A Nice COMC Problem

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Question B12 of the 1999 Canadian Open Mathematics Challenge was:

Q: *Triangle ABC is any one of the set of triangles having base BC equal to a and height from A to BC equal to h, with $h < \frac{\sqrt{3}}{2}a$. P is a point inside the triangle such that the value of*

$$\measuredangle PAB = \measuredangle PBA = \measuredangle PCB = \alpha .$$

Show that the measure of α is the same for every triangle in the set.

The problem is one implication of the following locus problem:

L: *Let BC be parallel to the line ℓ such that the distance from BC to ℓ is less than $\frac{\sqrt{3}}{2}$ of the length of BC. Let A be a variable point on ℓ. Find the locus of all points P inside $\triangle ABC$ such that*

$$\measuredangle PAB = \measuredangle PBA = \measuredangle PCB. \qquad (*)$$

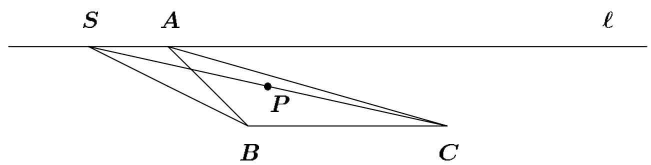
The two constraints: "the distance from BC to ℓ is less than $\frac{\sqrt{3}}{2}$ of the length of BC", and "P inside $\triangle ABC$" restrict the more general question:

G: *Let BC be parallel to the line ℓ. Let A be a variable point of ℓ. Find the locus of all points P that satisfy $(*)$.*

These are nice problems for computer investigation. The locus for **L** is hard to guess. When **G** is considered, the locus consists of several branches, some of which are unfamiliar and difficult curves. A computer geometry package, such as Geometer's Sketchpad, easily allows one to find solutions P associated with each A. (Note that P must lie on the perpendicular bisector of AB.) Such an investigation should reveal the answer to **L**, and indicate that the answer to **G** is more difficult.

To use the geometry package to produce a trace of the locus, a compass and straight edge type construction for P must be found. That is, given A, a compass and straight edge construction must be found for the various P that satisfy $(*)$. Such constructions are (of course) tied closely to a synthetic proof of the problem. Finally, a computer package such as **MAPLE** can be used to find an equation for the unfamiliar branches of the trace. (After eliminating the radicals, an 8^{th} degree polynomial, with highest order term x^4y^4, is the result.)

In this article, we outline a solution of **G**, leaving much of the work to the reader. To avoid tiresome details, special cases are ignored: cases such as when P lies on the lines that form the edges of the triangle. The reader can verify (easily) that there is nothing exciting about them.



Theorem 1. Let BC be parallel to ℓ. Let A be on ℓ and P inside $\triangle ABC$ with $\measuredangle PAB = \measuredangle PBA = \measuredangle PCB = \alpha$. If the line PC meets ℓ at S, then the points B, P, A and S lie on a circle. As a consequence, $SB = BC$ and $\measuredangle ASC = \measuredangle BCS = \measuredangle BSC$.

Proof. $\measuredangle BCS = \measuredangle ASC = \alpha$ (interior opposite angles), showing that $\measuredangle ASC = \measuredangle ASP = \measuredangle ABP$, and that $ASBP$ is concyclic. Hence, $\measuredangle BSP = \measuredangle BAP = \alpha$ (angles on the same chord), so that $SB = BC$.

Remarks:

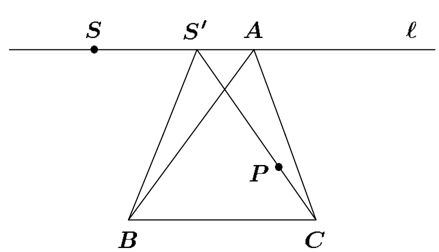
1. If $h > a$, it follows from Theorem 1 that there are no points P inside $\triangle ABC$ satisfying $(*)$.

2. If $h < a$ (we leave the case $h = a$ to the reader), there are two points (independent of A), call them S and S', on ℓ, with $SB = S'B = BC$ (we assume that S and S' are labelled so that $\measuredangle SBC > 90°$ and $\measuredangle S'BC < 90°$). Theorem 1 shows that the locus of P (inside $\triangle ABC$) lies on $SC \cup S'C$.

3. The point P is either at the intersection of the line SC, the circle SAB and the perpendicular bisector of AB, or at the intersection of the line $S'C$, the circle $S'AB$ and the perpendicular bisector of AB. Thus, it is easily constructed with compass and straight edge.

