = 140^\circ$. However, points $P$ and $R$ are in different half-planes with respect to the line $AQ$. Also, $\angle ARQ = 180^\circ - \angle QRC = 140^\circ$ and $RA = RQ$. Therefore, point $R$ coincides with $O$. Thus, $R$ is the circumcentre of $\triangle APQ$ and $RP = RQ$. Since $\angle PQR = 60^\circ$, $\triangle PQR$ is equilateral. Hence, $PQ = QR$, and since $QR = QC$, we obtain $PQ = QC$. Thus,

$$AQ = 2BQ \implies PQ = QC.$$


Given a permutation $\tau$, determine all pairs of permutations $\alpha$ and $\beta$, such that $\tau = \beta \circ \alpha$ and $\alpha^2 = \beta^2 = \iota$ (the identity permutation). That is, determine all factorizations of $\tau$ into two permutations, each composed of disjoint transpositions.
Solution by the proposer.

Define the order of an element \( i \) in \( \tau \) to be the least positive integer \( m \) such that \( \tau^m(i) = i \); that is, the length of the cycle containing \( i \).

Construct \( \alpha \) by selecting any two elements of the permutation, \( i \) and \( j \), of the same order in \( \tau \) and not necessarily distinct, and make these assignments to \( \alpha \):

For each integer \( t \), let \( \alpha \) transpose \( \tau^t(i) \) and \( \tau^{-t}(j) \). \hfill (1)

In particular, for \( t = 0 \), this gives \( \alpha(i) = j \). A number of other elements are assigned transpositions with \( i \) and \( j \), but many elements may remain unassigned. Continue the construction of \( \alpha \) by repeating the above selection (from unassigned elements) and assignment until \( \alpha \) is completely defined. Because the \( i \) and \( j \) of each selection are of the same order, the transpositions specified in (1) are consistent even though they are redundant when \( t \) exceeds the order of \( i \) and \( j \). It is also clear that the transpositions defined for each selection use elements distinct from those used for other selections, so, when the construction of \( \alpha \) is complete, its definition is consistent and composed solely of disjoint transpositions.

When \( \alpha \) is complete, \( \beta \) is determined:

For each element \( i \), \( \beta(i) = \tau(\alpha(i)) \). \hfill (2)

We will see that each such construction of an \( \alpha \) and a \( \beta \) satisfies the conditions of the problem and that each solution takes the form of such a construction.

In (2), substitute \( \alpha(i) \) for \( i \) to get

\[
\beta(\alpha(i)) = \tau(\alpha(\alpha(i))) = \tau(i),
\]

showing that \( \tau = \beta \circ \alpha \), so it remains only to show that \( \beta \) is composed solely of transpositions.

In (1) we may substitute 1 for \( t \) and \( \alpha(i) \) for \( j \) to obtain the property \( \alpha(\tau(i)) = \tau^{-1}(\alpha(i)) \). We can apply \( \alpha \circ \tau \) to both sides and use \( \alpha^2(i) = i \) to obtain:

\[
\alpha(\tau(\alpha(\tau(i)))) = i. \hfill (3)
\]

To see that \( \beta \) is composed of disjoint transpositions, we consider whether \( \beta(\beta(i)) = i \):

\[
\beta(\beta(i)) = \tau(\alpha(\tau(\alpha(i)))), \quad \text{by two applications of (2)}
\]
\[
\alpha(\beta(\beta(i))) = \alpha(\tau(\alpha(\tau(\alpha(i))))),
\]
\[
\alpha(\beta(\beta(i))) = \alpha(i), \quad \text{by (3)}
\]
\[
\beta(\beta(i)) = i.
\]

Thus any \( \alpha \) and \( \beta \) constructed as described give a desired factorization of \( \tau \). To see that all such factorizations are the results of such constructions, we
show that $\alpha(i) = j$ requires $i$ and $j$ to be of the same order and determines the factorization.

