THE SKOLIAD CORNER

No. 50

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Last issue we gave some of the solutions to the problems of the Preliminary Round of the British Columbia Colleges Senior High School Contest for 2000. Here, thanks to Jim Totten, The University College of the Cariboo, one of the organizers, are the rest of the "official" solutions.

6. While 10 pin bowling, Sam left 3 pins standing which formed the vertices of an equilateral triangle. How many such equilateral triangles are possible?

(a) 15  (b) 14  (c) 12  (d) 10  (e) none of these

Solution. The answer is (a). Number the pins as shown in the diagram on the left. There is then one large equilateral triangle with 4 pins on a side, namely the one with vertices numbered (1, 7, 10). There are also three equilateral triangles with 3 pins on a side, namely the ones whose vertices are numbered (1, 4, 6), (2, 7, 9), and (3, 8, 10). The equilateral triangles with 2 pins on a side come in two distinct orientations, one with a single vertex above the horizontal base and one with a single vertex below the horizontal base. For the first type we have six such: (1, 2, 3), (2, 4, 5), (3, 5, 6), (4, 7, 8), (5, 8, 9) and (6, 9, 10). For the second type we have only three such: (2, 3, 5), (4, 5, 8), and (5, 6, 9). This gives us a total of 15 equilateral triangles so far. However, there are two others which are skewed somewhat to the edges of the outer triangle: (2, 6, 8) and (3, 4, 9), which gives us a total of 17 equilateral triangles.

7. If I place a 6 cm × 6 cm square on a triangle, I can cover up to 60% of the triangle. If I place the triangle on the square, I can cover up to $\frac{2}{3}$ of the square. What is the area, in cm$^2$, of the triangle?

(a) 22$\frac{1}{2}$  (b) 24  (c) 36  (d) 40  (e) 60

Solution. The answer is (d). The critical idea here is to recognize that when the square covers as much of the triangle as possible, the triangle will also cover as much of the square as possible, and that at this point the amount of triangle covered is the same as the amount of square covered. Let $A$ be the area of the triangle. Then $0.6A = \frac{2}{3} \cdot 36$, or $A = 40$ cm$^2$. 
8. Two circles, each with radius 10 cm, are placed so that they are tangent to each other and a straight line. A smaller circle is nestled between them so that it is tangent to the larger circles and the line. What is the radius, in centimetres, of the smaller circle?

(a) \(\sqrt{10}\)  (b) 2.5  (c) \(\sqrt{2}\)  (d) 1  (e) none of these

Solution. The answer is (b). Let \(A\) be the centre of one of the large circles, let \(B\) be the point of contact between the two large circles and let \(C\) be the centre of the small circle. Then \(AB \perp BC\), \(AC = 10 + r\) and \(BC = 10 - r\). From the Theorem of Pythagoras, we have

\[
(10 + r)^2 = 10^2 + (10 - r)^2, \\
100 + 20r + r^2 = 100 + 100 - 20r + r^2, \\
40r = 100, \\
r = 2.5.
\]

9. Arrange the following in ascending order:

\[2^{5555}, 3^{3333}, 6^{2222}.\]

(a) \(2^{5555}\), \(3^{3333}\), \(6^{2222}\)  (b) \(2^{5555}\), \(6^{2222}\), \(3^{3333}\)  (c) \(6^{2222}\), \(3^{3333}\), \(2^{5555}\)  (d) \(3^{3333}\), \(6^{2222}\), \(2^{5555}\)  (e) \(3^{3333}\), \(2^{5555}\), \(6^{2222}\)

Solution. The answer is (e). See #10 of the Junior paper – solution on [2000 : 347].

10. Given that \(0 < x < y < 20\), the number of integer solutions \((x, y)\) to the equation \(2x + 3y = 50\) is:

(a) 25  (b) 16  (c) 8  (d) 5  (e) 3

Solution. The answer is (e). Clearly \(y\) must be even in order to get integer solutions. The largest possible value for \(y\) is 16 since we must have \(x > 0\). When \(y = 16\), we have \(x = 1\). Thus, \((x, y) = (1,16)\) is a solution. Let us consider successively smaller (even) values for \(y\): \((x, y) = (4,14), (7,12), (10,10), etc. However, the solution (10,10) and any further ones do not satisfy \(y > x\). Thus, we are left with the solutions \((x, y) = (1,16), (4,14), (7,12)\).
11. Suppose \( A, B, \) and \( C \) are positive integers such that
\[
\frac{24}{5} = A + \frac{1}{B + \frac{1}{C+1}}.
\]

