SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

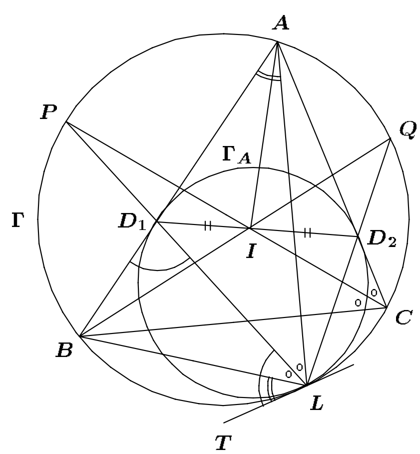
2464. [1999 : 366] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Given triangle ABC with circumcircle Γ, the circle Γ_A touches AB and AC at D_1 and D_2, and touches Γ internally at L. Define E_1, E_2, M, and F_1, F_2, N in a corresponding way. Prove that

(a) AL, BM, CN are concurrent;

(b) D_1D_2, E_1E_2, F_1F_2 are concurrent, and that the point of concurrency is the incentre of $\triangle ABC$.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*



Let P and Q be the second points of intersection of LD_1 and LD_2 with Γ, respectively. As shown in the figure above, let LT be the common tangent to Γ and Γ_A. Then

$$\angle BD_1L = \angle TLD_1 \quad \text{and} \quad \angle TLB = \angle BAL .$$

Hence, $\angle BLP = \angle TLD_1 - \angle TLB = \angle BD_1L - \angle BAL = \angle ALP$ so that $\angle ACP = \angle ALP = \angle BLP = \angle BCP$. Therefore, CP is the bisector of $\angle ACB$. Similarly, BQ is the bisector of $\angle ABC$.

Let I be the intersection of CP and BQ. Then I is the incentre of $\triangle ABC$. Since hexagon $ABQLPC$ is inscribed in Γ, by Pascal's Theorem D_1, I and D_2 are collinear. Thus, D_1D_2 passes through the incentre I of $\triangle ABC$. Similarly, E_1E_2 and F_1F_2 pass through I. Therefore, D_1D_2, E_1E_2 and F_1F_2 are concurrent at the incentre of $\triangle ABC$, and part (b) is proved.

Since I is the incentre of $\triangle ABC$, we have $\angle D_1AI = \angle D_2AI$. Since AD_1 and AD_2 are tangent to Γ_A, we have $AD_1 = AD_2$. Thus, $D_1I = D_2I$ and $AI \perp D_1D_2$. Since LD_1 and LD_2 are bisectors of $\angle ALB$ and $\angle ALC$, respectively, we have

$$\frac{BL}{BD_1} = \frac{AL}{AD_1} = \frac{AL}{AD_2} = \frac{CL}{CD_2}.$$

Thus, we get

$$\frac{BL}{CL} = \frac{BD_1}{CD_2}. \tag{1}$$

Since $\frac{1}{2}\angle BAC + \frac{1}{2}\angle ABC + \frac{1}{2}\angle ACB = 90°$, we have $\angle D_1AI + \angle D_1BI + \angle D_2CI = 90°$, so that

$$\angle D_1IB = \angle AD_1I - \angle D_1BI = (90° - \angle D_1AI) - \angle D_1BI = \angle D_2CI.$$

Similarly, we have $\angle D_1BI = \angle D_2IC$. Hence, we have $\triangle BD_1I \sim \triangle ID_2C$. Thus,

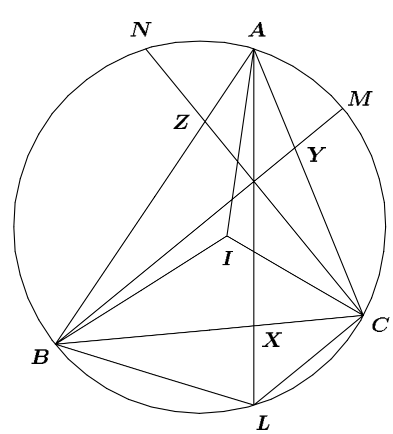
$$\frac{BD_1}{ID_2} = \frac{ID_1}{CD_2} = \frac{BI}{CI}.$$

Thus, we have

$$\frac{BD_1}{CD_2} = \frac{BD_1}{ID_2} \cdot \frac{ID_1}{CD_2} = \frac{BI^2}{CI^2}. \tag{2}$$

