SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The name of Walther Janous, Ursulinengymnasium, Innsbruck, Austria was inadvertently omitted from the list of solvers of 2495.

2490. [1999 : 505, 2000 : 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $\alpha > 1$. Denote by x_n the only positive root of the equation:

$$(x+n^2)(2x+n^2)(3x+n^2)\cdots(nx+n^2) \;=\; \alpha\, n^{2n}\,.$$

Find $\lim\limits_{n\to\infty} x_n$.

Comment by Nikolaos Dergiades, Thessaloniki, Greece.

In the comments after the solutions, it was stated that:

Konečný gave a one-line "proof" based on the "fact" that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{kx}{n^2}\right) \;=\; e^{x/2}$$

which he believed "must be well known", but could not find a reference.

Now, it is well known that if $\lim\limits_{n\to\infty} a_n = 0$, then $\lim\limits_{n\to\infty}\dfrac{\ln(1+a_n)}{a_n} = 1$, and since $(1+a_n)^{b_n} = e^{a_n b_n \frac{\ln(1+a_n)}{a_n}}$, we have

$$\lim_{n\to\infty}(1+a_n)^{b_n} \;=\; e^{\lim\limits_{n\to\infty}(a_n b_n)}\,. \qquad (1)$$

Using the generalized Bernoulli's Inequality, we have, for $1 \le k \le n$,

$$\left(1+\frac{x}{n}\right)^{\frac{k}{n}} \;\le\; \left(1+\frac{kx}{n^2}\right) \;\le\; \left(1+\frac{x}{n^2}\right)^{k}, \qquad \text{or}$$

$$\left(1+\frac{x}{n}\right)^{\frac{n+1}{2}} \;\le\; \prod_{k=1}^{n}\left(1+\frac{kx}{n^2}\right) \;\le\; \left(1+\frac{x}{n^2}\right)^{\frac{n(n+1)}{2}}\,.$$

From (1), we have that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n+1}{2}} \;=\; e^{\lim\limits_{n\to\infty}\left(\frac{x(n+1)}{2n}\right)} \;=\; e^{x/2} \qquad \text{and}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n^2}\right)^{\frac{n(n+1)}{2}} \;=\; e^{\lim\limits_{n\to\infty}\left(\frac{xn(n+1)}{2n^2}\right)} \;=\; e^{x/2}\,,$$

giving that $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(1+\dfrac{kx}{n^2}\right) \;=\; e^{x/2}$.

2501. [2000 : 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$ and $\angle ABC$ meet BC and AC at D and E respectively. Suppose that $AB + BD = AE + EB$. Characterize $\triangle ABC$.

I. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

On the extension of AB we take $BF = BD$. [Because BE bisects $\angle ABD$, which is an external angle of $\triangle BFD$], $\angle BFD = \angle BDF = \angle ABE = \angle EBD$. If F' on AC is symmetric with F about the bisector AD, then

$$\angle EF'D \;=\; \angle EBD\,. \qquad (1)$$

We now have $AB + BD = AE + EB$ is equivalent to $AE + EB = AF = AF' = AE + EF'$, or

$$EB \;=\; EF'\,. \qquad (2)$$

If the points B, D, F' are collinear, then F' coincides with C and from (1) we have

$$\angle ABC \;=\; 2\angle ACB\,.$$

If the points B, D, F' are not collinear, then since $\triangle EBF'$ is isosceles (from (2)) we conclude that $\triangle BDF'$ is also isosceles; that is, $BD = DF' = DF$. Hence, $\triangle BFD$ is equilateral and

$$\angle ABC \;=\; 120^\circ\,.$$

[*Comment*. Although Dergiades does not say so explicitly, it is clear that his argument is reversible, so that, conversely, if ABC is a triangle with either $\angle ABC = 2\angle ACB$ or $\angle ABC = 120^\circ$, then $AB + BD = AE + EB$.]

