THE OLYMPIAD CORNER

No. 209

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As a first set of problems, we give the 11^{th} Grade of the XXIII All Russian Olympiad of the Secondary Schools. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina, for collecting them.

XXIII ALL RUSSIAN OLYMPIAD OF THE SECONDARY SCHOOLS

11^{th} Grade

First Day

1. Solve, in integers, the equation $(x^2 - y^2)^2 = 1 + 16y$.

2. The Council of Wizards is tested in the following way: The King lines the wizards up in a row and places on the head of each of them either a white hat or a blue hat or a red hat. Each wizard sees the colours of hats of the people standing in front of him, but he neither sees the colour of his hat nor the colours of hats of the people standing behind. Every minute some of the wizards must announce one of the three colours (it is allowed to speak out just once). After completion of this procedure the King executes all the wizards who failed to guess the right colour of their hats. Prior to this ceremony all 100 members have agreed to minimize the number of executions. How many of them are definitely secure against the punishment?

3. Two circles intersect at the points A and B. A line is drawn through the point A. This line crosses the first circle again at the point C and it crosses the second circle again at the point D. Let M and N be the mid-points of the arcs BC and BD, respectively (these arcs do not contain A). Let K be the mid-point of the segment CD. Prove that the angle MKN is a right angle. (It may be assumed that A lies between C and D).

4. An $n \times n \times n$ cube is constituted of unit cubes. You are given a closed broken loop without self-crossings such that each link of it joins the centres of the two neighbouring cubes (the neighbours have common faces). We say the faces of the cubes that are crossed by this loop are marked. Prove that the edges of the cubes can be painted in two colours in such a way that marked faces would have an odd number of edges of both colours, and any unmarked faces would have an even number of edges of both colours.

Second Day

5. Given all possible quadratic trinomials of the type $x^2 + px + q$, with integer coefficients p and q, $1 \le p \le 1997$, $1 \le q \le 1997$, consider the sets of the trinomials:

(a) having integer zeros,

(b) not having real zeros.

Which of those sets is larger?

6. Suppose a polygon, a line l and a point P on the line l are in general position (that is, all the lines which are extensions of the sides of the polygon intersect l in different points that are different from P). We mark those of the vertices of the polygon for which the extensions of the sides leaving the vertex cross l at points with P between them. Prove that P lies inside this polygon if and only if there are odd numbers of marked vertices lying on both sides of l.

7. A sphere is inscribed in a tetrahedron. It touches the first of the pyramid's faces at the incentre, the second face at the orthocentre, and the third face at the point of intersection of the medians. Prove that this is a right tetrahedron.

8. Dominoes of size 2×1 are arranged in a rectangular box sized $m \times n$, where m and n are odd numbers. They cover almost the whole box except for a corner where there is a 1×1 hole. If a domino has a short common edge with this hole it is allowed to shift along itself for one unit and cover this hole (while doing this another hole opens). Prove that using some sequence of those shifts it is possible to move the hole into any corner of the box.

As a second Olympiad set for this number, we give the problems of the Fourth National Mathematical Olympiad of Turkey. My thanks go to Richard Nowakowski, Canadian Team Leader at the IMO in Argentina, for collecting them for us.

FOURTH NATIONAL MATHEMATICAL OLYMPIAD OF TURKEY

Second Round, First Day

December 6, 1996 — Time: 4.5 hours

1. Let $\{A_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be sequences of positive integers. Assume that, for each positive integer x there exist a unique positive integer N and a unique N-tuple $(x_1, x_2, \ldots, x_N)$ of integers such that

$$x = \sum_{n=1}^{N} x_n A_n \,, \quad 0 \le x_n \le \alpha_n \quad (n = 1, 2, \ldots, N) \quad \text{and} \quad x_N \ne 0\,.$$

Prove that

(i) $A_{n_0} = 1$ for some n_0;

(ii) if $k \neq j$, then $A_k \neq A_j$,

(iii) if $A_k \leq A_j$, then A_k divides A_j.

2. Given a square $ABCD$ of side length 2, let M and N be points on the edges $[AB]$ and $[CD]$, respectively. The lines CM and BN meet at P, while the lines AN and MD meet at Q. Show that $|PQ| \geq 1$.

3. n integers on the real axis are coloured. Determine for which positive integral values of k there exists a family K of closed intervals satisfying the following conditions:

(i) The union of all closed intervals in K contains all the coloured integers.

(ii) Any two distinct closed intervals in K are disjoint.

(iii) For each $I \in K$, $\frac{b_I}{a_I} = \frac{1}{k}$, where a_I is the number of integers in I, and b_I the number of coloured integers in I.

Second Round, Second Day

December 7, 1996 — Time: 4.5 hours

4. Given a quadrangle $ABCD$, the circle which is tangential to $[AD]$, $[DC]$ and $[CB]$ touches these edges at K, L and M, respectively. Denote the point at which the line which passes through L and is parallel to AD meets $[KM]$ by N, and the point at which $[LN]$ and $[KC]$ meet by P. Prove that

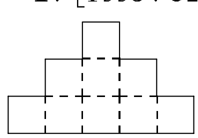
$$|PL| = |PN| .$$

5. Show that $\prod_{k=0}^{n-1} (2^n - 2^k)$ is divisible by $n!$ for each positive integer n.

6. Let $\mathbb{R}$ stand for the set of all real numbers. Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) > f(x)(1 + yf(x))$ for all positive real x, y.

