MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M65 3G3. The electronic address is

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The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University)

From the Editor-in-Chief

First, the Board of ***Crux Mathematicorum with Mathematical Mayhem*** extends a warm welcome to the new MAYHEM Editor, Shawn Godin, and to the new MAYHEM Assistant Editor, Chris Cappadocia. Shawn has been a member of the ***CRUX with MAYHEM*** family for many years. You can read his first editorial below.

Our sincere thanks go to the retiring editors, Naoki Sato and Cyrus Hsia. Both have moved on to other things — Naoki is an Actuarial Student and Cyrus is a Medical Student. We wish them every success in their new careers. They were instrumental in the smooth transition from two independent journals to the combined journal, and, as Editor-in-Chief, I thank them most sincerely for all their efforts and achievements.

Bruce Shawyer

Editorial

It is the beginning of a new volume and the start of my term as editor of Mayhem. As a brief introduction, I am a high school math and physics teacher in Orleans, east of Ottawa. I have been a reader of Crux for a number of years and am looking forward to working with all of the staff of Crux Mathematicorum with Mathematical Mayhem.

At this point I want to thank our outgoing staff. Naoki Sato has been on the staff of Mathematical Mayhem for a number of years. As editor, he has seen the marriage of Mayhem with Crux and has kept the original vision of Mayhem intact. He has been a great help, so far, to my transition to the editor's chair (and I hope he will be there to answer those frantic emails for a while yet!). Cyrus Hsia is also stepping down from his post as Mayhem's assistant editor. Together, these two guys have kept Mayhem the quality journal that it is. They will be greatly missed.

With Cyrus stepping down we have a new assistant editor, Chris Cappadocia. Chris is a former student of mine, from when I taught in North Bay, who is now in his first year of studies at the University of Waterloo. Chris is a very creative person, passionate about mathematics. He will be an asset to Mayhem, and I am looking forward to working with him.

A couple of months ago, some of the staff of Mayhem, old and new, sat down at the Fields Institute to discuss the direction of Mayhem. Our main conclusion was that Mayhem was to keep its focus as a journal for **students**. Since Mayhem now resides within Crux, we have decided to trim some of the areas where Crux and Mayhem overlap, to allow space for other material that will be of interest to the readers of Mayhem. Keep your eyes open for editorials in upcoming issues describing some of these new features.

Over the next year we will be phasing in some changes to Mayhem. Let us know what you think of them, and tell us about anything **you** would like to see. Remember, Mayhem is **your** journal. Help us to make it as good as it can be.

Shawn Godin

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 2 of 2002.

High School Problems

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H281 *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Suppose the monic polynomial $A(z) = \sum_{k=0}^{n} a_k z^k$ can be factored into $(z-z_1)(z-z_2)\cdots(z-z_n)$, where $z_1, z_2, \cdots, z_n$ are positive real numbers. Prove that $a_1 a_{n-1} \geq n^2 a_0$.

H282. Let $ABCD$ be a cyclic quadrilateral such that its diagonals are perpendicular. Let E be the intersection of AC and BD. It is known that $AE + ED = BE + EC$. Show that $ABCD$ is a trapezoid.

H283. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers in arithmetic progression. Prove that

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} > \frac{4n}{(a_1 + a_n)^2}.$$

H284. Prove that for any positive integer n,

$$1 \geq \frac{n^n}{(n!)^2} \geq \frac{(4n)^n}{(n+1)^{2n}}.$$

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A257. Given a quadrilateral $ABCD$, show that

$$|AB| \cdot |CD| + |BC| \cdot |AD| \geq |AC| \cdot |BD|.$$

When does equality hold?

A258. Is it possible to partition all positive integers into disjoint sets A and B such that

(i) no three numbers of A form an arithmetic progression, and

(ii) no infinite non-constant arithmetic progression can be formed by numbers of B?

(1996 Baltic Way)

A259. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(n^2) = f(n+m)f(n-m) + m^2$$

for all $m, n \in \mathbb{Z}$.