From (1) and (2) we have

$$\frac{BL}{CL} = \frac{BI^2}{CI^2}. \tag{3}$$



Now let X be the intersection of AL with BC (see diagram above). Since $\angle ABL + \angle ACL = 180^\circ$, we have from (3)

$$\begin{aligned}\frac{BX}{XC} &= \frac{[ABL]}{[ACL]} = \frac{\frac{1}{2}AB \cdot BL \cdot \sin\angle ABL}{\frac{1}{2}AC \cdot CL \cdot \sin\angle ACL} \\ &= \frac{AB}{AC} \cdot \frac{BL}{CL} = \frac{AB}{AC} \cdot \frac{BI^2}{CI^2}\,,\end{aligned}$$

where $[PQR]$ denotes the area of triangle PQR. Let Y and Z be the points of intersection of BM and CN with AC and AB, respectively. Then we can similarly show:

$$\frac{CY}{YA} = \frac{BC}{BA} \cdot \frac{CI^2}{AI^2} \qquad \text{and} \qquad \frac{AZ}{ZB} = \frac{CA}{CB} \cdot \frac{AI^2}{BI^2}\,.$$

Therefore,

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{AB}{AC} \cdot \frac{BC}{BA} \cdot \frac{CA}{CB} \cdot \frac{BI^2}{CI^2} \cdot \frac{CI^2}{AI^2} \cdot \frac{AI^2}{BI^2} = 1\,.$$

By Ceva's Theorem AX, BY and CZ are concurrent. This implies that AL, BM and CN are concurrent.

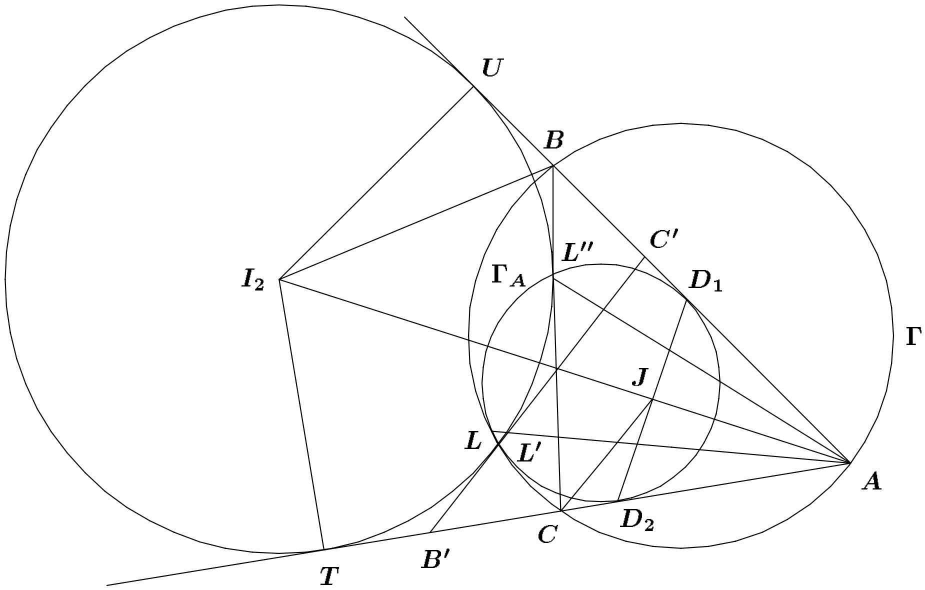
II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let $AB'C'$ be the symmetric triangle of triangle ABC relative to the bisector of angle A. Then both triangles have the same incircle and excircle in the angle A. Let I_a be the centre of the excircle which touches AB at U, BC at L'', CA at T, and $B'C'$ at L'.

We apply inversion with pole A and power $AB \cdot AC$. The inverse point of B is C' and the inverse point of C is B'. The inverse of the circle Γ is therefore the line $B'C'$ and hence, since the circle Γ_A touches AB, AC and Γ, then its inverse is a circle that touches the sides of triangle $AB'C'$. (See figure on page 435.)

If Γ_A were outside of Γ, then its inverse would be the incircle of $\triangle ABC$, but now the inverse of Γ_A is the excircle in the angle A. Thus, the inverse of D_1 is the point U, the inverse of D_2 is the point T, and the inverse of L is the point L'. Since L' is the symmetric point of L'', it follows that AL is the isogonal line of AL''. But it is known (F.G.-M. 1242, 1242a Gergonne-like theorems) that AL'' (and BM'' and CN'', which correspond to the other angles of $\triangle ABC$) passes through Nagel's point. Thus, the isogonals AL, BM and CN are also concurrent.

Now, AI_a is the bisector of angle A and intersects D_1D_2 at J. The inverse of the line D_1D_2 is a circle which passes through A, U and T, and since the points A, U, I_a, T are concyclic, then the inverse of J is the point I_a.



Hence,

$$\begin{aligned} AJ \cdot AI_a = AB \cdot AC &\implies \frac{AJ}{AC} = \frac{AB}{AI_a} \\ &\implies \triangle AJC \sim \triangle ABI_a \\ &\implies \angle ACJ = \angle AI_aB = \frac{C}{2}. \end{aligned}$$

Thus, the point J is the incentre of $\triangle ABC$, and D_1D_2, E_1E_2, F_1F_2 are concurrent.