II. *Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Since $BD = \frac{ac}{b+c}$, $AE = \frac{bc}{a+c}$, and $EB = \frac{2ac}{a+c}\cos\left(\frac{B}{2}\right)$, the given condition $AB + BD = AE + EB$ is equivalent to

$$c + \frac{ac}{b+c} \;=\; \frac{bc}{a+c} + \frac{2ac}{a+c}\cos\left(\tfrac{B}{2}\right)\,,$$

which simplifies to

$$\frac{a^2 - b^2 + c^2 + 2ac + ab}{b+c} \;=\; 2a\cos\left(\tfrac{B}{2}\right)\,.$$

In the last equation we replace $a^2 - b^2 + c^2$ by $2ac\cos B$ and divide both sides by a. The result is

$$\frac{2c(1+\cos B) + b}{b+c} = 2\cos\left(\tfrac{B}{2}\right)\,,$$

or

$$\frac{4c\cos^2\frac{B}{2} + b}{b+c} \;=\; 2\cos\left(\tfrac{B}{2}\right)\,.$$

The roots of this quadratic equation in $\cos\left(\frac{B}{2}\right)$ are $\frac{1}{2}$ and $\frac{b}{2c}$. If $\cos\left(\frac{B}{2}\right) = \frac{1}{2}$, then $B = 120°$. If $\cos\left(\frac{B}{2}\right) = \frac{b}{2c}$, then we use the Law of Sines, $\frac{b}{c} = \frac{\sin B}{\sin C}$, and replace $\sin B$ by $2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)$ to deduce that $\sin\left(\frac{B}{2}\right) = \sin C$, which is equivalent to $B = 2C$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (a second solution); MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Several solvers used the fact that $ac - b^2 + c^2$ is equivalent to the condition $\angle B = 2\angle C$; Amengual reports that this equivalence also appears in [1976 : 74], [1984 : 287], *and* [1996 : 265-267].

2502. [2000 : 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$, $\angle ABC$ and $\angle BCA$ meet BC, AC and AB at D, E and F respectively. Let p and q be the perimeters of $\triangle ABC$ and $\triangle DEF$ respectively.

Prove that $p \geq 2q$, and that equality holds if and only if $\triangle ABC$ is equilateral.

Solution by Michel Bataille, Rouen, France.

Because AD is the internal bisector of $\angle BAC$, $\dfrac{DB}{DC} = \dfrac{c}{b}$, so that $\dfrac{DB}{c} = \dfrac{DC}{b} = \dfrac{DB+DC}{b+c} = \dfrac{a}{b+c}$, and $DB = \dfrac{ac}{b+c}$. Similarly, $FB = \dfrac{ac}{a+b}$.

The Law of Cosines in $\triangle BDF$ gives $DF^2 = FB^2 + DB^2 - 2FB\cdot DB\cdot\cos B$. Using also $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$ and rearranging [note: $(b+a)^2+(b+c)^2 = 2(b+a)(b+c)+(a-c)^2$], we obtain:

$$DF^2 = \frac{ac}{(b+a)^2(b+c)^2}\left(b(a+b+c)(b^2-(a-c)^2)+ab^2c\right).$$

We therefore have $DF^2 \leq \dfrac{ac}{(b+a)^2(b+c)^2}\left(b(a+b+c)b^2+ab^2c\right)$, or

$$DF \leq b\sqrt{\frac{ac}{(b+a)(b+c)}}. \qquad (1)$$

(We note that equality holds if and only if $a = c$.) Using the AM–GM Inequality, we see that

$$\begin{aligned} DF &\le \frac{b\sqrt{ac}}{\sqrt{2\sqrt{ba}\cdot 2\sqrt{bc}}} = \frac{1}{2}\sqrt{b}\cdot\sqrt{\sqrt{ac}} \le \frac{1}{2}\cdot\frac{b+\sqrt{ac}}{2} \\ &\le \frac{1}{4}\left(b+\frac{a+c}{2}\right) = \frac{b}{4}+\frac{a}{8}+\frac{c}{8}. \end{aligned}$$

Analogous inequalities can be obtained for FE and ED leading to:

$$\begin{aligned} q &= DF+FE+ED \\ &\le \left(\frac{b}{4}+\frac{a}{8}+\frac{c}{8}\right)+\left(\frac{a}{4}+\frac{b}{8}+\frac{c}{8}\right)+\left(\frac{c}{4}+\frac{a}{8}+\frac{b}{8}\right) = \frac{p}{2}. \end{aligned}$$

Thus $p \ge 2q$ as desired.