A260.

Characterize the set of Pythagorean triples (integers (a, b, c) such that $a^2 + b^2 = c^2$) which do not contain a multiple of 5.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C99. Find all collections of polynomials p_{11}, p_{12}, p_{21}, p_{22} with complex coefficients satisfying the relation

$$\begin{pmatrix} p_{11}(XY) & p_{12}(XY) \\ p_{21}(XY) & p_{22}(XY) \end{pmatrix} = \begin{pmatrix} p_{11}(X) & p_{12}(X) \\ p_{21}(X) & p_{22}(X) \end{pmatrix} \cdot \begin{pmatrix} p_{11}(Y) & p_{12}(Y) \\ p_{21}(Y) & p_{22}(Y) \end{pmatrix} .$$

C100. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let x_1, x_2, $\ldots$, x_n be positive real numbers, let $S = \sum_{k=1}^{n} x_k$, and suppose that $(n-1)x_k < S$ for all k. Prove that

$$\prod_{j=1}^{n} (S - (n-1)x_k) \leq \prod_{j=1}^{n} x_j .$$

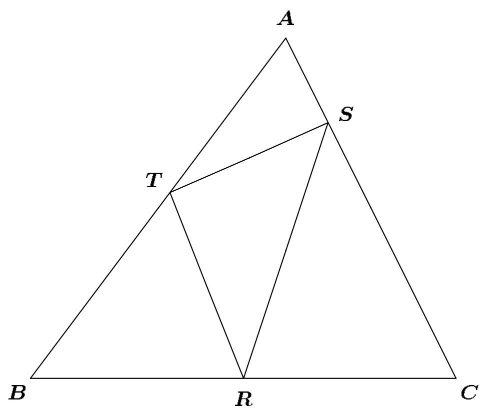
When does equality occur?

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. In triangle ABC, the points R, S, and T lie on the line segments BC, CA, and AB, respectively, such that R is the mid-point of BC, $CS = 3SA$, and $\frac{AT}{TB} = \frac{p}{q}$. If w is the area of $\triangle CRS$, x is the area of $\triangle RBT$, z is the area of $\triangle ATS$, and $x^2 = wz$, then what is the value of $\frac{p}{q}$?

(1997 Fermat, Problem 25)



Solution. The area of a triangle is one half of *base* times *height*. Let A be the area of $\triangle ABC$ and let $r = \frac{p}{q}$.

The ratio of the bases of $\triangle RBT$ and $\triangle ABC$ is $\frac{BR}{BC} = \frac{1}{2}$, and the ratio of the heights is $\frac{BT}{BA} = \frac{q}{p+q} = \frac{q/q}{p/q+q/q} = \frac{1}{r+1}$. Thus, the area of $\triangle RBT$ is $x = \frac{1}{2} \cdot \frac{1}{r+1} A$.

The ratio of the bases of $\triangle CRS$ and $\triangle ABC$ is $\frac{RC}{BC} = \frac{1}{2}$, and the ratio of the heights is $\frac{CS}{CA} = \frac{3}{4}$. Thus, the area of $\triangle CRS$ is $w = \frac{1}{2} \cdot \frac{3}{4} A = \frac{3}{8} A$.

Now reorient the triangle so that CA is the base of ABC. The ratio of the bases of $\triangle ATS$ and $\triangle ABC$ is $\frac{AS}{AC} = \frac{1}{4}$. The ratio of the heights is $\frac{AT}{AB} = \frac{p}{p+q} = \frac{p/q}{p/q+q/q} = \frac{r}{r+1}$. Thus, the area of $\triangle ATS$ is $z = \frac{1}{4} \cdot \frac{r}{r+1} A$.

Therefore, $x^2 = \frac{1}{4(r+1)^2} A^2$ and $wz = \frac{3}{8} \cdot \frac{r}{4(r+1)} A^2$. We must then have $\frac{1}{4(r+1)^2} = \frac{3}{8} \cdot \frac{r}{4(r+1)}$, or, after simplifying, $3r^2 + 3r - 8 = 0$. Solving this quadratic equation leads to the two values $r = \frac{-3 \pm \sqrt{105}}{6}$. Choosing the negative sign leads to a negative value of r, which makes no sense. Thus, we must have $r = \frac{-3+\sqrt{105}}{6}$. This is our value for $\frac{p}{q}$.

