THE ACADEMY CORNER

No. 36

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In this issue, we present the problems of the 7^{th} International Mathematics Competition for University Students, held at University College, London, UK, on 26 July and 31 July 2000. Our thanks to Moubinool Omarjee, Paris, France, for sending us the problems. We invite our readers to send in their nice solutions.

The International Mathematics Competition for University Students

First Day Problems, 26 July 2000

1. Is it true that if $f : [0,1] \to [0,1]$ is

   (a) monotone increasing

   (b) monotone decreasing

   then there exists an $x \in [0,1]$ for which $f(x) = x$?

2. Let $p(x) = x^5 + x$ and $q(x) = x^5 + x^2$. Find all pairs (w, z) of complex numbers with $w \neq z$ for which $p(w) = p(z)$ and $q(w) = q(z)$.

3. Suppose that A and B are square matrices of the same size with

$$\operatorname{rank}(AB - BA) \;=\; 1\,.$$

   Show that $(AB - BA)^2 = 0$.

4. (a) Show that, if $\{x_k\}$ is a decreasing sequence of positive numbers, then

$$\left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} \;\leq\; \sum_{k=1}^{n} \frac{x_k}{\sqrt{k}}\,.$$

   (b) Show that there is a constant C such that, if $\{x_k\}$ is a decreasing sequence of positive numbers, then

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left(\sum_{k=m}^{\infty} x_k^2\right)^{\frac{1}{2}} \;\leq\; C \sum_{k=1}^{\infty} x_k\,.$$

5. Let R be a ring of characteristic zero (not necessarily commutative). Let e, f and g be idempotent elements of R satisfying $e+f+g=0$. Show that $e=f=g=0$.

   (R is of characteristic zero means that, if $a \in R$ and n is a positive integer, then $na \neq 0$ unless $a=0$. An idempotent x is an element satisfying $x=x^2$.)

6. Let $f:\mathbb{R}\to(0,\infty)$ be an increasing differentiable function for which $\lim_{x\to\infty} f(x)=\infty$ and f' is bounded.

   Let $F(x)=\int_0^x f(t)\,dt$. Define the sequence $\{a_n\}$ inductively by

$$a_0=1\,, \quad a_{n+1}=a+n+\frac{1}{f(a_n)}\,,$$

   and the sequence $\{b_n\}$ simply by $b_n=F^{-1}(n)$.

   Prove that $\lim_{n\to\infty}(a_n-b_n)=\infty$.

Second Day Problems, 31 July 2000

1. (a) Show that the unit square can be partitioned into n smaller squares if n is large enough.

   (b) Let $d \geq 2$. Show that there is a constant $N(d)$ such that, whenever $n \geq N(d)$, a d-dimensional unit cube can be partitioned into n smaller cubes.

2. Let f be continuous and nowhere monotone on $[0,1]$. Show that the set of points on which f attains local minima is dense in $[0,1]$.

   (A function is nowhere monotone if there exists no interval where the function is monotone. A set is dense if each non-empty open interval contains at least one element of the set.)

3. Let $p(z)$ be a polynomial of degree n with complex coefficients. Prove that there exist at least $n+1$ complex numbers z for which $p(z)$ is 0 or 1.

4. Suppose that the graph of a polynomial of degree 6 is tangent to a straight line at the 3 points A_1, A_2, A_3, where A_2 lies between A_1 and A_3.

   (a) Prove that if the lengths of the segments A_1A_2 and A_2A_3 are equal, then the areas of the figures bounded by these segments and the graph of the polynomial are also equal.

(b) Let $k = \dfrac{A_2A_3}{A_1A_2}$, and let K be the ratio of the areas of the appropriate figures. Prove that