If $\triangle ABC$ is not equilateral, say $a \neq c$, inequality (1) is strict (as noted) and $p > 2q$. If $\triangle ABC$ is equilateral, $\triangle DEF$ is its median triangle, so that $p = 2q$. This completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

We received two generalization, one from Janous and one from Woo. Janous proves,

> For a triangle with semiperimeter s, if its internal angle bisectors meet the opposite sides in the vertices of a triangle whose side lengths are a_1, b_1, c_1, then
>
> $$a_1^2+b_1^2+c_1^2 \le \frac{s^2}{3},$$
>
> with equality if an only if the initial triangle is equilateral.

Unfortunately his proof requires a computer to manipulate the unwieldy formulas he obtains. By applying the inequality between the arithmetic and square-root means to his inequality he gets $a_1+b_1+c_1 \le s$, which is the inequality of Seimiya.

Woo's result seems to require that the original triangle be acute, namely:

> Let $\triangle ABC$ have $A < B < C < \frac{\pi}{2}$. Let AA', BB', CC' be its medians and let AA'', BB'', CC'' be its altitudes. For each point P inside the triangle let P_1, P_2, P_2, be the points on the sides such that the cevians AP_1, BP_2, CP_3 concur at P. Then $\triangle P_1P_2P_3$ will have perimeter less than half that of $\triangle ABC$ if P is inside the region bounded by the lines AA', AA'', CC', CC''.

Semiya's inequality (restricted to acute triangles) therefore follows from Woo's because the angle bisector from a vertex lies between the median and altitude from that vertex.

2503. [2000 : 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The incircle of $\triangle ABC$ touches BC at D, and the excircle opposite to B touches BC at E. Suppose that $AD = AE$. Prove that

$$2\angle BCA - \angle ABC = 180^\circ .$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece; and Gerry Leversha, St. Paul's School, London, England.

Let a, b, c be the side lengths and A, B, C be the angles of $\triangle ABC$. If s is the semiperimeter, then we know that $BD = s - b$, $BE = s$ and using the Cosine Rule with triangles ABD and ABE, we obtain that

$$\begin{aligned} AD^2 &= c^2 + (s-b)^2 - 2c(s-b)\cos B , \\ AE^2 &= c^2 + s^2 - 2cs\cos B , \end{aligned}$$

and since $AD = AE$, on subtracting, we get

$$-2sb + b^2 + 2cb\cos B = 0$$

or

$$-(a+b+c) + b + 2c\cos B = 0$$

or

$$2c\cos B = a + c .$$

The last equality can be rewritten, using the Sine Law, as

$$2\sin C\cos B = \sin(B+C) + \sin C$$

or

$$\sin(C - B) = \sin C .$$

Since $C - B \neq C$, we have

$$C - B = 180^\circ - C ;$$

that is, $2\angle BCA - \angle ABC = 180^\circ$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Cotham School, Bristol, UK; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2504. [2000 : 45] *Proposed by Hayo Ahlburg, Benidorm, Spain, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that A, B and C are the angles of a triangle. Determine the best lower and upper bounds of $\prod_{\text{cyclic}} \cos(B-C)$.

Solution by Kee-Wai Lau, Hong Kong.

The answer is

$$-\frac{1}{8} \le \prod_{\text{cyclic}} \cos(B-C) \;\le\; 1\,.$$

Firstly,

$$\prod_{\text{cyclic}} \cos(B-C) \;\le\; \left|\prod_{\text{cyclic}} \cos(B-C)\right| \;=\; \prod_{\text{cyclic}} |\cos(B-C)| \;\le\; 1\,.$$

Secondly, using the identity

$$\cos X \cos Y \;=\; \frac{1}{2}[\cos(X+Y)+\cos(X-Y)]\,,$$

we get

$$\begin{aligned}
\prod_{\text{cyclic}} \cos(B-C) &= \frac{1}{2}\Big(\cos(B-A)+\cos(A+B-2C)\Big)\cos(A-B) \\
&= \frac{1}{2}\left(\cos(A-B)+\frac{\cos(A+B-2C)}{2}\right)^2 \\
&\qquad -\frac{1}{8}\cos^2(A+B-2C) \\
&\ge -\frac{1}{8}\cos^2(A+B-2C) \;\ge\; -\frac{1}{8}\,.
\end{aligned}$$

The equilateral triangle shows that the upper bound 1 cannot be improved, and the degenerate triangle with $A = 2\pi/3$, $B = \pi/3$, $C = 0$ shows that the lower bound $-1/8$ cannot be improved either.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HENRY LIU, student, University of Cambridge, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One other reader misread the product for a sum.

