THE SKOLIAD CORNER

No. 49

R.E. Woodrow

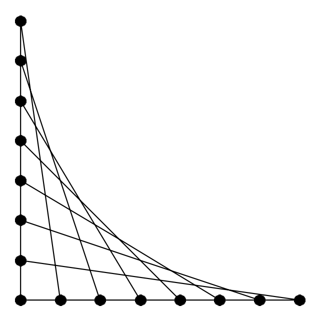
This issue we continue the theme with the problems of the Final Round of the Junior High School Mathematics Contest of the British Columbia Colleges. On the basis of the Preliminary Round, students are invited to write this contest as part of a visit and tour of a college campus. My thanks go to Jim Totten, The University College of the Cariboo, one of the organizers, for forwarding the materials to us.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Part A — Final Round — May 5, 2000

1. The last (ones) digit of a perfect square cannot be:

(a) 1 (b) 4 (c) 5 (d) 6 (e) 8

2. Suppose a string art design is constructed by connecting nails on a vertical axis and on a horizontal axis by line segments as follows: The nail furthest from the origin on the vertical axis is connected to the nail nearest the origin on the horizontal axis. Then proceed toward the origin on the vertical axis and away on the horizontal axis as shown in the diagram. If this were done on a project with 10 nails on each axis, the number of points of intersection of line segments would be:



(a) 45 (b) 46 (c) 47 (d) 48 (e) none of these

3. Assume there is an unlimited supply of pennies, nickels, dimes, and quarters. An amount (in cents) which cannot be made using exactly 6 of these coins is:

(a) 91 (b) 87 (c) 78 (d) 51 (e) 49

4. Given $x^2 + y^2 = 28$ and $xy = 14$, the value of $x^2 - y^2$ equals:

(a) -14 (b) 0 (c) 14 (d) 28 (e) 42

5. Given that $0 < x < y < 20$, the number of integer solutions (x, y) to the equation $2x + 3y = 50$ is:

(a) 16 (b) 9 (c) 8 (d) 5 (e) 3

6. The numbers $1, 3, 6, 10, 15\ldots$ are known as triangular numbers. Each triangular number can be expressed as $\frac{n(n+1)}{2}$ where n is a natural number. The largest triangular number less than 500 is:

(a) 494 (b) 495 (c) 496 (d) 497 (e) 498

7. An 80 m rope is suspended at its two ends from the tops of two 50 m flagpoles. If the lowest point to which the mid-point of the rope can be pulled is 36 m from the ground, then the distance, in metres, between the flagpoles is:

(a) $6\sqrt{39}$ (b) $36\sqrt{13}$ (c) $12\sqrt{39}$ (d) $18\sqrt{13}$ (e) $12\sqrt{26}$

8. At a certain party, the first time the door bell rang one guest arrived. On each succeeding ring two more guests arrived than on the previous ring. After 20 rings, the number of guests at the party was:

(a) 39 (b) 58 (c) 210 (d) 361 (e) 400

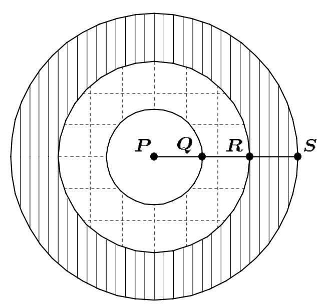
9. An operation $*$ is defined such that

$$A * B = A^B - B^A .$$

The value of $2 * (-1)$ is:

(a) -3 (b) -1 (c) $-\frac{1}{2}$ (d) 0 (e) $\frac{3}{2}$

10. Three circles with a common centre P are drawn as shown with $PQ = QR = RS$. The ratio of the area of the region between the inner and middle circles (shaded with squares) to the area of the region between the middle and outer circles (shaded with lines) is:



(a) $\frac{1}{3}$ (b) $\frac{4}{9}$ (c) $\frac{1}{2}$ (d) $\frac{3}{5}$ (e) $\frac{2}{3}$

Part B — Final Round — May 5, 2000

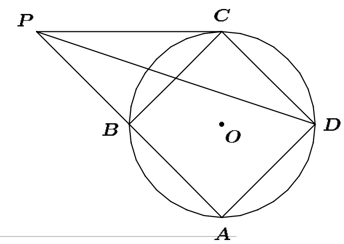
1. (a) How many three-digit numbers can be formed using only the digits 1, 2, and 3 if both of the following conditions hold:

(i) repetition is allowed;

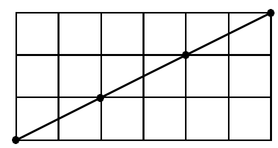
(ii) no digit can have a larger digit to its left.

(b) Repeat for a four-digit number using the digits 1, 2, 3, and 4.

2. The square $ABCD$ is inscribed in a circle of radius one unit. ABP is a straight line, PC is tangent to the circle. Find the length of PD. Make sure you explain thoroughly how you got **all** the things you used to find your solution!



3. If a diagonal is drawn in a 3×6 rectangle, it passes through four vertices of smaller squares. How many vertices does the diagonal of a 45×30 rectangle pass through?



4. Let a and b be any real numbers. Then $(a-b)$ is also a real number, and consequently $(a-b)^2 \geq 0$. Expanding gives $a^2 - 2ab + b^2 \geq 0$. If we add $2ab$ to both sides of the inequality, we get $a^2 + b^2 \geq 2ab$. Thus, for any real numbers a and b, we have $a^2 + b^2 \geq 2ab$.

Prove that for any real numbers a, b, c, d

(a) $2abcd \leq b^2c^2 + a^2d^2$.