Other consequences:

1. It is obvious that if Γ_A were outside of Γ, then the results are the same, but the point of concurrency would be the excentre instead of the incentre.
2. The inverse of the circumcircles of $\triangle ABD_2$ and $\triangle ACD_1$ are the lines $C'T$ and $B'U$, respectively (which are concurrent with AL', F.G.-M. 1242), and hence, these circumcircles intersect AL at a point that is the inverse of the symmetric of the adjoint of Gergonne's point of $\triangle ABC$ (F.G.-M. 1242).
3. If r, r_1, r_2, r_3 are the radii of the incircle and the excircles of $\triangle ABC$, s is the semiperimeter of $\triangle ABC$ and R_1, R_2, R_3 are the radii of Γ_A, Γ_B, Γ_C, respectively, then since the power of A to the excircle is $AT^2 = s^2$, then

$$R_1 = \frac{bc}{s^2} \cdot r_1 = \frac{bc}{s^2} \cdot \frac{sr}{s-a} = \frac{r}{\cos^2(A/2)},$$

and hence,

$$R_1 + R_2 + R_3 = r\left(\frac{1}{\cos^2(A/2)} + \frac{1}{\cos^2(B/2)} + \frac{1}{\cos^2(C/2)}\right)$$
$$\geq \frac{3r}{\cos^2((A+B+C)/6)} = 4r\,,$$

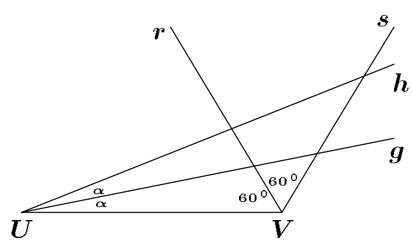
by Jensen's inequality, since the function $f(x) = 1/\cos^2 x$ is convex.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Janous observes that Bankoff (Mixtilinear Adventure, CRUX 9 (1983), pp. 2–7) *has shown that the incentre I is even the mid-point of all three line segments; he further mentions a forthcoming note by Paul Yiu (to appear in the* American Mathematical Monthly), *entitled* Mixtilinear Incircles, *in which he shows that AL, BM and CN are concurrent at the external centre of similitude of the circumcircle and the incircle of $\triangle ABC$.*

2467. [1999 : 367] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given is a line segment UV and two rays, r and s, emanating from V such that $\angle(UV, r) = \angle(r, s) = 60^\circ$, and two lines, g and h, on U such that $\angle(UV, g) = \angle(g, h) = \alpha$, where $0 < \alpha < 60^\circ$.



The quadrilateral $ABCD$ is determined by g, h, r and s. Let P be the point of intersection of AB and CD.

Determine the locus of P as α varies in $(0, 60^\circ)$.

(Editor's note: As was pointed out by several solvers, P should be the intersection of AC and BD.)

Solution by Nikolaos Dergiades, Thessaloniki, Greece and Michael Lambrou, University of Crete, Crete, Greece (amalgamated and adapted by the editors)

We solve a more general problem by replacing the condition:

"$\angle(UV, r) = \angle(r, s) = 60^\circ$"

by

"UV is the external bisector of $\angle(r, s)$",

and replacing the condition

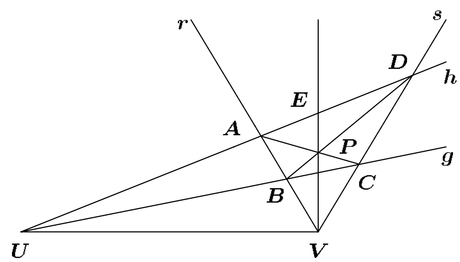
"$\angle(UV, g) = \angle(g, h) = \alpha$"

by

"the lines g and h each meet the rays r and s".

In this case the locus is (part of) the internal bisector of $\angle(r, s)$. Thus, for the problem as given originally, the locus is (part of) the perpendicular to UV at V.

Let $A = h \cap r$, $B = g \cap r$, $C = g \cap s$, $D = h \cap s$, and let $P = AC \cap BD$. Let VP meet the line h at E.



In triangle VAD we have from Ceva's Theorem:

$$\frac{\overline{EA}}{\overline{ED}} \cdot \frac{\overline{BV}}{\overline{BA}} \cdot \frac{\overline{CD}}{\overline{CV}} = -1\,, \tag{1}$$

and from Menelaus' Theorem (relative to U, B, C):

$$\frac{\overline{UA}}{\overline{UD}} \cdot \frac{\overline{BV}}{\overline{BA}} \cdot \frac{\overline{CD}}{\overline{CV}} = 1\,. \tag{2}$$

From (1) and (2) we have

$$\frac{\overline{UA}}{\overline{UD}} = -\frac{\overline{EA}}{\overline{ED}}\,,$$

which means that the points U, E are harmonic conjugates to A, D. Since VU is the external bisector of $\angle AVD$, (that is, $\angle(r, s)$), we conclude that VE is the internal bisector.