Klamkin points out that the result is true for arbitrary *angles A, B, C, and in fact Lau's proof works in that generality.*

Most readers obtained strict inequality in the lower bound, which is correct if degenerate triangles are not allowed.

2506. [2000 : 46] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

In the lattice plane, determine all lattice straight lines that are tangent to the unit circle. (A lattice straight line is a straight line containing two lattice points.)

Solution by Skidmore College Problem Group, Skidmore College, Saratoga Springs, New York.

Theorem. $ax + by = c$ is the equation of a lattice straight line tangent to the unit circle if and only if $a^2 + b^2 = c^2$, where a, b, $c \in \mathbb{Z}$.

Proof: ($\Longleftarrow$) Let S^1 be the unit circle, and suppose that a, b, $c \in \mathbb{Z}$ with $a^2 + b^2 = c^2$. Let ℓ be the line with equation $ax + by = c$. It is a standard result from number theory that, since $\gcd(a, b)|c$, the Diophantine equation $ax + by = c$ has infinitely many integral solutions (compare, for example, *Elementary Number Theory* by Underwood Dudley, 2nd Edition, p. 25, Theorem 1). Thus, ℓ is a lattice straight line. Also, $C(a/c, b/c) \in S^1 \cap \ell$. Moreover, $m_\ell = -a/b$ and $m_{OC} = b/a$, so that $\ell \perp OC$, implying that ℓ is tangent to S^1.

($\Longrightarrow$) Suppose that ℓ is a lattice straight line through the points $A(r, s)$ and $B(t, u)$, where r, s, t, $u \in \mathbb{Z}$, and suppose that ℓ is tangent to S^1. Then ℓ has equation $ax + by = c$, where $a = s - u$, $b = t - r$, and $c = st - ru$. Since ℓ is tangent to S^1, we know that the distance from ℓ to the origin is 1, so that, using the formula for the distance between a point and a line in the coordinate plane, we get

$$\frac{|a(0) + b(0) - c|}{\sqrt{a^2 + b^2}} = 1.$$

Simple algebra then yields $a^2 + b^2 = c^2$.

Note: It is easy to see that this result generalizes to Euclidean 3–space $\mathbb{R}^3$. First, we say that a plane π is a *lattice plane* if π contains three non-collinear lattice points. If S^2 is the unit sphere, then a lattice plane π is tangent to S^2 if and only if π has equation $ax + by + cz = d$, where a, b, c, $d \in \mathbb{Z}$ and $a^2 + b^2 + c^2 = d^2$. With appropriate definitions, this result can be further generalized in the obvious way to $\mathbb{R}^n$, using the unit hypersphere S^{n-1} and lattice $(n - 1)$–dimensional subspaces.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; and the proposer.

Bradley, uses t to parameterize the unit circle:

$$x = \frac{1 - t^2}{1 + t^2}, \qquad y = \frac{2t}{1 + t^2}$$

which gives him the equation of the tangent at 't' as

$$x(1 - t^2) + 2ty = 1 + t^2.$$

He remarks that resolving this with the knowledge that t is rational (which he proves is the case for this problem) shows that for each such pair (x, y), there exists an integer z such that $x^2 + y^2 = 1 + z^2$. He has written a paper for the Mathematical Gazette (Note 80.34, pp. 49-51) relating all solutions of the Diophantine equation $x^2 + y^2 = 1 + z^2$ to Pythagorean triplets and conversely.

2507. [2000 : 46] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Show that there are infinitely many pairs of distinct natural numbers, n and k such that $\gcd(n! + 1, k! + 1) > 1$;

I. *Solution by the Austrian IMO Team-2000; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Let k be any positive integer such that $k + 1$ is not a prime, and let p be a prime such that $p \mid k! + 1$. Then $p > k$ and thus $p \geq k + 2$ for otherwise $p = k + 1$, a contradiction. Let $n = p - 1$. Then $n > k$. Since $p \mid n! + 1$ by Wilson's Theorem, we have $\gcd(n! + 1, k! + 1) \geq p > 1$.