Next, some catching up. Somehow, we misplaced Edward T.H. Wang's solution to problem #1 of the XI Italian Mathematical Olympiad. Here is his solution.

1. [1998 : 323] *XI Italian Mathematical Olympiad 1995.*



Determine for which values of the integer n it is possible to cover up, without overlapping, a square of side n with tiles of the type shown in the picture where each small square of the tile has side 1.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Call a tile of the given shape a "nonatile". We show that a perfect cover of an $n \times n$ board by nonatiles is possible if and only if n is a multiple of 6. First note that a nonatile consists of 9 unit squares. Hence, if a perfect cover exists, then $9 \mid n^2$ and thus, $3 \mid n$. We claim that n must be *even*. Suppose that an $n \times n$ board has been covered up by non-overlapping nonatiles where n is *odd*. We colour all the n^2 unit squares of the board alternatingly black and white, as on an ordinary chessboard. Then clearly: $|$number of black squares $-$ number of white squares$| = 1$. We now classify all the nonatiles into two kinds: black or white, depending on whether the top (protruded) square is coloured black or white. Note that a black nonatile has 6 black squares and 3 white squares, while a white nonatile has 6 white squares and 3 black squares.

Hence, if there are b black nonatiles and w white nonatiles, then $|$number of black squares $-$ number of white squares$| = 3|b - w| \neq 1$ and this is a contradiction. Hence, n is even and $6 \mid n$ follows. To finish the proof, it suffices to demonstrate that a 6×6 board admits a perfect cover (using 4 nonatiles) as shown by the diagram below.

1	1	1	1	1	2
3	1	1	1	2	2
3	3	1	2	2	2
3	3	3	4	2	2
3	3	4	4	4	2
3	4	4	4	4	4

Next we give a different generalization and solution of a problem of the Dutch Mathematical Olympiad 1993 [1997 : 197; 1998 : 389].

5. [1997 : 197; 1998 : 389–390] *Dutch Mathematical Olympiad, 1993.*

$P_1, P_2, \ldots, P_{11}$ are eleven distinct points on a line, $P_iP_j \leq 1$ for every pair P_i, P_j. Prove that the sum of all (55) distances P_iP_j, $1 \leq i, j \leq 11$ is smaller than 30.

Solution by Achilleas Sinefakopoulos, student, University of Athens, Greece.

More generally, consider n distinct adjacent points $P_1, P_2, \ldots, P_n$ on a line ($n \geq 2$) such that $P_iP_j \leq 1$ for every pair P_i, P_j. We shall prove by induction on m that if $n = 2m + 1 > 3$, then $\sum_{1 \leq i < j \leq n} P_iP_j < m(m+1)$.

First, note that, if $n = 3$, then $P_1P_2 + P_2P_3 + P_1P_3 = 2P_1P_3 \le 2$, with equality if and only if $P_1P_3 = 1$. Now, if $n = 5$, then $\sum_{1<i<j<5} P_iP_j < 2$ (since $P_2P_43 < P_1P_5 \le 1$). Therefore,

$$\begin{aligned}
\sum_{1\le i<j\le 5} P_iP_j &= \sum_{1<i<j<5} P_iP_j + P_1P_5 + \sum_{1<k<5} (P_1P_k + P_kP_5) \\
&= \sum_{1<i<j<5} P_iP_j + P_1P_5 + \sum_{1<k<5} P_1P_5 \\
&= \sum_{1<i<j<5} P_iP_j + 4P_1P_5 < 2 + 4 \;=\; 6\,.
\end{aligned}$$

Suppose that the result is true for some $n = 2m + 1 > 3$. We show that it is also true for $n + 2$. Indeed, by the inductive hypothesis, we have $\sum_{1<i<j<n+1} P_iP_j < m(m + 1)$. Hence,

$$\begin{aligned}
\sum_{1\le i<j\le n+2} P_iP_j &= \sum_{1<i<j<n+2} P_iP_j + P_1P_{n+2} + \sum_{1<k<n+2} (P_1P_k + P_kP_{n+2}) \\
&= \sum_{1<i<j<n+2} P_iP_j + P_1P_{n+2} + \sum_{1<k<n+2} P_1P_{n+2} \\
&= \sum_{1<i<j<n+2} P_iP_j + (n + 1)P_1P_{n+2} \\
&< m(m + 1) + (n + 1) \;=\; (m + 2)(m + 1)\,.
\end{aligned}$$

The proof is complete.

To round things out with respect to the February 1999 number here is an additional solution.

4. [1999 : 6–7] *Vietnamese Mathematical Olympiad 3 / 1996, Category B.*

Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying simultaneously two conditions:

(i) $f(1995) = 1996$

(ii) for every $n \in \mathbb{Z}$, if $f(n) = m$, then $f(m) = n$ and $f(m + 3) = n - 3$, ($\mathbb{Z}$ is the set of integers).

Solution by Pierre Bornsztein, Courdimanche, France.

Let $f : \mathbb{Z} \to \mathbb{Z}$ satisfy the two conditions. From (ii), we deduce: $f(f(n)) = n$ for all $n \in \mathbb{Z}$; that is, f is involutory.

The equality $f(m+3) = n-3$ then gives $m+3 = f(n-3)$ or $f(n-3) = f(n) + 3$. From this, we easily obtain, by induction, $f(n - 3k) = f(n) + 3k$ for all positive integers k.