Suppose $\alpha(i) = j$. This is $\alpha(\tau^t(i)) = \tau^{-1}(j)$ for $t = 0$. We assume this holds for some $t$ and prove it holds for $t+1$. Since $\tau = \beta \circ \alpha$, $\tau(x) = \beta(\alpha(x))$. Applying $\beta$ to both sides yields:

$$\beta(\tau(x)) = \alpha(x). \quad (4)$$

Alternately, substituting $\alpha(\tau(t^i(i)))$ for $x$ yields:

\[
\begin{align*}
\tau(\alpha(\tau(t^i(i)))) &= \beta(\alpha(\alpha(\tau(t^i(i))))), \\
\tau(\alpha(\tau(t^i(i)))) &= \beta(\alpha(\tau(t^i(i)))), \\
\tau(\alpha(\tau(t^i(i)))) &= \alpha(\tau(t^i(i))), \quad \text{by (4)} \\
\alpha(\tau(t^i(i))) &= \tau^{-1}(\alpha(\tau(t^i(i))), \quad \text{applying } \tau^{-1} \text{ to both sides} \\
\alpha(\tau(t^i(i))) &= \tau^{-1}(\tau^{-t}(j)), \quad \text{by induction hypothesis} \\
\alpha(\tau(t^{t+1}(i))) &= \tau^{-t}(j).
\end{align*}
\]

So induction holds, and $\alpha(\tau(t^i(i))) = \tau^{-t}(j)$ for all non-negative integers $t$. The negative integers follow similarly from $\alpha(j) = i$, itself a consequence of $\alpha$'s composition of transpositions. This property also demonstrates $i$ and $j$ have the same order: since $i = \tau^m(i)$ implies

$$j = \alpha(i) = \alpha(\tau^m(i)) = \tau^{-m}(j),$$

which implies $\tau^m(j) = j$, and the converse holds similarly.

Also solved by MICHEL BATAILLE, Rouen, France; and HENRY LIU, student, Trinity College Cambridge, UK. There was one incomplete solution.

---


Suppose that $a$, $b$ and $c$ are positive real numbers. Prove that

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{1 + a} + \frac{1}{1 + b} + \frac{1}{1 + c}\right) \geq \frac{9}{1 + abc}.$$
Solution by Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.

Applying the Arithmetic-Geometric-Mean Inequality twice, we have

\[
\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \\
\geq 3 \cdot \sqrt[3]{\frac{1}{1+a} \cdot \frac{1}{1+b} \cdot \frac{1}{1+c}} \\
= \frac{3 \cdot 1}{\sqrt[3]{1+1/\sqrt{a}} \cdot 1+1/\sqrt{b}} \\
\geq \frac{3}{\sqrt[3]{abc} \cdot 3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}},
\]

and so by the Geometric-Harmonic-Mean Inequality we have

\[
\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right) \geq \frac{9}{3 \sqrt[3]{abc} \cdot \sqrt[3]{\frac{1}{a+1/\sqrt{a}} \cdot \frac{1}{b+1/\sqrt{b}} \cdot \frac{1}{c+1/\sqrt{c}} + 1}} \\
\geq \frac{9}{\sqrt[3]{abc}} \left(\sqrt[3]{abc} + 1\right).
\]

We also have, for \(x \in \mathbb{R}^+\), that

\[
(1 + x^3) - x(x+1) = (x-1)(x^2 - 1) = (x-1)^2(x+1) \geq 0,
\]

whence \(x(x+1) \leq 1 + x^3\) with equality if and only if \(x = 1\).

The result now follows immediately from (1) and (2) with \(x = \sqrt[3]{abc}\). It is easily seen by the conditions on the preceding inequalities that equality holds if and only if \(a = b = c = 1\).

Also solved by the Austrian IMO-Team 2000, GEORGE BALOGLOU, SUNY at Oswego, NY, USA; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Práxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADIS, Thessaloniki, Greece; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; RICHARD J. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong; and PETER Y. WOO, Biola University, La Mirada, CA, USA. Partial or incomplete solution were submitted by MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; and VEDULA N. MURTY, Visakhapatnam, India. There was also one incorrect solution.