II. *Solution by Kee-Wai Lau, Hong Kong; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $k \geq 3$ be any *odd* integer and p be a prime divisor of $k! + 1$. Then $p > k$ implies $p \geq k + 2$ as both p and k are odd. Let $n = p - 1$. Then $n > k$ and the rest follows as in I above.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHARLES DIMINNIE and TREY SMITH, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was one partially incorrect solution submitted.

Janous remarked that a solution would follow immediately from the following known theorem which can be found in Problems in Number Theory *(in Bulgarian), Sofia, 1985 by St. M. Dodvnekov and K.B. Chakarian.*

> ***Theorem**: There exist infinitely many primes p with the property that there is a unique natural number $q < p$ such that $p \mid (q-1)! + 1$. For the present problem, simply take $n = p - 1$ and $k = q - 1$.*

2508. [2000 : 46] *Proposed by J. Chris Fisher, University of Regina, Regina, Saskatchewan.* (Corrected) In problem 2408 [1999 : 49; 2000 : 55] we defined a point P to be *Cevic* with respect to $\triangle ABC$ if the vertices D, E, F of its pedal triangle determine concurrent cevians; more precisely, D, E, F are the feet of perpendiculars from P to the respective sides BC, CA, AB, while AD, BE, CF are concurrent.

1. Show that a point D on the line BC can determine 0, 1, 2, or infinitely many positions for E on AC for which P is Cevic.

2. Describe the possible locations of E if D divides the segment BC in the ratio $\lambda : 1 - \lambda$ (when P is Cevic and λ is an arbitrary real number).

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA (abridged by the editor).

1. When D is the mid-point of BC and $AB = AC$, then [because the figure is symmetric about the line AD] every point P on AD would be Cevic, and so there would be infinitely many positions for E. We now prove that *when D is not the mid-point of BC there can be at most two positions for E*. Assume that $BD < DC$. [We shall see that when D is the mid-point of BC and $AC \neq AB$, there is always a unique position for E on AC; the argument that follows shows that the second position of E moves out to infinity as D approaches the mid-point.] Let D' be the point on line BC such that $BD : DC = BD' : D'C$, with B between D and D'; D' is called the *harmonic conjugate of D with respect to B and C*. Let H, K be points on lines AB, AC such that $AHD'K$ is a parallelogram. Let O be the point for which $OH \perp HA$ and $OK \perp KA$. If E, F are variable points on AC, AB such that AD, BE, CF are concurrent, we shall call $\triangle DEF$ a *cevian* triangle. For any point E on AC, $\triangle DEF$ cevian implies that D' lies on EF. [Indeed, this is a consequence of Ceva's Theorem; see, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, Theorem 220 on p. 149.] As the line EF moves about D', define P to be the point such that $PE \perp AC$ and $PF \perp AB$. We assert that the locus of P is the hyperbola Z through A with OH and OK as asymptotes:

Let AP cut OH at Q, and OK at R. Let A', P' be points on OH and A'', P'' be points on OK such that $OA'AA''$ and $OP'PP''$ are parallelograms. Then $\dfrac{AA''}{PP''} = \dfrac{RA}{RP} = \dfrac{KA}{KE} = \dfrac{D'F}{D'E} = \dfrac{FH}{AH} = \dfrac{PQ}{AQ} = \dfrac{PP'}{AA'}$. Hence, the parallelograms $OA'AA''$ and $OP'PP''$ have equal areas, which implies the assertion that P lies on the hyperbola Z through A with OK and OH as asymptotes.

Finally, it is exactly when $DP \perp BC$ that the cevian $\triangle DEF$ is also the pedal triangle of P. The line through D that is perpendicular to BC will intersect Z in at most two points, which proves 1.