Astonishing Pairs of Numbers

Richard Hoshino, student, University of Waterloo

During one dull psychology class, I mindlessly scribbled in my notes: $1+2+3+4+5=15$. It was nothing overly profound, but I thought it was rather interesting that the sum of the numbers from 1 to 5, inclusive, resulted in the digit 1 followed by the digit 5. Curious to see what other combinations exhibited similar properties, I fooled around with some numbers and soon found out that $2+\ldots+7=27$, and $4+\ldots+29=429$. Thus, this motivated the following problem.

*We say that an ordered pair of positive integers (a,b) with $a<b$ is **astonishing** if the sum of the integers from a to b, inclusive, is equal to the digits of a followed by the digits of b. Determine all astonishing ordered pairs.*

Surprisingly, there are many beautiful patterns that arise from this problem, and we shall describe them in this article. I was curious to see if there was an ordered pair (a,b) where b has exactly 1999 digits, and while attempting to solve this problem, I discovered some extremely neat results. Hopefully this article will illustrate that often the simplest of ideas can lead to surprising and extraordinary results. In my case, however, this was completely by accident.

Our problem can be reformulated as follows: find all solutions in positive integers a and b, with $a+(a+1)+\ldots+(b-1)+b=a\cdot 10^n+b$, where b is an integer with exactly n digits.

So our equation becomes:

$$\begin{aligned}
\frac{b(b+1)}{2}-\frac{(a-1)a}{2} &= a\cdot 10^n+b\,,\\
b(b+1)-a(a-1) &= 2a\cdot 10^n+2b\,,\\
b^2+b-a^2+a &= 2a\cdot 10^n+2b\,,\\
b^2-b-(a^2+(2\cdot 10^n-1)a) &= 0\,.
\end{aligned}$$

And by the quadratic formula, we find that

$$\begin{aligned}
b &= \frac{1\pm\sqrt{1+4(a^2+(2\cdot 10^n-1)a)}}{2}\\
&= \frac{1\pm\sqrt{(2a+(2\cdot 10^n-1))^2-((2\cdot 10^n-1)^2-1)}}{2}\,.
\end{aligned}$$

We want b to be an integer, so we require that the discriminant be a perfect square, specifically an odd perfect square. Thus, let

$$(2a + (2 \cdot 10^n - 1))^2 - ((2 \cdot 10^n - 1)^2 - 1) = m^2 ,$$

where m is a positive integer. Then, $b = \frac{1+m}{2}$. Note that we can get rid of the $\pm$ sign because we require $b > 0$. Thus we have:

$$\begin{aligned}
(2a + (2 \cdot 10^n - 1))^2 - ((2 \cdot 10^n - 1)^2 - 1) &= m^2 , \\
(2a + (2 \cdot 10^n - 1))^2 - m^2 &= (2 \cdot 10^n - 1)^2 - 1 , \\
(2a + (2 \cdot 10^n - 1) + m)(2a + (2 \cdot 10^n - 1) - m) & \\
&= ((2 \cdot 10^n - 1) + 1)((2 \cdot 10^n - 1) - 1) , \\
(2a + 2 \cdot 10^n + m - 1)(2a + 2 \cdot 10^n - m - 1) & \\
&= 2 \cdot 10^n \cdot (2 \cdot 10^n - 2) , \\
(2a + 2 \cdot 10^n + m - 1)(2a + 2 \cdot 10^n - m - 1) & \\
&= 2^{n+2} \cdot 5^n \cdot (10^n - 1) .
\end{aligned}$$

Let $x = 2a + 2 \cdot 10^n + m - 1$ and $y = 2a + 2 \cdot 10^n - m - 1$. Then we have $xy = 2^{n+2} \cdot 5^n(10^n - 1)$ and $x + y = 4a + 4 \cdot 10^n - 2$. But then $x + y$ is a multiple of 2, but not 4, so both x and y cannot be multiples of 4. Furthermore, $x + y$ is even, so we cannot have x being odd and y being even, or vice-versa. It follows that the only possibilities are to have $(x, y) = (2p, 2^{n+1}q)$ or $(x, y) = (2^{n+1}q, 2p)$, where $pq = 5^n(10^n - 1)$.