Let m be any integer; m may be written $m = f(n)$ for an n (for $n = f(m)$, actually). Condition (ii) gives $f(m+3) = f(m) - 3$, and by induction: $f(m+3k) = f(m) - 3k$ for all positive integers k. All this can be summed up by $f(n+3k) = f(n) - 3k$ for all $n, k \in \mathbb{Z}$.

It follows that for every $k \in \mathbb{Z}$,

$$\begin{aligned} f(3k) &= f(0) - 3k\,, \\ f(3k+1) &= f(1) - 3k\,, \\ f(3k+2) &= f(2) - 3k\,. \end{aligned}$$

Let $k = 665$. We have $f(1995) = f(0) - 1995 = 1996$ and $f(1996) = 1995 = f(1) - 1995$. Then $f(0) = 3991$ and $f(1) = 3990$.

Thus, for every $k \in \mathbb{Z}$,

$$\begin{aligned} f(3k) &= 3991 - 3k\,, \\ f(3k+1) &= 3990 - 3k\,, \\ f(3k+2) &= f(2) - 3k\,. \end{aligned}$$

We remark that

$$\begin{aligned} f(3\mathbb{Z}) &= 3\mathbb{Z} + 1\,, \\ f(3\mathbb{Z}+1) &= 3\mathbb{Z}\,, \\ \text{and so,}\quad f(3\mathbb{Z}+2) &\subset 3\mathbb{Z} + 2\,. \end{aligned}$$

Moreover f is bijective, so $f(3\mathbb{Z}+2) = 3\mathbb{Z}+2$. Denote $f(2) = 3a + 2$, where $a \in \mathbb{Z}$. Then:

$$\text{if } f \text{ is a solution then } f(m) = \left\{ \begin{array}{ll} 3991 - m & \text{if } m \not\equiv 2 \pmod 3 \\ 3a + 4 - m & \text{if } m \equiv 2 \pmod 3\,. \end{array} \right\}$$

Conversely, a straightforward calculation shows that any such f satisfies the conditions.

Now, we turn to solutions from our readers to problems given in the March 1999 number of the *Corner* with problems of the 19$^{\text{th}}$ Austrian-Polish Mathematics Competition 1996 [1999 : 70–71].

1. Let $k \geq 1$ be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties:

(a) The decimal representation of n consists of exactly k digits.

(b) All digits of n are odd.

(c) The number n is divisible by 5.

(d) The number $m = \frac{n}{5}$ has k odd (decimal) digits.

Solution by Jacqueline Freeman and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Call a natural number "ideal" if it satisfies conditions (a)–(d). We show that $n = a_k a_{k-1} \cdots a_1 \in \mathbb{N}$ is ideal if and only if $a_i \in \{5, 7, 9\}$ for all $i = 1, 2, \ldots, k$ and $a_1 = 5$. This is clearly true when $k = 1$, and thus, we assume that $k \geq 2$.

If n is ideal, then (c) implies that $a_1 = 5$. We show that none of the a_i's can equal 1 or 3. Let $n/5 = b_k b_{k-1} \cdots b_1$ and consider the process of long division of n by 5. Clearly, $a_k \neq 1, 3$ for otherwise $n/5$ would have only $k-1$ digits. If $a_i = 1$ or 3 for any i, $2 \leq i \leq k-1$, then clearly $b_i = 0, 2, 4, 6$, or 8, depending on whether the remainder "carried over" when dividing a_{i+1} by 5 is 0, 1, 2, 3, or 4, respectively. This is a contradiction to (d).

Conversely, suppose $n = a_k a_{k-1} \cdots a_1$, where $a_i \in \{5, 7, 9\}$ for all $i = 1, 2, \ldots, k$ and $a_1 = 5$. To show that n is ideal, it clearly suffices to verify (d). Since $a_k \geq 5$, $m = n/5$ has k digits, and we can write $m = b_k b_{k-1} \cdots b_1$. Clearly, b_1 is odd. If b_i is even for some i, $2 \leq i \leq k$, then from $n = 5m$ we see that $a_i = 0, 1, 2, 3$, or 4 depending on whether the "carry" (when b_{i-1} is multiplied by 5) is 0, 1, 2, 3, or 4, respectively. This is a contradiction. Hence, all the b_i's are odd. This completes the proof of our claim.

Finally, since $a_1 = 5$ and each of the a_i's ($i = 2, 3, \ldots, k$) can take on any one of the 3 values 5, 7, or 9, the total number of ideal integers is 3^{k-1}.

Remark. In fact, it is not difficult to show that if n is ideal then all the digits of $n/5$ must be 1, 5, or 9.

2. A convex hexagon $ABCDEF$ satisfies the following conditions:

(a) The opposite sides are parallel (that is, $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$).

(b) The distances between the opposite sides are equal (that is, $d(AB, DE) = d(BC, EF) = d(CD, FA)$, where $d(g, h)$ denotes the distance between lines g and h).

(c) $\angle FAB$ and $\angle CDE$ are right angles.

Show that diagonals BE and CF intersect at an angle of $45°$.

Solution by Toshio Seimiya, Kawasaki, Japan.

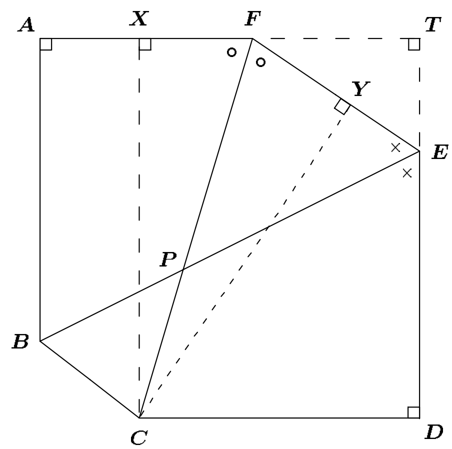
Let X, Y be the feet of the perpendiculars from C to AF, EF, respectively. (See figure.)