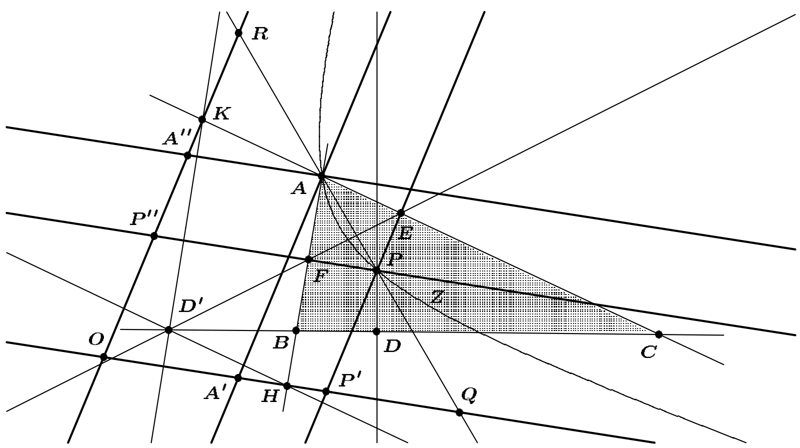
Editor's comment. Bataille presents the same argument from a projective point of view; it is very concise: Start with a point D on BC. As in Woo's argument, for each E on AC one determines the cevian triangle DEF, from which P is defined as the intersection point of the perpendiculars to the sides from E and F; define D^* to be the foot of perpendicular from P to BC. The mapping that takes D to D^* is a composition of perspectivities, and is thus a projectivity of line BC. Consequently, unless it is the identity map, it has at most two invariant (= double) points; at those positions where $D = D^*$ the resulting point P is Cevic. We return to Woo and his treatment of part 2.

2. We will now see how to construct for a given point D on BC, those points E and E', F and F' for which the corresponding point P is Cevic (when such points exist). First, we solve a general construction problem: Given $\triangle OXY$ and a point A within $\angle XOY$ but not on XY, *construct points P*

and P' on XY such that line AP cuts lines OX, OY at Q, R with $AQ = PR$, and line AP' cuts lines OX, OY at Q', R' with $AQ' = P'R'$.

Solution. Draw line $X'AY' \| XY$, cutting lines OX, OY at X', Y'. Draw $AN \perp X'AY'$, cutting the semicircle on $X'Y'$ as diameter at N. Then $AN = \sqrt{X'A \times AY'}$. Next draw a line parallel to XY at distance AN from it, cutting the semicircle on XY as diameter possibly at two points J, J'. (Of course, J and J' may coincide or fail to exist.) Let P, P' be the projections of J, J' on XY. Let line AP cut lines OX, OY at Q, R. Let line AP' cut lines OX, OY at Q', R'. We now prove $AQ = PR$: Since $JP = AN$, $X'A \times AY' = XP \times PY$. Hence $\dfrac{QA}{QP} = \dfrac{X'A}{XP} = \dfrac{PY}{AY'} = \dfrac{RP}{RA}$. Therefore $QA = PR$. Similarly, $Q'A = P'R'$. However, if $X'A \times AY'$ is too large, making $AN > \dfrac{XY}{2}$, then J, J' and P, P' do not exist. This ends the construction.

Returning to the main problem, we refer to the diagram.



Given A, B, C, D we can construct D' and quadrilateral $OHAK$. We can then construct up to two possible positions of P lying on the line through D perpendicular to BC such that AP cuts OH at Q and OK at R, with $PQ = AR$. Then $\dfrac{A''A}{P''P} = \dfrac{RA}{RP} = \dfrac{PQ}{AQ} = \dfrac{PP'}{AA'}$, which proves that P lies on the hyperbola. Then DEF is the pedal triangle of P as well as a cevian triangle of $\triangle ABC$, and P is Cevic as desired. However, for some positions of D it is possible that P cannot be constructed, in which case no Cevic point exists.

Editor's comment. Most solvers used algebra to obtain a quadratic equation for $\mu = \dfrac{CE}{CA}$, whose zeros give the positions of E for which P is Cevic. Dergiades wrote his equation in terms of $\lambda = \dfrac{BD}{BC}$ and the sides a, b,

c of $\triangle ABC$:

$$p\mu^2 + q\mu + r \;=\; 0\,,$$

where $\left\{\begin{array}{l} p = 2b^2 - 4b^2\lambda \\ q = -a^2 - 3b^2 + c^2 + 4(a^2+b^2)\lambda - 4a^2\lambda^2 \\ r = a^2 + b^2 - c^2 + (-3a^2 - b^2 + c^2)\lambda + 2a^2\lambda^2 \end{array}\right.$.