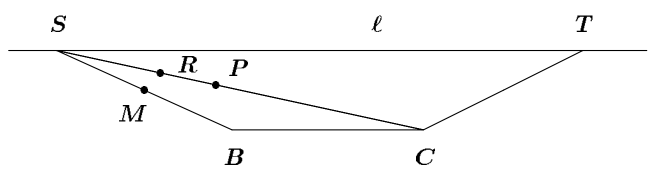
4. With the orientation of the above diagram, both S and S' must be to the left of A, since the line CP must pass though the segment AB if P is inside $\triangle ABC$.

To answer **Q**, note that if $h < \frac{\sqrt{3}}{2}a$, then P cannot be inside $\triangle ABC$ and lie on $S'C$. For then we would have $\measuredangle S'BC < 60°$, giving that $\measuredangle BCA > \measuredangle BCS' = \measuredangle BS'C > 60°$ (remember that S' must be to the left of A) and that the sum of the angles of $\triangle ABC$ is greater than $\measuredangle PCB + \measuredangle PAB + \measuredangle PBA = 3\measuredangle BCS' > 180°$.



To complete the discussion of **L**, we must decide which points on SC can be the point P associated with some choice of A. As in the proof of Theorem 1, $SAPB$ is concyclic. Thus, given A on ℓ, we can find P on SC, and *vice versa*, using that circle.

Let M be the mid-point of SB, and let R be the point on SC such that $RM \perp SB$. Since $\measuredangle SBC > 90°$, the point R exists.



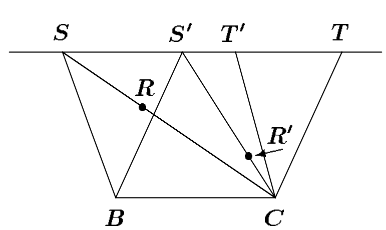
Theorem 2. The locus of question **L** is RC.

Proof. Let T be the point on ℓ with $\measuredangle SBC = \measuredangle BCT$. Suppose that P is on RC. Let A be the point (other than S) where circle SBP meets ℓ. (Consider the case that circle SBP is tangent to ℓ separately. Then, $A = S$, $P = R$.) If A is to the left of S, then $\measuredangle ASP + \measuredangle ABP = 180°$ (opposite angles of a cyclic quadrilateral). However, $\measuredangle TSP = \measuredangle SBR < \measuredangle SBP < \measuredangle ABP$, so that $\measuredangle ASP + \measuredangle ABP > \measuredangle ASP + \measuredangle TSP = 180°$, a contradiction. Thus, A must be to the right of S. Since $ASBP$ is concyclic, it follows that P is inside $\triangle ABC$.

We have already shown that if A and P satisfy the conditions of problem **L**, then P is on SC, $SAPB$ is concyclic, and A is to the right of S. We leave it to the reader to complete this theorem by showing that P must be on the segment RC, and also to show that A is on the segment ST.

We now turn to problem **G**. First, consider the case in which P is (again) inside $\triangle ABC$.

If $\frac{\sqrt{3}}{2}a < h < a$, the above ideas can be used to show that there is a segment $R'C$ on $S'C$ such that $RC \cup R'C$ is the locus of P. The point R' is the intersection of the perpendicular bisector of $S'B$ with $S'C$. For there to be a solution P on $R'C$, the point A must lie on the segment $S'T'$, and S' must be to the left of T'. In this case, there are two solutions for P inside $\triangle ABC$: one on RC and one on $R'C$. Note that S' is to the left of T' if and only if $h > \frac{\sqrt{3}}{2}a$.



For a given A, allowing P to be either inside or outside $\triangle ABC$, results in more solutions. Indeed, as A moves to the left of S, and P is the intersection of the circle ASB with the line SC, the point P moves on the line SC past R, then past S. It can be shown that any point P on the ray $\overrightarrow{CS}$ satisfies (∗). Likewise, P can be any point on the ray $\overrightarrow{CS'}$ (even if there are no solutions on CS' that lie inside $\triangle ABC$). Thus, the locus of **G** includes the rays $\overrightarrow{CS}$ and $\overrightarrow{CS'}$.

If A is to the right of T, the circle ASB intersects SC at a point P below BC. Then, $\measuredangle PCB$ is supplementary to the equal angles $\measuredangle PAB$ and $\measuredangle PBA$. Calling such a P a supplementary solution, we have that any point P on lines SC or $S'C$ is either a solution or a supplementary solution to **G**; that is, there is an A on ℓ with $\measuredangle PAB = \measuredangle PBA = \measuredangle PCB$, or $\measuredangle PAB = \measuredangle PBA = 180^\circ - \measuredangle PCB$.