Since $AF \parallel CD$, it follows that $CX = d(AF, CD)$.

Since $BC \parallel EF$, it follows that $CY = d(BC, EF)$.

Since $d(AF, CD) = d(BC, EF)$, we have $CX = CY$. Thus, CF bisects $\angle AFE$. Similarly, BE bisects $\angle DEF$. Let T be the intersection of AF and DE. Since $AB \parallel DE$ and $\angle A = 90°$, we get

$$\angle FTE = 90° .$$



Then,

$$\begin{aligned} &\angle AFE + \angle DEF \\ &\quad = (\angle T + \angle TEF) + (\angle T + \angle TFE) \\ &\quad = 2\angle T + (\angle TEF + \angle TFE) \\ &\quad = 90^\circ \times 2 + 90^\circ = 270^\circ . \end{aligned}$$

Thus, we have

$$\angle PFE + \angle PEF = \tfrac{1}{2}(\angle AFE + \angle DEF) = 135^\circ .$$

Hence, we get

$$\begin{aligned} \angle FPE &= 180^\circ - (\angle PFE + \angle PEF) \\ &= 180^\circ - 135^\circ = 45^\circ . \end{aligned}$$

3. The polynomials $P_n(x)$ are defined recursively by $P_0(x) = 0$, $P_1(x) = x$ and

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \quad \text{for} \quad n \geq 2 .$$

For every natural number $n \geq 1$, find all real numbers x satisfying the equation $P_n(x) = 0$.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's write-up.

We will prove that for $n \geq 1$, the only real solution of $P_n(x) = 0$ is $x = 0$.

For $n \geq 2$,

$$P_n(x) - P_{n-1}(x) = (x-1)(P_{n-1}(x) - P_{n-2}(x)) .$$

Then an easy induction leads to

$$P_n(x) - P_{n-1}(x) \;=\; (x-1)^{n-1}(P_1(x) - P_0(x)) \;=\; x(x-1)^{n-1}\,.$$

That is, $P_n(x) = P_{n-1}(x) + x(x-1)^{n-1}$ for $n \geq 2$, and we note that it remains true for $n = 1$.

We deduce that, for $n \geq 1$,

$$\begin{aligned} P_n(x) &= x(x-1)^{n-1} + x(x-1)^{n-2} + \cdots + x + P_0(x) \\ &= x((x-1)^{n-1} + (x-1)^{n-2} + \cdots + 1)\,. \end{aligned}$$

Then, if $x = 2$, $P_n(2) = 2n \neq 0$ and if $x \neq 2$, $P_n(x) = x \cdot \frac{((x-1)^n - 1)}{x-2}$. Thus, $P_n(x) = 0$ if and only if $x = 0$ or $(x-1)^n = 1$ for $x \neq 2$.

If n is even, $(x-1)^n = 1$ if and only if $x = 0$ or $x = 2$. Then $P_n(x) = 0$ if and only if $x = 0$.

If n is odd, $(x-1)^n = 1$ if and only if $x = 2$. Then $P_n(x) = 0$ if and only if $x = 0$.

Thus, for $n \geq 1$, $P_n(x) = 0$ if and only if $x = 0$.

4. The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that $-1 \leq xy + yz + zt + tx \leq 0$.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's solution.

First, we have:

$$xy + yz + zt + tx \;=\; (x+z)(y+t) \;=\; -(x+z)^2 \;\leq\; 0$$

(since $y + t = -(x+z)$). Then,

$$|xy + yz + zt + tx| \;\leq\; (x^2 + y^2 + z^2 + t^2)^{1/2}(y^2 + z^2 + t^2 + x^2)^{1/2} \;=\; 1$$

(by the Cauchy-Schwarz inequality). The conclusion follows.

Remark. Equality $xy + yz + zt + tx = 0$ holds if and only if $x + z = y + t = 0$. Therefore, inequality $xy + yz + zt + tx \leq 0$ becomes an equality for the quadruplets $(x, y, z, t) = (a, b, -a, -b)$ where a, b are real numbers such that $a^2 + b^2 = \frac{1}{2}$.

If equality $xy + yz + zt + tx = -1$ holds, then we have equality in the Cauchy-Schwarz inequality so that (x, y, z, t) and (y, z, t, x) are proportional. Since at least one of the numbers x, y, z, t must be non-zero, this leads to $x = y = z = t$ or $y = -x$, $z = x$, $t = -x$. The first case is incompatible with the hypotheses, and the second case provides only the quadruplets:

$$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) \quad \text{and} \quad \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) .$$

Conversely, these two quadruplets satisfy $xy + yz + zt + tx = -1$ and we may conclude: $xy + yz + zt + tx = -1$ holds if and only if $(x, y, z, t) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ or $\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

6. Natural numbers k, n are given such that $1 < k < n$. Solve the system of n equations

$$x_i^3 \cdot (x_i^2 + x_{i+1}^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \quad \text{for} \quad 1 \leq i \leq n ,$$

with n real unknowns x_1, x_2, ..., x_n. Note: $x_0 = x_n$, $x_{n+1} = x_1$, $x_{n+2} = x_2$, and so on.

Solution by Pierre Bornsztein, Courdimanche, France.