He gave explicit examples of the various possibilities using the right triangle whose sides are $a = 6$, $b = \sqrt{15}$, $c = \sqrt{21}$. The reader can check for himself that when

$\lambda = \frac{1}{3}$, $10\mu^2 - 8\mu + 4 = 0$
has no real solution, and there can be *no* Cevic point.

$\lambda = \frac{1}{2}$, $6\mu - 3 = 0$
has one real solution, and there is a *unique* Cevic point.

$\lambda = \frac{7}{16}$, $120\mu^2 + 54\mu - 27 = 0$
has two real solutions, and there are *two* Cevic points.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; and the proposer.

Indeed, the proposer's solution finally arrived with an apology, a correction, and a formula that purports to give the probability that he gets the theorem wrong on the first try. Alas, the formula can take values greater than 1.

2509. [2000 : 46] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Show that there are infinitely many pairs of distinct natural numbers, n and k such that $\gcd(n! - 1, k! - 1) > 1$.

Solution by Charles Diminnie and Trey Smith, Angelo State University, San Angelo, TX, USA.

First note that if p is a prime, then by Wilson's Theorem we have $(p-1)(p-2)! = (p-1)! \equiv -1 \equiv p - 1 \pmod{p}$ and hence, $(p-2)! \equiv 1 \pmod{p}$, since $\gcd(p, p-1) = 1$.

[Ed: This is actually well known and in fact is the result usually established first in the proof of Wilson's Theorem.]

Let $k > 3$ be any even integer. Then $k! - 1 > 1$ and $k + 2$ is not a prime. Let p be any prime divisor of $k! - 1$. Then $k \neq p - 2$ and $\gcd((p-2)! - 1, k! - 1) \geq p > 1$.

Also solved by the AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, KS, USA; and the proposer.

2510. [2000 : 46] *Proposed by Ho-joo Lee, student, Kwangwoon University, South Korea.*

In $\triangle ABC$, $\angle ABC = \angle ACB = 80°$ and P is on the line segment AB such that $AP = BC$. Find $\angle BPC$.

I. *Solution by Geoffrey A. Kandall, Hamden, CT, USA.*

Let $AP = BC = r$, $PC = s$ and $\angle BPC = \theta$. We have $\angle PAC = 20°$ and $\angle ACP = \theta - 20°$. Using the Law of Sines twice, we have

$$\frac{\sin(\theta - 20°)}{\sin(20°)} = \frac{r}{s} = \frac{\sin(\theta)}{\sin(80°)}.$$

Hence,

$$\frac{\sin(\theta - 20°)}{\sin(\theta)} = \frac{\sin(20°)}{\cos(10°)}.$$

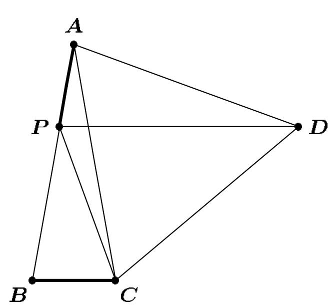
It is easy to see that $\theta = 30°$ is a solution of this equation. It is the only solution since the function

$$f(\theta) = \frac{\sin(\theta - 20°)}{\sin(\theta)} = \cos(20°) - \sin(20°)\cot(\theta)$$

is strictly increasing for $0° < \theta < 180°$.

II. Solution by Jens Windelband, Hegel-Gymnasium, Magdeburg, Germany and Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

We draw a point D such that $\triangle DAP$ is congruent to given $\triangle ABC$. Since $\measuredangle BAC = 20\,°$ and $\measuredangle DAP = \measuredangle ABC = 80\,°$, we find $\measuredangle DAC = 60\,°$.



Moreover, $DA = AC$, from which we conclude that $\triangle DAC$ is an isosceles triangle with an included angle of $60\,°$; that is, it is equilateral. Hence points A, P, C lie on a circle with centre D. For chord AP of this circle we find by the central angle theorem $\measuredangle ADP = 20\,° = 2\measuredangle ACP$, or $\measuredangle ACP = 10\,°$. Finally, in $\triangle ACP$ the sum of the measures of the two non-adjacent interior angles $\measuredangle PAC + \measuredangle ACP = 30\,°$ equals the measure of the exterior angle $\measuredangle BPC = 30\,°$.