Hence, $4a + 4 \cdot 10^n - 2 = x + y = 2^{n+1}q + 2p$, and so

$$a = \frac{2^{n+1}q + 2p + 2 - 4 \cdot 10^n}{4} = 2^{n-1}q - 10^n + \frac{p+1}{2} .$$

Now, $b = \frac{1+m}{2}$ and $2m = x - y$, so that $b = \frac{2+x-y}{4}$. But b is a positive integer. Thus, we require $x - y$ to be positive; that is, $x - y = |2^{n+1}q - 2p|$. Note that the formula is correct for both of our possibilities for (x, y) detailed above. Thus, we have:

$$b = \frac{1 + |2^n q - p|}{2} .$$

Hence, for each n, we need to cycle through all possible ordered pairs (p, q) so that $pq = 5^n(10^n - 1)$, and see if this ordered pair (p, q) will give a solution (a, b), where $0 < a < b$ and b has exactly n digits.

For example, for $n = 1$, we have $pq = 45$, so that the only possibilities for (p, q) are $(1, 45)$, $(3, 15)$, $(5, 9)$, $(9, 5)$, $(15, 3)$, $(45, 1)$. Substituting into our formula for a and b, we find that the corresponding pairs (a, b) are $(36, 45)$, $(7, 14)$, $(2, 7)$, $(0, 1)$, $(1, 5)$, $(14, 22)$, respectively. Only $(1, 5)$ and $(2, 7)$ satisfy our requirements, so these are the only astonishing pairs with $n = 1$.

For any $n > 1$, the number $pq = 5^n(10^n - 1)$ has many factors, so that it is quite time consuming to look for each possible pair (p, q). However, with the power of a computer algebra system, such as MAPLE, the calculations are very easy.

With the aid of MAPLE, we list all astonishing pairs (a, b), where $n \leq 5$.

$$
\begin{array}{lcll}
n & = & 1 & : \quad (1,5),\ (2,7) \\
n & = & 2 & : \quad (4,29),\ (13,53),\ (18,63),\ (33,88),\ (35,91) \\
n & = & 3 & : \quad (7,119),\ (78,403),\ (133,533),\ (178,623) \\
n & = & 4 & : \quad (228,2148),\ (273,2353),\ (388,2813),\ (710,3835), \\
 & & & \quad\ \ (1333,5333),\ (1701,6076),\ (1778,6223), \\
 & & & \quad\ \ (2737,7889),\ (3273,8728),\ (3563,9163) \\
n & = & 5 & : \quad (3087,25039),\ (3478,26603),\ (12488,51513), \\
 & & & \quad\ \ (13333,53333),\ (14208,55168),\ (17778,62223), \\
 & & & \quad\ \ (31463,85338),\ (36993,93633).
\end{array}
$$

Look carefully at these numbers, we have some very interesting patterns here. The pairs $(1,5)$, $(13,53)$, $(133,533)$, $(1333,5333)$, $(13333,53333)$ are all astonishing. It is very likely that this pattern continues indefinitely and, as we shall see, this is indeed the case. We shall also find some other incredible sequences of astonishing pairs, and discover some remarkable properties of such sequences.