We prove that there are two solutions

$$x_1 = x_2 = \cdots = x_n = 0$$

and

$$x_1 = x_2 = \cdots = x_n = \frac{1}{\sqrt[3]{k}} .$$

First, suppose that $(x_1, \ldots, x_n)$ is a solution.

First Case. There is $i \in \{1, \ldots, n\}$ such that $x_i = 0$.

From the cyclic symmetry, we suppose, without loss of generality, that $x_1 = 0$. Then

$$x_1^3(x_1^2 + \cdots + x_k^2) = 0 = x_n^2 .$$

Thus, $x_n = 0$.

Continuing, we deduce that $x_i = 0$ for all i.

Second Case. $x_i \neq 0$ for all i.

Since $x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2$, we deduce that $x_i > 0$ for all i. From cyclic symmetry, without loss of generality, we have

$$x_1 = \min\{x_i : i = 1, 2, \ldots, n\} .$$

Then $x_1^3 \leq x_2^3$ and $x_1^2 \leq x_{k+1}^2$. Thus,

$$x_1^2 + x_2^2 + \cdots + x_k^2 \leq x_2^2 + x_3^2 + \cdots + x_{k+1}^2 .$$

We deduce that

$$\begin{aligned} x_n^2 &= x_1^3(x_1^2 + \cdots + x_k^2) \\ &\leq x_2^3(x_2^2 + \cdots + x_{k+1}^2) = x_1^2 . \end{aligned}$$

Thus, $x_n \leq x_1$.

What we have actually shown is that if $x_i = \min\{x_1, \ldots, x_n\}$, then $x_{i-1} = x_i$ $(= \min\{x_1, \ldots, x_n\})$. An easy induction leads to $x_1 = x_2 = \cdots = x_n$.

Let a denote the common value. Then $a^3(ka^2) = a^2$; that is, $a = \frac{1}{\sqrt[3]{k}}$.

Conversely, it is easy to verify that $(0, 0, \ldots, 0)$ and $\left(\frac{1}{\sqrt[3]{k}}, \ldots, \frac{1}{\sqrt[3]{k}}\right)$ are solutions.

7. Show that there do not exist non-negative integers k and m such that $k! + 48 = 48(k+1)^m$.

Solution by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Suppose the given equation holds for some non-negative integers k and m. Then $48 \mid k!$. Since $48 = 2^4 \times 3$, we must have $k \geq 6$. If $k = 6$ or 7, the equation becomes $16 = 7^m$ or $106 = 8^m$, respectively. Clearly, neither is possible. Hence, $k \geq 8$ and the given equation can be rewritten as

$$3 \times 5 \times 7 \times 8 \times \cdots \times (k-1) \times k + 1 \;=\; (k+1)^m\,. \qquad (1)$$

Suppose $k + 1$ is a composite. Then it has a prime divisor q. Since $q \leq k$, we have $q \mid k!$, which implies $q \mid 48$. Since $k \geq 8$, the left side of (1) is odd and thus, q must be odd. Hence, $q = 3$, which is clearly impossible in view of (1), since $3 \nmid 1$.

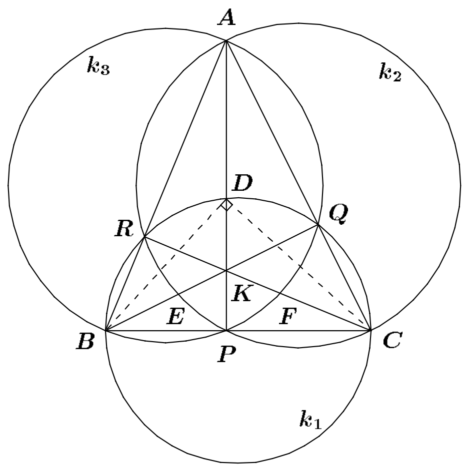
Therefore, $k + 1 = p$ is a prime. Then $k! = (p-1)! \equiv -1 \pmod{p}$ by Wilson's Theorem; that is, $p \mid k! + 1$.

Rewriting the given equation as $k! + 1 + 47 = 48p^m$, we infer that $p \mid 47$ and so $p = 47$. Then we have $46! + 48 = 48 \times 47^m$ or $46! = 48(47^m - 1)$. Since the prime divisors of $46!$ include 5, 7, 11 which are all coprime with 48, we have $47^m \equiv 1 \pmod{5, 7, 11}$. Now, straightforward checking reveals that $\text{ord}_5(47) = 4$ (that is, the least positive integer n such that $47^n \equiv 1 \pmod 5$ is $n = 4$), $\text{ord}_7(47) = 6$ and $\text{ord}_{11}(47) = 5$. Hence, m is divisible by $\text{lcm}\{4, 6, 5\}$; that is, $60 \mid m$. So $m \geq 60$ and we have $48 \times 47^m \geq 48 \times 47^{60}$. Clearly, $48 \times 47^{60} > 46! + 48$ and we have a contradiction.

Next we turn to solutions to problems of the 3rd Turkish Mathematical Olympiad [1999 : 72].

2. For an acute triangle ABC, k_1, k_2, k_3 are the circles with diameters $[BC]$, $[CA]$, $[AB]$, respectively. If K is the radical centre of these circles, $[AK] \cap k_1 = \{D\}$, $[BK] \cap k_2 = \{E\}$, $[CK] \cap k_3 = \{F\}$ and $\text{area}(ABC) = u$, $\text{area}(DBC) = x$, $\text{area}(ECA) = y$, and $\text{area}(FAB) = z$, show that $u^2 = x^2 + y^2 + z^2$.

Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



Let P, Q, and R be the feet of the perpendiculars from A, B, and C, to BC, CA, and AB, respectively.

AP, BQ, and CR are common chords of k_2, k_3; k_3, k_1; and k_1, k_2, respectively. Thus, K is the orthocentre of $\triangle ABC$. Since $\angle BDC = 90°$ and $DP \perp BC$, we get

$$DP^2 = BP \cdot CP . \tag{1}$$

Since $BK \perp AC$ and $AP \perp BC$, we have $\angle BKP = \angle ACP$. Further, we have $\triangle BKP \sim \triangle ACP$. It follows that $BP : AP = KP : CP$; that is

$$AP \cdot KP = BP \cdot CP . \tag{2}$$

From (1) and (2) we have

$$DP^2 = AP \cdot KP .$$

Hence, we have

$$DP^2 \cdot BC^2 = (AP \cdot BC) \times (KP \cdot BC) .$$

From this we get

$$x^2 = u \times \text{area } (KBC) . \tag{3}$$

Similarly, we have

$$y^2 = u \times \text{area } (KCA) , \tag{4}$$

and

$$z^2 \;=\; u \times \text{area }(KAB)\,. \qquad (5)$$

Therefore, we obtain from (3), (4), and (5),

$$\begin{aligned} x^2+y^2+z^2 &= u \times \{\text{area }(KBC)+\text{area }(KCA)+\text{area }(KAB)\} \\ &= u \times \text{area }(ABC) \\ &= u^2\,. \end{aligned}$$

3. Let $\mathbb{N}$ denote the set of positive integers. Let A be a real number and $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 1$ and

$$1 < \frac{a_{n+1}}{a_n} \;\le\; A \quad \text{for all} \quad n \in \mathbb{N}\,.$$

(a) Show that there is a unique non-decreasing surjective function $k : \mathbb{N} \to \mathbb{N}$ such that $1 < \frac{A^{k(n)}}{a_n} \le A$ for all $n \in \mathbb{N}$.

(b) If k takes every value at most m times, show that there exists a real number $C > 1$ such that $C^n \le Aa_n$ for all $n \in \mathbb{N}$.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

(a) The condition $1 < \frac{A^{k(n)}}{a_n} \le A$ is equivalent to $A^{k(n)-1} \le a_n < A^{k(n)}$ or $k(n) - 1 \le \frac{\ln(a_n)}{\ln(A)} < k(n)$. Hence, k is necessarily the function given by $k(n) = 1 + \left\lfloor \frac{\ln(a_n)}{\ln(A)} \right\rfloor = 1 + \lfloor \log_A(a_n) \rfloor$ for all $n \in \mathbb{N}$. This shows the unicity of k. Since $\{a_n\}$ is increasing and $a_1 = 1$, we have $a_n \ge 1$. We also note $A > 1$. Hence, $\frac{\ln(a_n)}{\ln(A)}$ is a non-negative real number and $1 + \lfloor \log_A(a_n) \rfloor$ is a positive integer. Thus, we can define a function $k : \mathbb{N} \to \mathbb{N}$ by the formula $k(n) = 1 + \lfloor \log_A(a_n) \rfloor$. For all $n \in \mathbb{N} : a_n < a_{n+1}$, so that $\log_A(a_n) < \log_A(a_{n+1})$ and $k(n) \le k(n+1)$. Thus, k is non-decreasing. Now, let us remark that, for $s \in \mathbb{N}$, $k(n) = s$ is equivalent to $A^{s-1} \le a_n < A^s$. Therefore, if $\{a_n\}$ is bounded above, say $a_n \le M$ for all n, an s satisfying $A^{s-1} > M$ (such an s exists since $A > 1$) cannot be an image under k. Thus, if $\{a_n\}$ satisfies the hypothesis and is convergent (for instance $a_n = \frac{1}{2}\left(1 + \frac{1}{n}\right)^n\Big)$, the surjectivity of k cannot be obtained. Consequently, we will henceforth assume that $\{a_n\}$ is not bounded above.

Given $s \in \mathbb{N}$, we prove that the equation $k(n) = s$ has at least one solution. For $s = 1$, $n = 1$ is obviously a solution, so now we suppose that $s \ge 2$. From the supplementary hypothesis, there exists $n \in \mathbb{N}$ such that $a_n \ge A^{s-1}$. Let r be the least of these n's, so that $a_{r-1} < A^{s-1} \le a_r$. Then $a_r \le Aa_{r-1} < A^s$ so that $A^{s-1} \le a_r < A^s$ and r is a solution. The function k fulfils all the demands and (a) follows.

(b) For each $s \in \mathbb{N}$, denote by $N(s)$ the number of n such that $k(n) = s$; that is, such that $A^{s-1} \leq a_n < A^s$. By hypothesis, we have $N(s) \leq m$ for each s.

