THE ACADEMY CORNER

No. 36

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In this issue, we present the problems of the 7^{th} International Mathematics Competition for University Students, held at University College, London, UK, on 26 July and 31 July 2000. Our thanks to Moubinool Omarjee, Paris, France, for sending us the problems. We invite our readers to send in their nice solutions.

The International Mathematics Competition for University Students

First Day Problems, 26 July 2000

1. Is it true that if $f : [0,1] \to [0,1]$ is

   (a) monotone increasing

   (b) monotone decreasing

   then there exists an $x \in [0,1]$ for which $f(x) = x$?

2. Let $p(x) = x^5 + x$ and $q(x) = x^5 + x^2$. Find all pairs (w, z) of complex numbers with $w \neq z$ for which $p(w) = p(z)$ and $q(w) = q(z)$.

3. Suppose that A and B are square matrices of the same size with

$$\text{rank}\,(AB - BA) \;=\; 1\,.$$

   Show that $(AB - BA)^2 = 0$.

4. (a) Show that, if $\{x_k\}$ is a decreasing sequence of positive numbers, then

$$\left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} \;\leq\; \sum_{k=1}^{n} \frac{x_k}{\sqrt{k}}\,.$$

   (b) Show that there is a constant C such that, if $\{x_k\}$ is a decreasing sequence of positive numbers, then

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left(\sum_{k=m}^{\infty} x_k^2\right)^{\frac{1}{2}} \;\leq\; C \sum_{k=1}^{\infty} x_k\,.$$

5. Let R be a ring of characteristic zero (not necessarily commutative). Let e, f and g be idempotent elements of R satisfying $e + f + g = 0$. Show that $e = f = g = 0$.

   (R is of characteristic zero means that, if $a \in R$ and n is a positive integer, then $na \neq 0$ unless $a = 0$. An idempotent x is an element satisfying $x = x^2$.)

6. Let $f : \mathbb{R} \to (0, \infty)$ be an increasing differentiable function for which $\lim_{x\to\infty} f(x) = \infty$ and f' is bounded.

   Let $F(x) = \int_0^x f(t)\,dt$. Define the sequence $\{a_n\}$ inductively by

$$a_0 = 1\,, \quad a_{n+1} = a + n + \frac{1}{f(a_n)}\,,$$

   and the sequence $\{b_n\}$ simply by $b_n = F^{-1}(n)$.

   Prove that $\lim_{n\to\infty} (a_n - b_n) = \infty$.

Second Day Problems, 31 July 2000

1. (a) Show that the unit square can be partitioned into n smaller squares if n is large enough.

   (b) Let $d \geq 2$. Show that there is a constant $N(d)$ such that, whenever $n \geq N(d)$, a d-dimensional unit cube can be partitioned into n smaller cubes.

2. Let f be continuous and nowhere monotone on $[0, 1]$. Show that the set of points on which f attains local minima is dense in $[0, 1]$.

   (A function is nowhere monotone if there exists no interval where the function is monotone. A set is dense if each non-empty open interval contains at least one element of the set.)

3. Let $p(z)$ be a polynomial of degree n with complex coefficients. Prove that there exist at least $n + 1$ complex numbers z for which $p(z)$ is 0 or 1.

4. Suppose that the graph of a polynomial of degree 6 is tangent to a straight line at the 3 points A_1, A_2, A_3, where A_2 lies between A_1 and A_3.

   (a) Prove that if the lengths of the segments A_1A_2 and A_2A_3 are equal, then the areas of the figures bounded by these segments and the graph of the polynomial are also equal.

(b) Let $k = \dfrac{A_2A_3}{A_1A_2}$, and let K be the ratio of the areas of the appropriate figures. Prove that

$$\frac{2}{7}\,k^5 \;<\; K \;<\; \frac{7}{2}\,k^5\,.$$

5. Let $\mathbb{R}^+$ be the set of positive real numbers.
   Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x,\, y \in \mathbb{R}^+$,

$$f(x)\,f(yf(x)) \;=\; f(x+y)\,.$$

6. For an $m \times m$ real matrix A, define e^A to be $\sum_{n=0}^{\infty} \frac{1}{n!}A^n$.
   (The sum is convergent for all matrices.)

   Prove or disprove, that for all real polynomials p and $m \times m$ real matrices A and B, $p\left(e^{AB}\right)$ is nilpotent if and only if $p\left(e^{BA}\right)$ is nilpotent. (A matrix A is nilpotent if $A^k = 0$ for some positive integer k.)