Let us attempt to find formulas that generate astonishing pairs. Assume that $q = 5^n r$. Since $pq = 5^n(10^n - 1)$, we have $p = \frac{10^n - 1}{r}$. Then we can solve for a and b in terms of r.

$$
\begin{aligned}
a &= 2^{n-1}q - 10^n + \frac{p+1}{2} \\
&= 2^{n-1} \cdot 5^n r - 10^n + \frac{\frac{10^n-1}{r} + 1}{2} \\
&= 10^n \cdot \frac{r}{2} - 10^n + \frac{10^n + r - 1}{2r} \\
&= \frac{10^n}{2}\left(r + \frac{1}{r} - 2\right) + \frac{r-1}{2r}\,, \\
b &= \frac{1 + |2^n q - p|}{2} \\
&= \frac{1 + |2^n \cdot 5^n r - \frac{10^n-1}{r}|}{2} \\
&= \frac{1 + |10^n r - \frac{10^n-1}{r}|}{2} \\
&= \frac{1 + |10^n(r - \frac{1}{r}) + \frac{1}{r}|}{2}\,.
\end{aligned}
$$

Now we shall find values of r such that a and b are integers for all n. In order for (a, b) to be an astonishing ordered pair, we require b to have exactly n digits; that is, we require $10^{n-1} \leq b < 10^n$. If we can find an r such that this inequality holds for all n, then we will generate an infinite set of astonishing ordered pairs (a, b), since each integer n will give us one astonishing ordered pair.

For large n, $b = \frac{1+\left|10^n\left(r-\frac{1}{r}\right)+\frac{1}{r}\right|}{2}$ can be approximated as $\left|\frac{10^n}{2}\left(r-\frac{1}{r}\right)\right|$, since 1 and $\frac{1}{r}$ are extremely small quantities, compared with $10^n\left(r-\frac{1}{r}\right)$. Simplifying $10^{n-1} \leq \left|\frac{10^n}{2}\left(r-\frac{1}{r}\right)\right| < 10^n$, we get $\frac{1}{5} \leq \left|r-\frac{1}{r}\right| < 2$.

Solving this inequality, we find that we require $\sqrt{2}-1 < r \leq \frac{\sqrt{101}-1}{10}$ or $\frac{\sqrt{101}+1}{10} \leq r < \sqrt{2}+1$. (Note: we require $r > 0$, since otherwise p and q will be negative).

Since p and q must be (odd) integers, r must be a rational number of the form $\frac{x}{y}$, where y divides 5^n and x divides $10^n - 1$. If for a certain n, y does not divide 5^n or x does not divide $10^n - 1$, then we will not get an astonishing pair for that value of n.

For example, we can have $r = \frac{3}{5}$, since 3 divides $10^n - 1$ for all n, 5 divides 5^n for all n, and $\sqrt{2}-1 < \frac{3}{5} \leq \frac{\sqrt{101}-1}{10}$.

Substituting $\frac{3}{5}$ into our formula for a and b, we find, upon simplification, that:

$$\begin{aligned} a &= \frac{2 \cdot 10^n - 5}{15}, \\ b &= \frac{8 \cdot 10^n - 5}{15}. \end{aligned}$$

By our work above, we know that each ordered pair (a, b) generated by the formula above is astonishing. Substituting in $n = 1, 2, 3, \ldots,$ we generate an infinite number of astonishing ordered pairs:

$$(1, 5), (13, 53), (133, 533), (1333, 5333), (13333, 53333), \ldots.$$

So, as hypothesized earlier, this pattern does indeed continue indefinitely. Analyzing this problem in this manner, we can generate other sequences of astonishing ordered pairs.

Let $r = \frac{9}{5}$. This number r ensures that p and q are integers for $n \geq 2$, and $\frac{\sqrt{101}+1}{10} \leq \frac{9}{5} < \sqrt{2}+1$. Hence for every $n \geq 2$, we can find an astonishing ordered pair (a, b) where $r = \frac{9}{5}$. Substituting this value of r into our formula for a and b, we find that:

$$
\begin{aligned}
a &= \frac{2(4 \cdot 10^n + 5)}{45}, \\
b &= \frac{7(4 \cdot 10^n + 5)}{45}.
\end{aligned}
$$

For each $n \geq 2$, we get an astonishing ordered pair:

$$(18, 63), (178, 623), (1778, 6223), (17778, 62223), (177778, 622223), \ldots$$

As you notice, there is a really interesting pattern in this sequence as well. Notice that to generate astonishing pairs, we repeatedly add a 7 to each a and a 2 to each b. Is it not weird then that $\frac{b}{a} = \frac{7}{2}$? In addition, $(2, 7)$ is an astonishing pair. I wonder if that is a coincidence?