We have by no means solved **G**. We have seen that if P is required to be inside $\triangle ABC$, then the locus is $RC \cup R'C$ ($\frac{\sqrt{3}}{2}a < h < a$), or RC ($0 < h < \frac{\sqrt{3}}{2}a$). If P is also allowed to be outside $\triangle ABC$, then the locus includes $\overrightarrow{CS} \cup \overrightarrow{CS'}$.

For each A, the lines AB, BC and CA divide the plane into seven regions: the inside of $\triangle ABC$, the infinite regions adjacent to the vertices A, B, C (each bounded by two rays), and the infinite regions adjacent to the sides AB, BC and CA (each bounded by two rays and a line segment).

Suppose that P is in the infinite region adjacent to A. Then $\measuredangle PAB > \measuredangle PCB$. (Extend PA to meet BC, and use the result that an exterior angle of a triangle is greater than either of the two interior opposite angles.) Thus, P cannot be a solution to **G**. Similarly, there is no solution P to **G** inside the infinite regions adjacent to B or C.

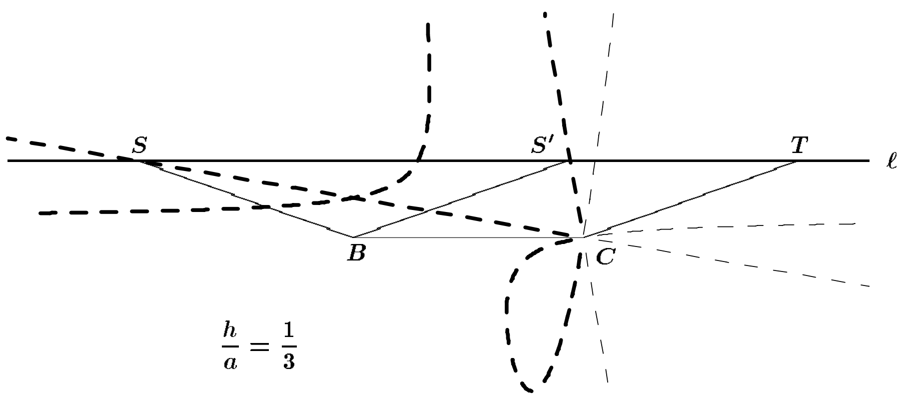
For any A on ℓ, the point P where the perpendicular bisector of AB meets the circle ABC to the left of AB is a solution to **G**. (Use the circle theorems!) Conversely, if P is a solution to **G** inside the infinite region adjacent to AB, then $PACB$ is concyclic, since $\measuredangle PCB = \measuredangle PAB$. Thus, P is at the intersection of the perpendicular bisector of AB and the circle ABC, the intersection to the left of AB.

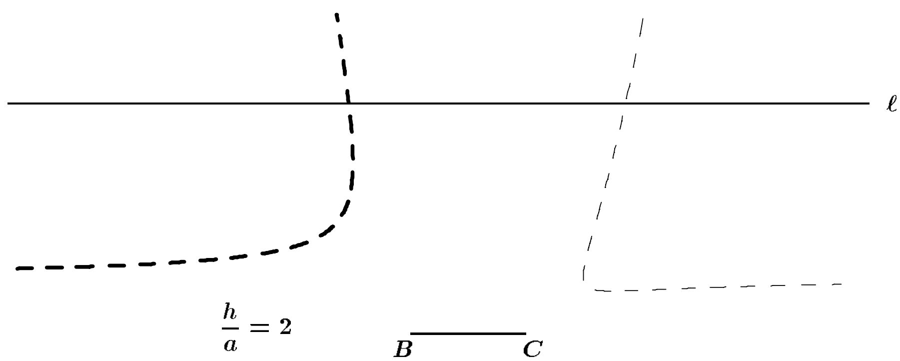
If P is a solution to **G** in the infinite region adjacent to BC, then P is again a point of intersection of the circle ABC and the perpendicular bisector of AB — this time, the intersection to the right of AB. However, if $h > a$, then there is no choice of A that leads to a solution P in this region.

Finally, we must consider P in the infinite region adjacent to AC . We have already seen some solutions in this region: if A is to the left of S (S'), then there is a solution P on $\overrightarrow{RS}$ ($\overrightarrow{R'S'}$) which will be in the region adjacent to AC. We leave it to the reader to prove that these are the only solutions in this region.

For some points A, there are supplementary solutions for **G** in the region adjacent to AC, namely, the point of intersection of the perpendicular bisector of AB and the circle ABC that lies to the right of AB.

Theorem 3. The locus of **G** is indicated in the figures below for $h < a$ and $h > a$. The solutions are indicated with thick dashed lines, while the supplementary solutions are indicated with thin dashed lines. Together, these loci consist of the lines SC and SC', and of those points where, for each A on ℓ, the perpendicular bisector of AB intersects the circle through A, B and C.





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