PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (*) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard 8½"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than 1 May 2001. They may also be sent by email to crux-editors@cms.math.ca. (It would be appreciated if email proposals and solutions were written in \LaTeX.) Graphics files should be in \LaTeX format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.

2589. Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.

For \( n = 2, 3, \ldots \), evaluate \( \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n}{k-1} \).

2590. Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.

For \( n = 1, 2, \ldots \), prove that \( \prod_{k=1}^{n} \left( \binom{n}{k} \right)^2 \leq \left( \frac{1}{n+1} \binom{2n}{n} \right)^n \).

2591. Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.

Two players, \( A \) and \( B \), each toss \( n \) fair coins, and two other players, \( C \) and \( D \), toss \( n - 1 \) and \( n + 1 \) fair coins, respectively.

For each \( n = 2, 3, \ldots \), prove that the two events:

\( A \) gets exactly one head more than \( B \)

and

\( C \) and \( D \) get exactly the same number of heads

are equally likely.

Find the probability of these events.

Describe all numbers, which can be represented in the form of \(a^3 + b^3 + c^3 + d^3\), where \(a, b, c, d\) are natural numbers.

2593. Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.

Let \(S(a)\) denote the sum of the digits of the natural number \(a\). Let \(k\) and \(n\) be natural numbers with \((n, 3) = 1\). Prove that there exists a natural number \(m\) which is divisible by \(n\) and \(S(m) = k\) if either

(a) \(k > n - 2\); or

(b) \(k > s^2(n) + 7S(n) - 9\).


Given a point \(M\) inside the triangle \(ABC\) (see diagram), prove that

\[
\min(MA, MB, MC) + MA + MB + MC < AB + BC + AC.
\]

2595. Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.

Given that \(M\) and \(N\) are points inside the triangle \(ABC\) such that \(\angle MAB = \angle NAC\) and \(\angle MBA = \angle NBC\), prove that

\[
\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.
\]

2596. Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA.

Write \(r \ll s\) if there is an integer \(k\) satisfying \(r < k < s\). Find, as a function of \(n\) \((n \geq 2)\) the least positive integer \(k\) satisfying

\[
\frac{k}{n} \ll \frac{k}{n-1} \ll \frac{k}{n-2} \ll \ldots \ll \frac{k}{2} \ll k.
\]
2597. Proposed by Michael Lambrou, University of Crete, Crete, Greece.

Let $P$ be an arbitrary interior point of an equilateral triangle $ABC$.

Prove that $|\angle PBC - \angle PCB| \leq \arcsin \left(2 \sin \left(\frac{|\angle PAB - \angle PAC|}{2}\right)\right) - \left(\frac{|\angle PAB - \angle PAC|}{2}\right) \leq |\angle PAB - \angle PAC|$.

Show that the left inequality cannot be improved in the sense that there is a position $Q$ of $P$ on the ray $AP$ giving an equality.

(Thus, the inequality in 2255 is improved.)

2598. Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.

Suppose that $AD$, $BE$ and $CF$ are the internal angle bisectors of $\triangle ABC$, with $D$ on $BC$, $E$ on $CA$ and $F$ on $AB$. Write $a = BC$, $b = CA$, $c = AB$, $x = AE$ and $y = AF$. We are given that $x + y = a$. Prove that:

(a) $a^2 = bc$;
(b) $\frac{1}{x} - \frac{1}{y} = \frac{1}{b} - \frac{1}{c}$;
(c) $\frac{1}{x} + \frac{1}{y} = \left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2$;
(d) $AD < c$.

2599. Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.

Let $P$ be a point inside the triangle $ABC$ and let $AP$, $BP$, $CP$ meet the sides $BC$, $CA$, $AB$ at $L$, $M$, $N$, respectively. Show that the following two conditions are equivalent:

$$\frac{1}{AP} + \frac{1}{PL} = \frac{1}{BP} + \frac{1}{PM} = \frac{1}{CP} + \frac{1}{CN};$$

$$\angle APN = \angle NPB = \angle BPL = \angle LPC = \angle CPM = \angle MPB = 60^\circ.$$ 

2600. Proposed by Svetlozar Doichev, Stara Zagora, Bulgaria.

Find all real numbers $x$ such that, if $a$ and $b$ are the lengths of sides of a triangle with medians from the mid-points of these sides of lengths $m_a$ and $m_b$, respectively, then the equalities $a + xm_a = b + xm_b$ and $a = b$ are equivalent.