The solution given by Woo was quite similar to the one above and was based on the following interesting lemma which can be proved easily by the AM-GM inequality:

Lemma: For any finite set \(S\) of positive real numbers, let \(\text{AM}(S)\) and \(\text{GM}(S)\) denote the arithmetic mean and the geometric mean of the elements in \(S\), respectively. Then for any \(p, q, r > 0\), we have

\[
\text{GM}\{p, q, r\} \leq \text{GM}\{1+p, 1+q, 1+r\} - 1 \leq \text{AM}\{p, q, r\}.
\]

Prove that, if \( t \geq 1 \), then

\[
\ln t \leq \frac{t - 1}{2(t + 1)} \left( 1 + \sqrt{\frac{2t^2 + 5t + 2}{t}} \right).
\]

Also, prove that, if \( 0 < t \leq 1 \), then

\[
\ln t \geq \frac{t - 1}{2(t + 1)} \left( 1 + \sqrt{\frac{2t^2 + 5t + 2}{t}} \right).
\]

1. Solution by Michel Bataille, Rouen, France.

First we remark that

\[
\frac{t - 1}{2(t + 1)} \left( 1 + \sqrt{\frac{2t^2 + 5t + 2}{t}} \right)
\]

is changed into its negative when \( t \) is replaced by \( 1/t \) (just as \( \ln t \) is). The second inequality of the problem is thus an immediate consequence of the first one which we will only consider. Thus we suppose \( t > 1 \) (there is equality for \( t = 1 \)).

Letting \( x = \ln t > 0 \), we readily see that our inequality is successively equivalent to

\[
x \leq \frac{1}{2} \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \left( 1 + \sqrt{4 \cosh x + 5} \right),
\]

\[
x \left( \sqrt{4 \cosh x + 5} - 1 \right) \leq 2 \sinh x,
\]

\[
\sqrt{4 \cosh x + 5} \leq 1 + \frac{2 \sinh x}{x},
\]

and finally,

\[
\cosh x + 1 \leq \frac{\sinh x}{x} + \frac{\sinh^2 x}{x^2}.
\]

(1)

Now, from the usual expressions of \( \cosh x \) and \( \sinh x \) as power series, we get

\[
\cosh x + 1 = 2 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}
\]
and (using $\text{sinh}^2 x = (\cosh 2x - 1)/2$)

$$\frac{\text{sinh} x}{x} + \frac{\text{sinh}^2 x}{x^2} = 2 + \sum_{n=1}^{\infty} \left( \frac{2^{2n+1}}{(2n + 2)!} + \frac{1}{(2n + 1)!} \right) x^{2n}. $$

Hence, to obtain (1), it suffices to prove that, for all integers $n \geq 1$:

$$\frac{1}{(2n)!} \leq \frac{2^{2n+1}}{(2n + 2)!} + \frac{1}{(2n + 1)!}. $$

But this is equivalent to $n(n + 1)/2 \leq 2^{2n-2}$, which is clearly true for $n = 1$ and, for $n \geq 2$, results from

$$1 + 2 + 3 + \cdots + n \leq 1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1 < 2^n \leq 2^{2n-2}. $$

The proof is now complete.

**II. Solution by Christopher J. Bradley, Clifton College, Bristol, UK.**

Consider

$$f(t) = \frac{t - 1}{2(t + 1)} \left( 1 + \sqrt{\frac{2t^2 + 5t + 2}{t}} \right) - \ln t. $$

Clearly $f(1) = 0$. Thus, to prove the proposition, it is sufficient to prove that $f'(t) \geq 0$ for $t > 0$. Now,

$$f'(t) = \frac{(t^4 + 4t^3 + 8t^2 + 4t + 1) - 2(t^2 + t + 1)\sqrt{t(2t^2 + 5t + 2)}}{2t(t + 1)^2 \sqrt{t(2t^2 + 5t + 2)}}, $$

and

$$(t^4 + 4t^3 + 8t^2 + 4t + 1)^2 - 4(t^2 + t + 1)^2 t(2t^2 + 5t + 2) = (t^2 - 1)^4 \geq 0$$

for $t > 0$, so that the result follows.

A curious and rather magnificent result, that I could not have contemplated solving without DERIVE.

**Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec, Canada; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Visakhapatnam, India; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.**

Bracken’s proof also used power series of hyperbolic functions. Most other solutions were similar to Solution II.
What conditions must the real numbers $x$, $y$ and $z$ satisfy so that
$$\cot x \cot y \cot z = \cot x + \cot y + \cot z,$$
where $x$, $y$, $z \neq n\pi$, with $n$ being an integer?

Solution by Eckard Specht, Otto-von-Guericke University Magdeburg, Germany.