Finally, as g turns so that $\angle(UV, g)$ runs from 0 until g is parallel to s and as h turns so that $\angle(UV, h)$ runs from 0 until h is parallel to r (as in the original problem), excepting the case when h is parallel to s, the point P sweeps the said bisector.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

2469. [1999 : 367] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a triangle ABC, consider the altitude and the angle bisector at each vertex. Let P_A be the intersection of the altitude from B and the bisector at C, and Q_A the intersection of the bisector at B and the altitude at C.

These determine a line P_AQ_A. The lines P_BQ_B and P_CQ_C are analogously defined. Show that these three lines are concurrent at a point on the line joining the circumcentre and the incentre of triangle ABC. Characterize this point more precisely.

[Ed. the solution is combined with that of the next problem.]

2470. [1999 : 368] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a triangle ABC, consider the median and the angle bisector at each vertex. Let P_A be the intersection of the median from B and the bisector at C, and Q_A the intersection of the bisector at B and the median at C. These determine a line P_AQ_A. The lines P_BQ_B and P_CQ_C are analogously defined. Show that these three lines are concurrent. Characterize this intersection more precisely.

Combined generalization of **2469** *and* **2470** *devised independently by Nikolaos Dergiades, Thessaloniki, Greece and by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Given $\triangle ABC$, its incentre I, and a point X not on the sides of the triangle, define

$$\begin{array}{lll} P_A = BX \bigcap CI & P_B = CX \bigcap AI & P_C = AX \bigcap BI \\ Q_a = BI \bigcap CX & Q_B = CI \bigcap AX & Q_C = AI \bigcap BX, \end{array}$$

and prove that

(i) P_AQ_A, P_BQ_B, P_CQ_C are concurrent at a point S, and

(ii) S lies on the line joining I to the isogonal conjugate of X.

Solution to (i) by Peter Y. Woo, Biola University, La Mirada, CA, USA.

The concurrency is an immediate consequence of Pappus' theorem: Q_A, P_B, X are three points on the line CX while Q_B, P_A, I are three points on CI. The cross-joins intersect at $S = P_AQ_A \bigcap Q_BP_B$, $P_C = IQ_A \bigcap Q_BX$, and $Q_C = P_AX \bigcap IP_B$, whose collinearity implies that S is on the axis P_CQ_C, which proves (i).

Editor's comment. Note the appropriateness of an alternative statement of Pappus' theorem — "If two triangles are doubly perspective, they are triply perspective." Here we are given that the triangles of P-points and of Q-points are perspective from both I and X, so our conclusion is that they are perspective from a third point S.

Solution to (ii) by Michel Bataille, Rouen, France (whose separate solutions to **2469** and **2470** have been combined by the editor).

We shall use trilinear coordinates relative to $\triangle ABC$ with $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ and $I(1,1,1)$. Let the coordinates of the given point X be $(1/u, 1/v, 1/w)$ so that its isogonal conjugate is the point $X'(u,v,w)$. Let P_A have the unknown coordinates (x,y,z) that we want to evaluate in terms of the given u, v and w.

P_A is on BX so that $\det\begin{pmatrix} x & 0 & \frac{1}{u} \\ y & 1 & \frac{1}{v} \\ z & 0 & \frac{1}{w} \end{pmatrix} = 0$, which gives $\dfrac{x}{w} = \dfrac{z}{u}$.

P_A is on CI so that $\det\begin{pmatrix} x & 0 & 1 \\ y & 0 & 1 \\ z & 1 & 1 \end{pmatrix} = 0$, which gives $y = x$.

From this we get $P_A(w, w, u)$ and, by suitable permutations, $Q_A(v, u, v)$, $P_B(v, u, u)$, $Q_B(w, w, v)$, $P_C(v, w, v)$ and $Q_C(w, u, u)$. By adding P_A to Q_A, and so forth, we see that P_AQ_A, P_BQ_B, P_CQ_C concur at the point S that satisfies

$$S = (v + w, w + u, u + v) .$$

By adding the coordinates of S to $X'(u, v, w)$ we see that S, X' and the incentre $I(1, 1, 1)$ are collinear, as desired.

In **2469** we are given that X is the orthocentre $H(\sec A, \sec B, \sec C)$; its isogonal conjugate is the circumcentre $X'(u, v, w) = O(\cos A, \cos B, \cos C)$. Here S is the point on IO with coordinates $(\cos B + \cos C, \cos C + \cos A, \cos A + \cos B)$; this point is X_{65} in Kimberling's list [Ed. Clark Kimberling, Central Points and Central Lines in the Plane of a Triangle, *Math. Mag.* **67**:3 (June 1994) 163-187. Alternatively, one can consult his web page: http://cedar.evansville.edu/~ck6/encyclopedia/].