The value of \( A + 2B + 3C \) equals:

(a) 9  (b) 12  (c) 15  (d) 16  (e) 20

**Solution.** The answer is (c). Since \( B \) and \( C \) are positive integers, we see that \( B + 1/(C + 1) > 1 \), whence its reciprocal is smaller than 1. Therefore, \( A \) must represent the integer part of \( 24/5 \); that is, \( A = 4 \). Then we have
\[
\frac{4}{5} = \frac{1}{B + \frac{1}{C+1}} \quad \text{or} \quad \frac{5}{4} = B + \frac{1}{C+1}.
\]

For exactly the same reason as above we see that \( B \) must be the integer part of \( 5/4 \); that is, \( B = 1 \). Then \( 1/4 = 1/(C + 1) \), which implies that \( C = 3 \). Then
\[
A + 2B + 3C = 4 + 2(1) + 3(3) = 15.
\]

12. A box contains \( m \) white balls and \( n \) black balls. Two balls are removed randomly without replacement. The probability one ball of each colour is chosen is:

(a) \( \frac{mn}{(m+n)(m+n-1)} \)  (b) \( \frac{mn}{(m+n)^2} \)  (c) \( \frac{2mn}{(m+n-1)(m+n-2)} \)  (d) \( \frac{2mn}{(m+n)(m+n-1)} \)  (e) \( \frac{m(m+1)}{(m+n)(m+n-1)} \)

**Solution.** The answer is (d). The number of ways of choosing 2 balls (without replacement) from a box with \( m+n \) balls is \( \binom{m+n}{2} \). The number of ways of choosing 1 white ball is \( m \), the number of ways of choosing 1 black ball is \( n \). Thus, the probability that one ball of each colour is chosen is:
\[
\frac{mn}{\binom{m+n}{2}} = \frac{mn}{(m+n)(m+n-1)/2} = \frac{mn}{(m+n)(m+n-1)/2} = \frac{2mn}{(m+n)(m+n-1)}.
\]

13. If it takes \( x \) builders \( y \) days to build \( z \) houses, how many days would it take \( q \) builders to build \( r \) houses? Assume these builders work at the same rate as the others.

(a) \( \frac{qy}{xz} \)  (b) \( \frac{r}{qz} \)  (c) \( \frac{qz}{rxy} \)  (d) \( \frac{sz}{qz} \)  (e) \( \frac{r}{qz} \)

**Solution.** The answer is (d). Since \( x \) builders build \( z \) houses in \( y \) days, we see that \( x \) builders build \( z/y \) houses per day, or 1 builder builds \( z/(xy) \) houses per day. Hence \( q \) builders build \( qz/(xy) \) houses per day. Let \( s \) be the number of days needed for \( q \) builders to build \( r \) houses. Then \( q \) builders build \( r/s \) houses per day. Equating these we get \( qz/(xy) = r/s \), whence \( s = rxy/(qz) \) days.
14. If \( x^2 + xy + x = 14 \) and \( y^2 + xy + y = 28 \), then a possible value for the sum of \( x + y \) is:

(a) \(-7\)  (b) \(-6\)  (c) \(0\)  (d) \(1\)  (e) \(3\)

*Solution.* The answer is (a). Adding the two given equations gives:

\[
\begin{align*}
x^2 + 2xy + y^2 + x + y &= 42; \\
(x + y)^2 + (x + y) &= 42 \\
(x + y - 6)(x + y + 7) &= 0.
\end{align*}
\]

Thus, \( x + y = 6 \) or \( x + y = -7 \).

15. Two congruent rectangles each measuring 3 cm \( \times \) 7 cm are placed as in the figure. The area of overlap (shaded), in cm\(^2\), is:

(a) \( \frac{87}{7} \)  (b) \( \frac{20}{7} \)  (c) \( \frac{20}{7} \)  (d) \( \frac{21}{7} \)  (e) none of these

*Solution.* The answer is (a). All the unshaded triangles in the diagram below are right-angled and thus are congruent. By the Theorem of Pythagoras we have

\[
\begin{align*}
x^2 &= (7 - x)^2 + 3^2 = 49 - 14x + x^2 + 9, \\
14x &= 58 \quad \text{or} \quad x = \frac{29}{7}.
\end{align*}
\]

The area of the shaded parallelogram is \( 3x = \frac{87}{7} \) cm\(^2\).

That completes the *Skoliad Corner* for this issue. Send me suitable contest materials and suggestions for the future of the *Corner.*