We will show that (b) holds with $C = A^{1/m} > 1$. Let n be an arbitrary integer. If $k(n) = s$, then

$$n = N(1) + N(2) + \cdots + N(s-1) + j \quad \text{where} \quad j \in \{1, 2, \ldots, N(s)\}$$

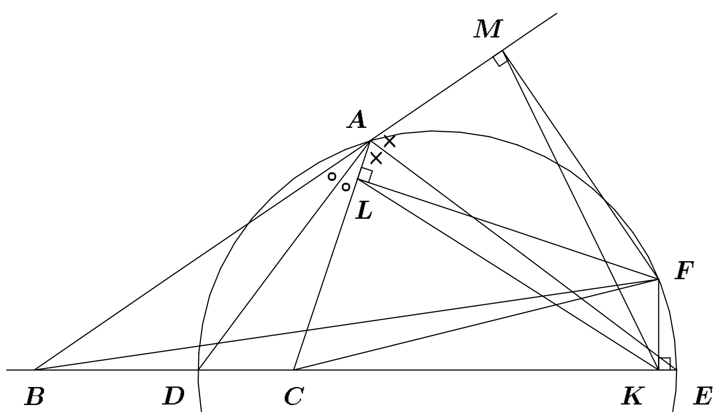
(because there are $N(1) + N(2) + \cdots + N(s-1)$ terms of the sequence $\{a_n\}$ which are $< A^{s-1}$). On the one hand, $a_n A \geq A^{s-1}A = A^s = C^{ms}$, and, on the other hand,

$$\begin{aligned} n &= N(1) + N(2) + \cdots + N(s-1) + j \\ &\leq N(1) + N(2) + \cdots + N(s-1) + N(s) \ \leq \ sm\,. \end{aligned}$$

Hence, $C^n \leq C^{ms}$ and we obtain $C^n \leq a_n A$. So (b) is proved.

4. In a triangle ABC with $|AB| \neq |AC|$, the internal and external bisectors of the angle A intersect the line BC at D and E, respectively. If the feet of the perpendiculars from a point F on the circle with diameter $[DE]$ to the lines BC, CA, AB are K, L, M, respectively, show that $|KL| = |KM|$.

Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



Since the circle with diameter DE is the Apollonius circle, we have

$$\frac{FB}{FC} = \frac{AB}{AC}\,. \tag{1}$$

By the Law of Sines, we have

$$\frac{AB}{AC} = \frac{\sin C}{\sin B}\,. \tag{2}$$

From (1) and (2), we get

$$\frac{FB}{FC} = \frac{\sin C}{\sin B};$$

that is,

$$FB \sin B = FC \sin C . \qquad (3)$$

Since the circle with diameter BF passes through K and M, we have

$$KM = FB \sin B .$$

Similarly, we have

$$KL = FC \sin C .$$

Therefore, we obtain from (3)

$$KL = KM .$$

5. Let $t(A)$ denote the sum of elements of A for a non-empty subset A of integers, and define $t(\phi) = 0$. Find a subset X of the set of positive integers such that for every integer k there is a unique ordered pair of subsets (A_k, B_k) of X with $A_k \cap B_k = \phi$ and $t(A_k) - t(B_k) = k$.

Solution by Pierre Bornsztein, Courdimanche, France.

We will prove that $X = \{3^i : i \in \mathbb{N}\}$ has the desired property.

LEMMA. Every integer n can be written $n = \sum \alpha_i 3^i$ where for all i, $\alpha \in \{-1, 0, 1\}$ and the sum has only a finite number of non-zero terms. Moreover, such a decomposition is unique.

Proof of the Lemma. First we prove the unicity:

If $\sum_{i=0} \alpha_i 3^i = \sum_{j=0} \beta_j 3^j$ with α_i, $\beta_j \in \{-1, 0, 1\}$, we recognize two possibilities.

Case 1. For all i, j, $\alpha_i = \beta_j = 0$, we are done.

Case 2. Otherwise, let i_0 be the smallest subscript such that $\alpha_{i_0} \neq 0$ or $\beta_{i_0} \neq 0$. Then

$$(\alpha_{i_0} - \beta_{i_0})3^{i_0} = -\sum_{i>i_0} \alpha_i 3^i + \sum_{j>i_0} \beta_j 3^j .$$

The right-hand side is divisible by 3^{i_0+1}. We deduce that $\alpha_{i_0} - \beta_{i_0}$ is divisible by 3. But $-2 \leq \alpha_{i_0} - \beta_{i_0} \leq 2$. Thus, $\alpha_{i_0} = \beta_{i_0}$. It follows that, for $0 \leq i \leq i_0$, $\alpha_i = \beta_i$ and $\sum_{i>i_0} \alpha_i 3^i = \sum_{j>i_0} \beta_j 3^j$.

If necessary, we denote by i_1 the smallest subscript such that $i_1 > i_0$ and $\alpha_{i_1} \neq 0$ or $\beta_{i_1} \neq 0$. Then if $0 \leq i < i_1$, $\alpha_i = \beta_i$, and with the same reasoning as above, we get $\alpha_{i_1} = \beta_{i_1}$, and so on. Unicity follows.

Now, let p be a fixed positive integer. The numbers of the form $N = \sum_{i=0}^{p} \alpha_i 3^i$ with $\alpha_i \in \{-1, 0, 1\}$ are all distinct, from above. For each $i \in \{0, \ldots, p\}$, there are 3 possible choices for α_i. Thus, we obtain exactly 3^{p+1} such integers. Moreover

$$-\sum_{i=0}^{p} 3^i \;\leq\; \sum_{i=0}^{p} \alpha_i 3^i \;\leq\; \sum_{i=0}^{p} 3^i\,;$$

that is,

$$-\left(\frac{3^{p+1}-1}{2}\right) \;\leq\; \sum_{i=0}^{p} \alpha_i 3^i \;\leq\; \left(\frac{3^{p+1}-1}{2}\right)\,.$$

But there are exactly 3^{p+1} integers in the interval $\left[-\left(\frac{3^{p+1}-1}{2}\right), \frac{3^{p+1}-1}{2}\right]$. It follows that each integer can be expressed in this form, proving the lemma.