Editor's comments. Janous and Yiu both show that I lies between O and S and divides that segment in the ratio $R : r$. Most solvers noted that S is the isogonal conjugate of the Schiffler point (X_{21} in Kimberling's list), which Kimberling named for the proposer of problem **1018** [1986: 150-152].

In **2470** we are given $X = G(1/a, 1/b, 1/c)$ whose isogonal conjugate is the Lemoine point $L(a, b, c)$. Here $S = (b + c, c + a, a + b)$, which is "the simplest unnamed centre" X_{37} in Kimberling's list.

[*Comment*. Janous and Yiu show instead that the centroid G lies between S and the isotomic conjugate of the incentre, and it divides that segment in the ratio $1 : 2$. This fact also appears in Kimberling's table 3.]

Both problems were also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

2471. [1999 : 368] *Proposed by Vedula N. Murty, Dover, PA, USA.*

For all integers $n \geq 1$, determine the value of $\displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}k}{k+1}\binom{n+1}{k}$.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

Let S_n, $n \geq 1$, be the sum in question. It is straightforward to check that the equality

$$\frac{k}{k+1}\binom{n+1}{k} = \binom{n+1}{k} - \frac{1}{n+2}\binom{n+2}{k+1}$$

holds for $k = 1, 2, \ldots, n$. Applying the above equality and the Binomial Theorem, we find

$$\begin{aligned} S_n &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k} - \frac{1}{n+2}\sum_{k=1}^{n}(-1)^{k-1}\binom{n+2}{k+1} \\ &= -(1-1)^{n+1} + 1 - (-1)^n \\ &\qquad -\frac{1}{n+2}\left((1-1)^{n+2} - 1 + (n+2) - (-1)^n\right) \\ &= \frac{1-(-1)^n(n+1)}{n+2}\,. \end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOSÉ LUIS DIAZ, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HO-JOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; DIGBY SMITH, Mount Royal College, Calgary, Alberta; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

2472. [1999 : 368] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

If A, B, C are the angles of a triangle, prove that

$$\begin{aligned} &\cos^2\left(\frac{A-B}{2}\right)\cos^2\left(\frac{B-C}{2}\right)\cos^2\left(\frac{C-A}{2}\right) \\ &\qquad \geq \left(8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)^3 . \end{aligned}$$

I. *Editorial comment.*

As some solvers pointed out, this follows immediately from problem **2382** [1999 : 440]. In fact, in [1999 : 441], it is given that

$$\cos^2\left(\frac{B-C}{2}\right) \;\geq\; 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \quad (=\;\frac{2r}{R}),$$

and, by symmetry, the same is true for $\cos^2[(A-B)/2]$ and $\cos^2[(C-A)/2]$, and so we are done.

II. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

In ***CRUX*** **585** [1981 : 303] it was shown that

$$\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right) \;\geq\; 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\,. \qquad (1)$$

By squaring (1) and multiplying the right hand side of the resulting inequality by $8\sin(A/2)\sin(B/2)\sin(C/2)$, which is ≤ 1 (see O. Bottema et al, *Geometric Inequalities*, item 2.12), the desired inequality follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HO-JOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Dergiades and Lambrou also derive the stronger inequality (1), *though without referring to **CRUX** 585 to do it. Janous finds the even stronger inequality*

$$\prod\cos\left(\frac{B-C}{2}\right) \;\geq\; \left(9-8\prod\sin\frac{A}{2}\right)\prod\sin\frac{A}{2} \;\geq\; 8\prod\sin\frac{A}{2}\,.$$

In the other direction, Janous also proves

$$\prod\cos\left(\frac{B-C}{2}\right) \;\leq\; \frac{1}{2}+3\prod\sin\frac{A}{2}+8\left(\prod\sin\frac{A}{2}\right)^2\,.$$

Lambrou further connected this problem to another earlier CRUX problem, as follows. Put $A=\pi-2A_1$, etc.; then A_1, B_1, C_1 are the angles of an acute triangle, and (1) *becomes*

$$\prod\cos(B_1-A_1) \;\geq\; 8\prod\cos A_1\,.$$

Rewriting A_1 as A, etc., we get that (1), *over all triangles, is equivalent to the inequality*

$$\prod\cos(B-A)\geq 8\prod\cos A \qquad (2)$$

over all acute triangles. Now let O be the circumcentre of triangle ABC, let AO meet the circle BOC again at A', and define B' and C' similarly. Then the radius R_1 of circle BOC satisfies