By the angle sum relationships for trigonometric functions we obtain
$$\cot(x + y + z) = \frac{\cos(x + y + z)}{\sin(x + y + z)} = \frac{\cot x \cot y \cot z - \cot x - \cot y - \cot z}{\cot y \cot z + \cot z \cot x + \cot x \cot y - 1}. \quad (1)$$

Hence the equation $\cot x \cot y \cot z = \cot x + \cot y + \cot z$ is satisfied if and only if the numerator of (1) vanishes. This leads to the zeros of the cotangent function (or cosine function): $$x + y + z = (2k + 1)\frac{\pi}{2}, \quad k \in \mathbb{Z}. \quad (2)$$

Clearly the denominator in (1) is non-zero in this case, so that (2) is the condition we are looking for.

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEKFET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO-TEAM 2000; MICHEL BATTLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Cifton College, Bristol, UK; NIKOLAOS DERGIADIES, Thessaloniki, Greece; JOSÉ LUIS DÍAZ, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUŠ, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Most of the solutions were similar to the one given above.

### 2525

March [2000 : 116]. Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.

Consider the recursions: $x_{n+1} = 2x_n + 3y_n$, $y_{n+1} = x_n + 2y_n$, with $x_1 = 2$, $y_1 = 1$. Show that, for each integer $n \geq 1$, there is a positive integer $K_n$ such that
$$x_{2n+1} = 2(K_n^2 + (K_n + 1)^2).$$
I. Solution by Michel Bataille, Rouen, France.
First, for all $n$ we have:
\[
x_{n+2} = 2x_{n+1} + 3y_{n+1} \\
= 2x_{n+1} + 3(x_n + 2y_n) \\
= 2x_{n+1} + 3x_n + 2(x_{n+1} - 2x_n).
\]
Hence
\[
x_{n+2} = 4x_{n+1} - x_n. \tag{1}
\]
With the help of $x_1 = 2$ and $x_2 = 7$, we classically obtain $x_n = \frac{1}{2} (u^n + v^n)$, where $u = 2 + \sqrt{3}$ and $v = 2 - \sqrt{3}$. Noticing $uv = 1$, we deduce that, for all $n$
\[
2x_n^2 = x_{2n} + 1 \quad \text{and} \quad x_{n+1} = \frac{1}{2} x_{2n+1} + 1. \tag{2}
\]
Now $x_{n+1} + x_n = 3(x_n + y_n) = 3(y_{n+1} - y_n)$, so that (by addition)
\[
x_1 + 2(x_2 + x_3 + \cdots + x_n) + x_{n+1} = 3y_{n+1} - 3y_1 = 3y_{n+1} - 3 = x_{n+2} - 2x_{n+1} - 3,
\]
which, using (1), easily yields
\[
x_1 + x_2 + \cdots + x_n = \frac{1}{2} (x_{n+1} - x_n - 1).
\]
Thus, denoting by $K_n$ the positive integer $x_1 + x_2 + \cdots + x_n$, we get
\[
K_n^2 + (K_n + 1)^2 = \frac{1}{4} [(x_{n+1} - x_n - 1)^2 + (x_{n+1} - x_n + 1)^2] \\
= \frac{1}{4} (2x_{n+1}^2 + 2x_n^2 + 2 - 4x_n x_{n+1}) \\
= \frac{1}{4} (x_{2n+2} + x_{2n} - 2x_{2n+1}) \quad \text{using (2)} \\
= \frac{1}{2} x_{2n+1} \quad \text{using (1)}.
\]
The result follows.

II. Solution by Walther Janous, Ursulengymnasium, Innsbruck, Austria.
[Editor's remark: As above the solver first established that $x_n = \frac{1}{2} (u^n + v^n)$ where $u = 2 + \sqrt{3}$ and $v = 2 - \sqrt{3}$.]
Then
\[
x_n = \frac{1}{2} \left[ \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^{2n} + \left( \frac{1 - \sqrt{3}}{\sqrt{2}} \right)^{2n} \right]
\]
for $n \in \mathbb{N}^*$. Thus, we have to verify the existence of $K_n \in \mathbb{N}^*$ such that $x_{2n+1} = 2 (2K_n^2 + 2K_n + 1)$, that is
\[
K_n = \frac{1}{2} \left( -1 + \sqrt{x_{2n+1} - 1} \right).
\]
Clearly, $K_n \neq 0$ for $n \in \mathbb{N}^*$; otherwise, we have $x_{2n+1} = 2$, in contradiction to the fact that $x_n$, $n = 1, 2, 3, \ldots$, is strictly increasing (as an easy induction will show). Now,