(b) $6abcd \leq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2$.

5. A circular coin is placed on a table. Then identical coins are placed around it so that each coin touches the first coin and its other two neighbours.

(a) If the outer coins have the same radius as the inner coin, show that there will be exactly 6 coins around the outside.

(b) If the radius of all 7 coins is 1, find the total area of the spaces between the inner coin and the 6 outer coins.

Last issue we gave the problems of the Preliminary Round of the British Columbia Colleges Senior High School Contest for 2000. Here, thanks to Jim Totten, The University College of the Cariboo, one of the organizers, are some of the "official" solutions. Look for the rest in the next issue.

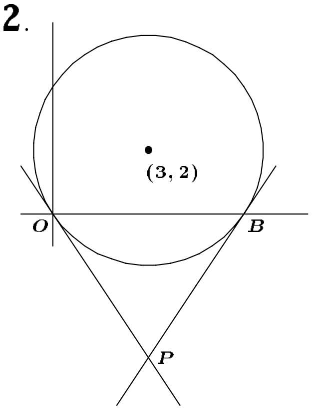
BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Preliminary Round — March 8, 2000

1. Antonino sets out on a bike ride of 40 km. After he has covered half the distance he finds that he has averaged 15 km/hr. He decides to speed up. The rate at which he must travel the rest of the trip in order to average 20 km/hr for the whole journey is:

(a) 25 km/hr **(b)** 30 km/hr **(c)** 35 km/hr **(d)** 36 km/hr **(e)** 40 km/hr

Solution. The answer is **(b)**. Antonino averages 15 km/hr for the first 20 km. This means it takes him $20/15 = 4/3$ hours to cover the first 20 km. In order to average 20 km/hr for a 40 km distance, he must cover the distance in 2 hours. He only has 2/3 hours remaining in which to cover the last 20 km. His speed over this last 20 km then must be (on average) $20/(2/3) = 30$ km/hr.

2.



A circle with centre at $(3, 2)$ intersects the x–axis at the origin, O, and at the point B. The tangents to the circle at O and B intersect at the point P. The y–coordinate of P is:

(a) $-3\frac{1}{2}$ **(b)** -4 **(c)** $-4\frac{1}{2}$ **(d)** -5 **(e)** none of these

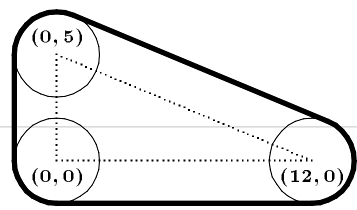
Solution. The answer is **(c)**. Let C be the centre of the circle. Since the points O and B are equidistant from the centre of the circle and also equidistant from the point P, and since both O and B lie on the x–axis, we see that P has coordinates $(3, y)$ with $y < 0$. The slope of OC is $2/3$. Since PO is the tangent line to the circle at O, we know that $PO \perp OC$. Therefore the slope of PO is $-3/2$. However, the slope of PO is computed to be $(y - 0)/(3 - 0) = y/3$. Together these imply that $y = -9/2$.

3. From five students whose ages are 6, 7, 8, 9, and 10, two are randomly chosen. The probability that the difference in their ages will be at least 2 years is:

(a) $\frac{1}{2}$ **(b)** $\frac{2}{5}$ **(c)** $\frac{3}{5}$ **(d)** $\frac{7}{10}$ **(e)** $\frac{3}{4}$

Solution. The answer is **(c)**. First of all the number of possible ways to choose a pair of distinct students from a set of five is $\binom{5}{2} = \frac{5!}{2!3!} = 10$. From this we need only eliminate those whose age difference is 1. Clearly there are exactly 4 such, namely $(6, 7)$, $(7, 8)$, $(8, 9)$, and $(9, 10)$. Thus, our probability of success is 6 out of 10, or $3/5$.

4. The centres of three circles of radius 2 units are located at the points $(0,0)$, $(12,0)$ and $(0,5)$. If the circles represent pulleys, what is the length of the belt which goes around all 3 pulleys as shown in the diagram?



(a) $30+\pi$ **(b)** $30+4\pi$ **(c)** $36+\pi$ **(d)** $60-4\pi$ **(e)** none of these

Solution. The answer is **(b)**. The straight sections of the belt are tangent to all 3 pulleys and thus perpendicular to the radius of each pulley at the point of contact. Thus the straight sections of the belt are the same lengths as the distances between the centres of the pulleys, which are 5, 12, and $\sqrt{5^2+12^2}=13$. Thus, the straight sections of belt add up to 30 units. The curved sections of belt, when taken together, make up one complete pulley, or a circle of radius 2. Thus the curved sections add up to $2\pi(2)=4\pi$. Thus, the full length of the belt is $30+4\pi$ units.

5. If Mark gets 71 on his next quiz, his average will be 83. If he gets 99, his average will be 87. How many quizzes has Mark already taken?

(a) 4 **(b)** 5 **(c)** 6 **(d)** 7 **(e)** 8

Solution. The answer is **(c)**. Let n be the number of quizzes Mark has already taken. Let x be his total score on all n quizzes. Then we have the following:

$$\frac{x+71}{n+1} = 83 \implies x = 83(n+1)-71\,,$$
$$\frac{x+99}{x+1} = 87 \implies x = 87(n+1)-99\,.$$

Solving this system yields $n=6$.

That completes the *Skoliad Corner* for this issue. The solutions to the problems of the British Columbia Colleges Senior High School Contest for 2000 will be completed in the next issue.

Send me suitable contest materials and suggestions for the future of the *Corner*.