$$R_1=\frac{R}{2\cos A}\,.$$

Letting O_1 be the centre of circle BOC, we find that $\angle O_1OA'=B-C$, so

$$OA'=2R_1\cos(B-C)=\frac{R\cos(B-C)}{\cos A}\,,$$

and cyclically for OB' and OC'. Thus (2) *is equivalent to*

$$OA'\cdot OB'\cdot OC'\geq 8R^3,$$

which is unused problem 13 from the 1996 IMO; see [1999 : 8] *for a solution.*

2473. [1999 : 368] *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

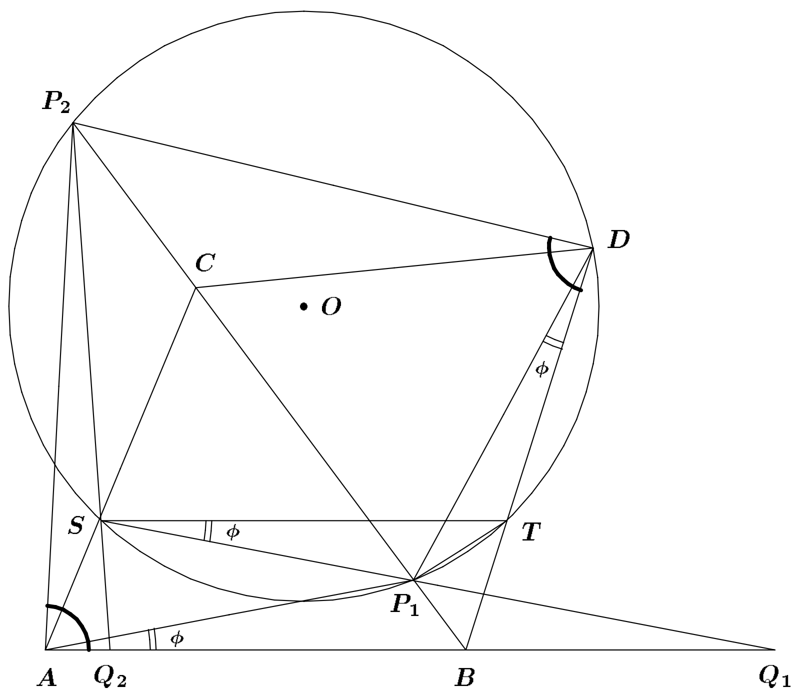
Given a point S on the side AC of triangle ABC, construct a line through S which cuts lines BC and AB at P and Q, respectively, such that $PQ = PA$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Let D be the reflection of A in BC, so that $\triangle DBC$ is congruent to $\triangle ABC$.

Let the line through S parallel to AB intersect BD at T. Let the circumcircle of $\triangle STD$ intersect BC at P_1 and P_2. Let SP_1 and SP_2 intersect AB and Q_1 and Q_2, respectively.

Then $P_1Q_1 = P_1A$ and $P_2Q_2 = P_2A$.



Proof. Note that BC is an axis of symmetry of quadrilaterals

$$ABCD\,,\quad ABDP_1\,,\quad \text{and}\quad ABDP_2\,. \tag{1}$$

Since quadrilateral SP_1TD is cyclic, we have

$$\angle P_1ST \;=\; \angle P_1DT \;=\; \phi\,, \text{ say.} \tag{2}$$

Now, (1) and (2) imply that $\angle P_1AB = \angle P_1DB = \phi$. Since $ST \| AB$, we have

$$\angle P_1Q_1A \;=\; \angle P_1ST \;=\; \phi\,. \tag{3}$$

Now, (2) and (3) imply that $P_1Q_1 = P_1A$. ■

From (1), we obtain that

$$\angle P_2AB = \angle P_2DB . \qquad (4)$$

Since quadrilateral P_2STD is cyclic, we have

$$\angle P_2ST = \angle P_2Q_2B = 180^\circ - \angle P_2AB . \qquad (5)$$

Now, (4) and (5) imply that $\angle P_2AQ_2 = \angle P_2Q_2A$, so that $P_2Q_2 = P_2A$. ■

Also solved by MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Stavanger, Norway; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Smeenk commented that he found it to be a very interesting problem on which he spent much time, but found only a quadratic equation with unknown $\tan\phi$, *which was quite unsatisfactory. He then asked a friend, who asked another friend, who found this elegant solution in an old issue of* Euclides, *a Dutch mathematical periodical.*

Bataille, Bejlegaard and Lambrou all made reference to part of this problem being a well-known classical problem. Bataille refered to H. Dörrie, *100 Great Problems of Elementary Geometry, Dover, 1965, and Lambrou, to* B. Bold, *Famous Problems of Geometry, also published by Dover.*

2474*. [1999 : 368] *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing continuous function satisfying, for all x, $y \in \mathbb{R}^+$:

$$f(x+y) + f\left(f(x) + f(y)\right) = f\Big(f\left(x + f(y)\right) + f\left(y + f(x)\right)\Big) .$$

Obviously $f(x) = c/x$ $(c > 0)$ is a solution. Determine all other solutions.