Now, let k be an integer. From the lemma, $k = \sum_{i\geq 0} \alpha_i 3^i$, where $\alpha_i \in \{-1, 0, 1\}$, and $\alpha_i \neq 0$ for at most finitely many values of i. Denote $A_k = \{3^i : \alpha_i = 1\}$ and $B_k = \{3^i : \alpha_i = -1\}$. Then $A_k, B_k \subset X$, $A_k \cap B_k = \emptyset$ and $t(A_k) - t(B_k) = k$. Moreover, suppose that A'_k, B'_k are subsets of X with $A'_k \cap B'_k = \emptyset$ and $t(A'_k) - t(B'_k) = k$. Note that A'_k and B'_k must be finite. Then,

$$\sum_{3^i \in A_k} 3^i - \sum_{3^i \in B_k} 3^i = k = \sum_{3^i \in A'_k} 3^i - \sum_{3^i \in B'_k} 3^i\,.$$

The unicity of the form $\sum \alpha_i 3^i$ gives $A_k = A'_k$ and $B_k = B'_k$.

6. Let $\mathbb{N}$ denote the set of positive integers. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the condition

$$m \mid n \iff f(m) \mid f(n)$$

for all $m, n \in \mathbb{N}$.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

For such a function f, we have:

(a) $f(1) = 1$: Since $1 \mid n$, we have $f(1) \mid f(n)$ for all $n \in \mathbb{N}$. But f is surjective, so that $f(n)$ may be any positive integer; hence, $f(1)$ divides all positive integers and $f(1) = 1$.

(b) f is bijective: f is already surjective; moreover, if $f(m) = f(n)$, then $f(m) \mid f(n)$ and $f(n) \mid f(m)$ so that $m \mid n$ and $n \mid m$; that is, $m = n$. Hence, f is also injective.

Obviously, the condition on f also reads: $m \mid n \iff f^{-1}(m) \mid f^{-1}(n)$ for all $m, n \in \mathbb{N}$.

(c) $f(p)$ is prime whenever p is prime: If k divides $f(p)$, $f^{-1}(k)$ divides p, so that $f^{-1}(k) = 1$ or p and $k = 1$ or $f(p)$.

More generally, $f(p^a) = (f(p))^a$ for all positive integers a: Let q be a prime integer dividing $f(p^a)$. Then $f^{-1}(q)$ is a prime integer dividing p^a. It follows that $f^{-1}(q) = p$ and $q = f(p)$. Therefore, $f(p^a)$ is a power of $f(p)$, say $f(p^a) = (f(p))^b$. Now, the $a+1$ distinct divisors of p^a, namely 1, p, $\dots$, p^a, provide $a+1$ distinct divisors of $f(p^a) = (f(p))^b$, namely 1, $f(p)$, $\dots$, $f(p^a)$. As $(f(p))^b$ has $b+1$ divisors, we have $a \le b$. Similarly, using f^{-1}, $b \le a$. Thus, $a = b$ and $f(p^a) = (f(p))^a$.

(d) If u, v are coprime positive integers, then $f(u)$, $f(v)$ are also coprime and $f(uv) = f(u)f(v)$. That $f(u)$, $f(v)$ are coprime is immediate since any prime integer p dividing $f(u)$ and $f(v)$ would provide a prime integer, namely $f^{-1}(p)$, dividing u and v.

Since $u \mid uv$ and $v \mid uv$, $f(u) \mid f(uv)$ and $f(v) \mid f(uv)$. Taking into account the result we have just shown, we get $f(u) \cdot f(v) \mid f(uv)$. With f^{-1}, $f(u)$, $f(v)$ instead of f, u, v, we obtain $u \cdot v \mid f^{-1}(f(u) \cdot f(v))$ and consequently $f(uv) \mid f(u) \cdot f(v)$. Hence, $f(uv) = f(u)f(v)$. Let P be the set of prime integers. We have obtained the following concerning f: The restriction of f to P is a bijection from P onto P and, combining the last two results, if $n = \prod_{p \in P} p^{v_p(n)}$, then $f(n) = \prod_{p \in P} (f(p))^{v_p(n)}$. [Here, $v_p(n)$ denotes the exponent of p in the standard factorization of n into prime powers: of course, we have $v_p(n) = 0$ for all but a finite number of p and the sequence $(v_p(n))_{p \in P}$ is uniquely determined by n.]

Conversely, let $\phi : P \to P$ be any bijection from P onto P and let $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n) = \prod_{p \in P} (\phi(p))^{v_p(n)} \quad \text{when} \quad n = \prod_{p \in P} p^{v_p(n)}. \tag{1}$$

Clearly, f is surjective. Moreover,

$$\begin{aligned} m \mid n \quad &\Longleftrightarrow \quad v_p(m) \le v_p(n) && \text{for all } p \in P \\ &\Longleftrightarrow \quad v_{\phi(p)}(f(m)) \le v_{\phi(p)}(f(n)) && \text{for all } p \in P \\ &\Longleftrightarrow \quad v_q(f(m)) \le v_q(f(n)) && \text{for all } q \in P \\ &\Longleftrightarrow \quad f(m) \mid f(n). \end{aligned}$$

In conclusion, the solutions are the functions extending to $\mathbb{N}$ the bijections $\phi : P \to P$ by means of the formula (1).

That completes the *Olympiad Corner* for this issue. Send me your Olympiad contest materials and your nice solutions!