Let us try some other values of r that satisfy the requirements that we specified. Let us try $r = \frac{99}{125}$. Notice that 125 divides 5^n for each $n \geq 3$, and 99 divides $10^n - 1$ for each even n. In addition, $\sqrt{2} - 1 < \frac{99}{125} \leq \frac{\sqrt{101}-1}{10}$. Thus, our formula for a and b will only give us astonishing pairs for $n = 4$, 6, 8, 10, $\ldots$.

Our formula for $r = \frac{99}{125}$ is:

$$
\begin{aligned}
a &= \frac{13(26 \cdot 10^n - 125)}{12375}, \\
b &= \frac{13(224 \cdot 10^n - 125)}{12375}.
\end{aligned}
$$

For $n = 4$, 6, 8, 10, $\ldots$, we find that the corresponding astonishing pairs are:

$$
\begin{gathered}
(273, 2353), (27313, 235313), (2731313, 23531313), \\
(273131313, 2353131313), \ldots.
\end{gathered}
$$

In this sequence, we add 13 at the end of each a and b to generate the next astonishing pair. Is it not weird that the number 13 appears in the formula for both a and b? Is that just a coincidence, or is there a reason why this is true?

Let us try one more value of r. Let us try $r = \frac{999}{625}$. We have $\frac{\sqrt{101}+1}{10} \leq \frac{999}{625} < \sqrt{2} + 1$. Since 625 divides 5^n for each $n \geq 4$, and 999 divides $10^n - 1$ if and only if n is a multiple of 3, our formula for a and b will only give us astonishing pairs for $n = 6$, 9, 12, 15, $\ldots$.

Our formula for $r = \frac{999}{625}$ is:

$$\begin{aligned} a &= \frac{187(374 \cdot 10^n + 625)}{624375}, \\ b &= \frac{812(374 \cdot 10^n + 625)}{624375}. \end{aligned}$$

For $n = 6, 9, 12, 15, \ldots$, we find that the corresponding astonishing pairs are:

$$(112013, 486388),\ (112012813, 486387188),$$
$$(112012812813, 486387187188),$$
$$(112012812812813, 486387187187188),$$
$$(112012812812812813, 486387187187187188), \ldots.$$

In this sequence, we add the number 281 near the end of each a to get the next term and we add the number 718 near the end of each b to get the next term. Notice that $\frac{b}{a} = \frac{812}{187}$. Now is it just a remarkable coincidence, or is there a reason why 281 and 718 are cyclic permutations of the number 812 and 187? Also, would it not be simply incredible if $(281, 718)$ is an astonishing ordered pair? Unfortunately if we add up the numbers from 281 to 718 inclusive we get the number 218781, which is not astonishing, but remarkably, it is a permutation of the number 281718; that is, it is almost astonishing! Now is *this* a coincidence, or is there some deep mathematical explanation as to why this is the case?

Try some other rational values of r, and generate some more patterns of astonishing sequences. For example, what sequences do $r = \frac{111}{125}$ and $r = \frac{9999}{15625}$ generate? Are there any interesting mathematical observations that you can make from looking at the patterns?

I have left many questions unanswered in this article, as the mathematics involved in answering these questions almost definitely extends beyond what I currently know. Possibly you will be able to extend the ideas in this article further. Is it not quite strange that we were able to derive all these interesting results, and it was motivated by the simple observation that $1 + 2 + 3 + 4 + 5 = 15$. That is quite *astonishing*, is it not?

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A Classical Inequality

Vedula N. Murty

Problem.