Editor's remark: This problem should have been starred when posed, since it was proposed without a solution.

Comment by proposer: One can prove that if f satisfies the functional equation, then $f(f(x)) = x$ for all $x \in \mathbb{R}^+$ (this was problem 5 of the 1997 Iranian Mathematical Olympiad), but determining all the solutions seems to be a very challenging problem.

This problem remains open.

2475. [1999 : 368] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Prove that

$$\sum_{j=0}^{n}\sum_{k=1}^{n}(-1)^{j+k}\binom{2n}{2j}\binom{2n}{2k-1} \;=\; 0\,.$$

I. *Solution by David Doster, Choate Rosemary Hall, Wallingford, CT, USA.*

Let $A_n = \sum\limits_{j=0}^{n}(-1)^j\binom{2n}{2j}$ and $B_n = \sum\limits_{k=1}^{n}(-1)^k\binom{2n}{2k-1}$.

Then $\sum\limits_{j=0}^{n}\sum\limits_{k=1}^{n}(-1)^{j+k}\binom{2n}{2j}\binom{2n}{2k-1} = A_nB_n$ and $(1+i)^{2n} = A_n - iB_n$.

Thus, $(-4)^n = (2i)^{2n} = \left((1+i)^2\right)^{2n} = (1+i)^{4n} = (A_n^2 - B_n^2) - 2iA_nB_n$, from which we infer that $A_n^2 - B_n^2 = (-4)^n$ and $A_nB_n = 0$.

II. *Solution by Gerry Leversha, St. Paul's School, London, England.*

It is sufficient to note that the given expression factorizes:

$$\sum_{j=0}^{n}\sum_{k=1}^{n}(-1)^{j+k}\binom{2n}{2j}\binom{2n}{2k-1} = \left(\sum_{j=0}^{n}(-1)^j\binom{2n}{2j}\right)\left(\sum_{k=1}^{n}(-1)^k\binom{2n}{2k-1}\right)$$

and then to note that, by the symmetry of Pascal's triangle, the first bracket on the right side is zero for odd values of n and the second bracket is zero for even values of n.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; CHOONGYUP SUNG, Pusan Science High School, Pusan, Korea; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

All the submitted solutions are minor variations of either I *or* II *above, with the only exception of that by the proposer, which uses De Moivre's formula.*

2476*. [1999 : 429] *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let n be a positive integer and consider the set $\{1, 2, 3, \ldots, 2n\}$. Give a **combinatorial** proof that the number of subsets A such that

1. A has exactly n elements, and
2. the sum of all elements in A is divisible by n,

is equal to

$$\frac{1}{n}\sum_{d|n}(-1)^{n+d}\binom{2d}{d}\phi\left(\tfrac{n}{d}\right)\,,$$

where ϕ is the Euler function. [Ed. Note the correction in the line above.]

Note: When n is prime, proving the formula is problem 6 of the 1995 IMO. A non-combinatorial proof of the formula is due to Roberto Dvornicich and Nikolay Nikolov.

Editor's remark: This problem should have been starred when posed, since it was proposed without a solution.

This problem remains open.

2478. [1999 : 429] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n \in \mathbb{N}$, evaluate $\displaystyle\sum_{k=0}^{n}\frac{n-k}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k}$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (adapted by the editor).

[Ed: This solution assumes familiarity with Newton's generalized binomial coefficients together with some identities thereof.]

Let S_n denote the summation to be evaluated.

The following formulas regarding generalized binomial coefficients are known and easy to verify:

$$(-4)^k\binom{-\frac{1}{2}}{k} = \binom{2k}{k}\,, \tag{1}$$

$$\binom{-1}{n} = (-1)^n\,, \tag{2}$$

$$\binom{m}{k+1} = \frac{m}{k+1}\binom{m-1}{k}\,, \tag{3}$$

$$\sum_{k=0}^{n}\binom{l}{k}\binom{m}{n-k} = \binom{l+m}{n}\,. \tag{4}$$

[Ed: In all formulas, l and m denote arbitrary integers and n and k, non-negative integers. Formula (4) is usually referred to as the Vandermonde's Convolution Formula.]