Also solved (using trigonometry) by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY J. PAN; student, East York C.I., Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; JONATHAN STOREY, student, Nottingham

High School, Nottingham, UK; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; (using only geometry) by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; KEE-WAI LAU, Hong Kong, China; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and (two solutions, both methods) by the proposer. There was one incomplete solution.

Benito and Frenández made use of an eight sided polygon, which has all sorts of nice properties. Perz and Sinefakopoulous pointed out that this problem is known, being essentially problem 2 of the A-level Junior paper of the Autumn round of the 1991 Tournament of the Towns. There is a solution on page 123, by Andy Liu, in International Mathematics Tournament of the Towns, *1989–1993, edited by P.J. Taylor, Australian Mathematics Trust, 1994.*

2511. [2000 : 46] *Proposed by Ho-joo Lee, student, Kwangwoon University, South Korea.*

In $\triangle ABC$, $\angle ABC = 60^\circ$ and $\angle ACB = 70^\circ$. Point D is on the line segment BC such that $\angle BAD = 20^\circ$. Prove that $AB + BD = AD + DC$.

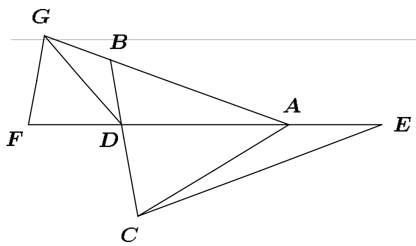
I. *Solution by David Loeffler, student, Cotham School, Bristol, UK.*

Let us scale the triangle so that $AB = 1$. Clearly, $\angle BDA = 100^\circ$. Thus, using the Sine Law on $\triangle ABD$, we have $BD = \dfrac{\sin(20^\circ)}{\sin(100^\circ)}$, and $AD = \dfrac{\sin(60^\circ)}{\sin(100^\circ)}$. A further application of the Sine Law to $\triangle ACD$ gives $DC = AD\dfrac{\sin(30^\circ)}{\sin(70^\circ)} = \dfrac{\sin(60^\circ)}{\sin(100^\circ)}\left(\dfrac{\sin(30^\circ)}{\sin(70^\circ)}\right)$. Now, we have

$$\begin{aligned}
AB + BD &= 1 + \frac{\sin(20^\circ)}{\sin(100^\circ)} = \frac{\sin(70^\circ)(\sin(100^\circ) + \sin(20^\circ))}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{2\sin(70^\circ)\sin(60^\circ)\cos(40^\circ)}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{2\cos(20^\circ)\sin(60^\circ)\sin(50^\circ)}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{\sin(60^\circ)(2\sin(50^\circ)\cos(20^\circ))}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{\sin(60^\circ)(\sin(70^\circ) + \sin(30^\circ))}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{\sin(60^\circ)}{\sin(100^\circ)}\left(1 + \frac{\sin(30^\circ)}{\sin(70^\circ)}\right) = AD + DC
\end{aligned}$$

as required.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*



On the extension of AD, we take $AE = DF = DC$, and on AB, we take $AG = AF$.

Since $\angle DAC = 30°$ and $\angle ADC = 80°$ from problem 2510, we conclude that $\triangle EDC$ is isosceles, with $\angle DEC = \angle GAF = 20°$, which means that $\triangle EDC$ is congruent to $\triangle AGF$.

Hence, $GF = DC = DF$, and further, $\angle GDF = 50°$, so that $\angle BDG = 30°$. Thus, $\angle BGD = 30°$, so that $BG = BD$.

Thus, $AB + BD = AG = AF = AD + DC$.

Also solved by AUSTRIAN IMO–TEAM 2000; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; HENRY J. PAN, student, East York C.I., Toronto, Ontario; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (two solutions, one †); HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JONATHAN STOREY, student, Nottingham High School, Nottingham, UK; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; ALBERT WHITE, St. Bonaventure University, St. Bonaventure, NY, USA; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA †; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer. All solvers, except those marked † used trigonometry.

Woo commented that "elegance means avoiding trigonometry".