\[
x_{2n+1} - 1 = \frac{1}{2} \left[ \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^{4n+2} - 2 + \left( \frac{1 - \sqrt{3}}{\sqrt{2}} \right)^{4n+2} \right]
\]

\[
= \left( \frac{1}{\sqrt{2}} \left[ \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^{2n+1} + \left( \frac{1 - \sqrt{3}}{\sqrt{2}} \right)^{2n+1} \right] \right)^2
\]

\[
= \left[ \frac{1}{2} \left( 2 + \sqrt{3} \right)^n + \frac{1 - \sqrt{3}}{2} \left( 2 - \sqrt{3} \right)^n \right]^2
\]

\[
= w_n^2,
\]

where we have used the last line to define the number $w_n$. Thus, $K_n = \frac{1}{2} (-1 + w_n)$. Thus, it remains to show only that $w_n$ is an odd integer for all $n \in \mathbb{N}^*$. Looking at the structure of $w_n$ we obtain (note $2 \pm \sqrt{3}$ are the characteristic roots) its recursion $w_{n+2} = 4w_{n+1} - w_n$ where $w_1 = 5$ and $w_2 = 19$. Because $w_{n+2} \equiv w_n \pmod{2}$, the proof is complete.

III. Solution by David R. Stone, Georgia Southern University, Statesboro, GA, USA.

The defining conditions and initial values characterize $(x_n, y_n)$ as the solutions to the Pell equation $x^2 - 3y^2 = 1$. If we set

\[
X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},
\]

the given recursions can be written as $X_{n+1} = PX_n$. Then the odd-subscripted terms can be obtained by left multiplication by

\[
P^2 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}.
\]

That is,

\[
x_{2n+1} = 7x_{2n-1} + 12y_{2n-1} \\
y_{2n+1} = 4x_{2n-1} + 7y_{2n-1}.
\] (1)

The first few examples are:

\[
x_1 = 2 \\
x_3 = 26 = 2(2^3 + 3^2) \\
x_5 = 362 = 2(9^2 + 16^2) \\
x_7 = 5042 = 2(35^2 + 36^2) \\
x_9 = 70226 = 2(132^2 + 133^2) \\
x_{11} = 978122 = 2(494^2 + 495^2).
\]
Finding $K_n$ so that $x_{2n+1} = 2 \left(K_n^2 + (K_n + 1)^2 \right) = 2 \left(2K_n^2 + 2K_n + 1 \right)$ is equivalent to solving:

$$4K_n^2 + 4K_n + (2 - x_{2n+1}) = 0,$$

which has the solution $K_n = \frac{1}{2} \left(-1 \pm \sqrt{x_{2n+1} - 1} \right)$. Thus, if we let $K_n = \frac{1}{2} \left(\sqrt{x_{2n+1} - 1} \right)$, we have $x_{2n+1} = 2 \left(K_n^2 + (K_n + 1)^2 \right)$. It remains to show that a thus-defined $K_n$ is an integer. From (1) it is clear by induction that $x_{2n+1}$ is even and $x_{2n+1} + 1$ is a multiple of 3. Thus, the consecutive integers $x_{2n+1} - 1$ and $x_{2n+1} + 1$ are relatively prime. Therefore, the square $y_{2n+1}^2$ factors into relatively prime factors:

$$y_{2n+1}^2 = \frac{x_{2n+1}^2 - 1}{3} = \frac{(x_{2n+1} - 1)(x_{2n+1} + 1)}{3}.$$

Hence $x_{2n+1} - 1$ is an (odd) perfect square, so that $K_n$ as defined is an integer.

Also solved by SEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; the AUSTRIAN IMO-TEAM 2000; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MURRAY S. KRAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, Trinity College Cambridge, UK; DAVID LOEFFLER, student, Cottenham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, KS, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Most of the solvers used an approach similar to II above. Diminnie actually solved the recurrence relation for $y_n$, also, and then showed that

$$K_n = \frac{y_n + y_{n+1} - 1}{2}.$$

He also pointed out other interesting properties of $x_n$ and $y_n$, namely:

1. \( \lim_{n \to \infty} \frac{x_n}{y_n} = \sqrt{3} \).
2. \( x_{2n} = 6y_n^2 + 1 \).
3. \( y_{n+1}^2 - y_n^2 = y_{2n+1} \).

### Crux Mathematicorum

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