Using (3) with $m = \frac{1}{2}$ and $-\frac{1}{2}$, respectively, we obtain

$$\binom{\frac{1}{2}}{k+1} = \frac{1}{2(k+1)}\binom{-\frac{1}{2}}{k} \tag{5}$$

$$\text{and} \quad \binom{-\frac{1}{2}}{n-k} = \frac{-1}{2(n-k)}\binom{-\frac{3}{2}}{n-k-1}. \tag{6}$$

Therefore, we have

$$\begin{aligned}
S_n &= \sum_{k=0}^{n-1} \frac{n-k}{k+1}(-4)^k \binom{-\frac{1}{2}}{k}(-4)^{n-k}\binom{-\frac{1}{2}}{n-k} && \text{(by (1))}\\
&= (-4)^n \sum_{k=0}^{n-1} 2\binom{\frac{1}{2}}{k+1}(-\frac{1}{2})\binom{-\frac{3}{2}}{n-k-1} && \text{(by (5), (6))}\\
&= -(-4)^n \sum_{k=0}^{n-1} \binom{\frac{1}{2}}{k+1}\binom{-\frac{3}{2}}{n-k-1} \\
&= -(-4)^n \left(\binom{-1}{n} - \binom{\frac{1}{2}}{0}\binom{-\frac{3}{2}}{n}\right) && \text{(by (4))}\\
&= -(-4)^n \left((-1)^n + 2(n+1)\binom{-\frac{1}{2}}{n+1}\right) && \text{(by (2), (6))}\\
&= -4^n + \frac{n+1}{2}\binom{2n+2}{n+1} && \text{(by (1))}\\
&= (2n+1)\binom{2n}{n} - 4^n .
\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and the proposer.

Most of the other submitted solutions involve differentiation and integration of the power series expansion of the function $f(x) = (1-4x)^{-\frac{1}{2}}$ *for* $|x| < \frac{1}{4}$.

2479. [1999 : 430] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Writing $\tau(n)$ for the number of divisors of n, and $\omega(n)$ for the number of distinct prime factors of n, prove that

$$\sum_{k=1}^{n} (\tau(k))^2 = \sum_{k=1}^{n} 2^{\omega(k)} \sum_{j=1}^{\lfloor n/k \rfloor} \left\lfloor \frac{\lfloor n/k \rfloor}{j} \right\rfloor .$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany. If $n, k \in \mathbb{N}$, then there exist $m \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, k-1\}$ such that $n - 1 = km + r$. Then

$$\left\lfloor \frac{n}{k} \right\rfloor = m + \left\lfloor \frac{r+1}{k} \right\rfloor \qquad \text{and} \qquad \left\lfloor \frac{n-1}{k} \right\rfloor = m .$$

Hence,

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor = \begin{cases} 1 & \text{if } k|n. \\ 0 & \text{otherwise.} \end{cases}$$

Let S_n, $n \in \mathbb{N}_0$, denote the sum on the right hand side of the desired equation (empty sums are understood to be 0). If $n \in \mathbb{N}$, then from the above result we have

$$S_n - S_{n-1} = \sum_{k|n} 2^{\omega(k)} \sum_{j=1}^{\lfloor n/k \rfloor} \left(\left\lfloor \frac{\lfloor n/k \rfloor}{j} \right\rfloor - \left\lfloor \frac{\lfloor (n-1)/k \rfloor}{j} \right\rfloor \right) ,$$

or, by applying the result once again,

$$S_n - S_{n-1} = \sum_{k|n} 2^{\omega(k)} \tau\left(\frac{n}{k}\right) .$$

With the notation of Dirichlet products (see, for example, T.M. Apostol, Introduction to Analytic Number Theory, 2nd Ed., Springer, 1984, 29-39), the latter equation may be written as $S_n - S_{n-1} = (f * \tau)(n)$, $n \in \mathbb{N}$, where the arithmetical function f is defined by $f(n) = 2^{\omega(n)}$, $n \in \mathbb{N}$. We claim that $f * \tau = \tau^2$ (where τ^2 means ordinary pointwise multiplication). Since f, τ, and τ^2 are all multiplicative, it suffices to show that $(f * \tau)(p^e) = (\tau(p^e))^2$ when p is a prime and $e \in \mathbb{N}$. We have

$$\begin{aligned} (f * \tau)(p^e) &= (\tau * f)(p^e) = \sum_{j=0}^{e} \tau(p^j) f(p^{e-j}) \\ &= \tau(p^e) + 2 \sum_{j=0}^{e-1} \tau(p^j) = (e+1) + 2 \sum_{j=0}^{e-1} (j+1) \\ &= (e+1)^2 = (\tau(p^e))^2 . \end{aligned}$$

It follows that $S_n - S_{n-1} = (\tau(n))^2$ for all $n \in \mathbb{N}$. Replacing n by k and summing over $k = 1, 2, \ldots, n$, $n \in \mathbb{N}$, gives the requested equation.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and the proposer.

Janous observes that the identity

$$(\tau(n))^2 = \sum_{k|n} 2^{\omega(k)} \tau\left(\frac{n}{k}\right)$$

(implied in the above proof, but not explicitly stated) may be new and is worth noting in its own right, and that applying the Möbius inversion formula to it yields another identity which may be new:

$$\sum_{k|n} 2^{\omega(k)} = \sum_{k|n} \mu(k) \tau^2\left(\frac{n}{k}\right)$$