Let A, B, and C denote the angles of a triangle ABC. The following inequality is well known:

$$1 \;<\; \cos A + \cos B + \cos C \;\leq\; \frac{3}{2}\,. \tag{1}$$

We present two solutions to the left-hand side of the inequality and three solutions to the right-hand side inequality and draw the attention of the students to a wrong proof usually given to prove the right-hand side inequality. One of the proofs presented for the left-hand side inequality is believed to be new.

Proof 1.

Consider the identity given below which is easily verified.

$$\sum a(b^2+c^2-a^2) \;\equiv\; (a+b-c)(b+c-a)(c+a-b) + 2abc\,. \tag{2}$$

Here, a, b, and c denote the side lengths of a triangle, and the sum is cyclical over a, b, and c. Dividing both sides of equation (2) by $2abc$ we obtain

$$\sum \frac{b^2+c^2-a^2}{2bc} \;=\; \frac{(a+b-c)(b+c-a)(c+a-b)}{2abc} + 1\,. \tag{3}$$

Noting that the left-hand side of (3) is $\cos A + \cos B + \cos C$, we immediately see that $\cos A + \cos B + \cos C > 1$, since the right-hand side of (3) is greater than 1. We believe this proof is new.

Let $x = a+b-c$, $y = b+c-a$, and $z = c+a-b$. Then x, y, and z are all positive. This implies that $a = (z+x)/2$, $b = (x+y)/2$, and $c = (y+z)/2$. The Arithmetic-Geometric Inequality gives us

$$(x+y)(y+z)(z+x) \;\geq\; 8xyz\,. \tag{4}$$

This implies that

$$\cos A + \cos B + \cos C \;\leq\; \frac{3}{2}\,.$$

This completes the proof of the classical inequality (1).

Proof 2.

We now present a trigonometric proof of the classical inequality. The following are well known trigonometric identities given in standard text books on trigonometry. The reference given at the end of this paper is an excellent text book on trigonometry.

$$\begin{aligned} \cos A + \cos B + \cos C &= 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right), \qquad (5) \\ r &= 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right), \qquad (6) \\ OI^2 &= R^2 - 2Rr. \qquad (7) \end{aligned}$$

Notice that (7) implies that $R^2 - 2Rr \geq 0$, which implies that

$$0 < \frac{r}{R} \leq \frac{1}{2}. \qquad (8)$$

Equations (5) and (6) imply that

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}. \qquad (9)$$

Equations (8) and (9) prove the classical inequality. The notation used above is standard and is familiar to readers of ***CRUX with MAYHEM***.

Proof 3.

We now use Jensen's inequality, which states that

$$w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) \leq f(w_1x_1 + w_2x_2 + w_3x_3), \qquad (10)$$

where the w_i are positive with a sum of 1, and $f(x)$ is concave down throughout its domain.

Quite a few students use (10) with $w_1 = w_2 = w_3 = 1/3$ and $f(x) = \cos x$. Unfortunately, $\cos x$ is not concave down throughout $(0, \pi)$.

To circumvent this difficulty we first prove the inequality

$$\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) \leq \frac{3}{2}. \qquad (11)$$

To see this, simply take $w_1 = w_2 = w_3 = 1/3$ and $f(x) = \sin(x/2)$ and use (10) with $x_1 = A$, $x_2 = B$, and $x_3 = C$. We immediately obtain (11).

We now establish the following inequality:

$$\cos A + \cos B + \cos C \leq \sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right). \qquad (12)$$

First,

$$\cos B + \cos C = 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$
$$= 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) \leq 2\sin\frac{A}{2}\,. \qquad (13)$$

Similarly,

$$\cos C + \cos A \leq 2\sin\frac{B}{2}\,, \qquad (14)$$
$$\cos A + \cos B \leq 2\sin\frac{C}{2}\,. \qquad (15)$$

Adding (13), (15), and (15) we obtain (12). This completes the proof of the classical inequality.

Reference

S. L. Loney, "Plane Trigonometry Part 1", Metric Edition, Radha Publishing House, Calcutta.

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