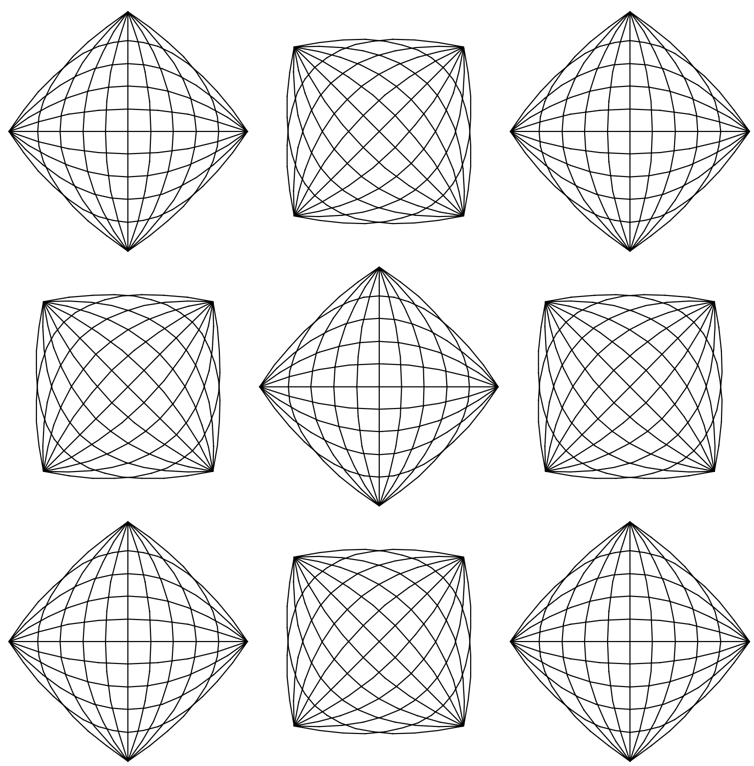
$$\frac{2}{7}\,k^5 \;<\; K \;<\; \frac{7}{2}\,k^5\,.$$

5. Let $\mathbb{R}^+$ be the set of positive real numbers.
Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x,\, y \in \mathbb{R}^+$,

$$f(x)\,f(yf(x)) \;=\; f(x+y)\,.$$

6. For an $m \times m$ real matrix A, define e^A to be $\sum_{n=0}^{\infty} \frac{1}{n!}A^n$.
(The sum is convergent for all matrices.)

Prove or disprove, that for all real polynomials p and $m \times m$ real matrices A and B, $p\left(e^{AB}\right)$ is nilpotent if and only if $p\left(e^{BA}\right)$ is nilpotent.
(A matrix A is nilpotent if $A^k = 0$ for some positive integer k.)



THE OLYMPIAD CORNER

No. 209

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a first set of problems, we give the 11^{th} Grade of the XXIII All Russian Olympiad of the Secondary Schools. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina, for collecting them.

XXIII ALL RUSSIAN OLYMPIAD OF THE SECONDARY SCHOOLS

11^{th} Grade

First Day

1. Solve, in integers, the equation $(x^2 - y^2)^2 = 1 + 16y$.

2. The Council of Wizards is tested in the following way: The King lines the wizards up in a row and places on the head of each of them either a white hat or a blue hat or a red hat. Each wizard sees the colours of hats of the people standing in front of him, but he neither sees the colour of his hat nor the colours of hats of the people standing behind. Every minute some of the wizards must announce one of the three colours (it is allowed to speak out just once). After completion of this procedure the King executes all the wizards who failed to guess the right colour of their hats. Prior to this ceremony all 100 members have agreed to minimize the number of executions. How many of them are definitely secure against the punishment?

3. Two circles intersect at the points A and B. A line is drawn through the point A. This line crosses the first circle again at the point C and it crosses the second circle again at the point D. Let M and N be the mid-points of the arcs BC and BD, respectively (these arcs do not contain A). Let K be the mid-point of the segment CD. Prove that the angle MKN is a right angle. (It may be assumed that A lies between C and D).

4. An $n \times n \times n$ cube is constituted of unit cubes. You are given a closed broken loop without self-crossings such that each link of it joins the centres of the two neighbouring cubes (the neighbours have common faces). We say the faces of the cubes that are crossed by this loop are marked. Prove that the edges of the cubes can be painted in two colours in such a way that marked faces would have an odd number of edges of both colours, and any unmarked faces would have an even number of edges of both colours.

Second Day

5. Given all possible quadratic trinomials of the type x^2+px+q, with integer coefficients p and q, $1 \leq p \leq 1997$, $1 \leq q \leq 1997$, consider the sets of the trinomials:

(a) having integer zeros,

(b) not having real zeros.

Which of those sets is larger?

6. Suppose a polygon, a line l and a point P on the line l are in general position (that is, all the lines which are extensions of the sides of the polygon intersect l in different points that are different from P). We mark those of the vertices of the polygon for which the extensions of the sides leaving the vertex cross l at points with P between them. Prove that P lies inside this polygon if and only if there are odd numbers of marked vertices lying on both sides of l.

7. A sphere is inscribed in a tetrahedron. It touches the first of the pyramid's faces at the incentre, the second face at the orthocentre, and the third face at the point of intersection of the medians. Prove that this is a right tetrahedron.

8. Dominoes of size 2×1 are arranged in a rectangular box sized $m \times n$, where m and n are odd numbers. They cover almost the whole box except for a corner where there is a 1×1 hole. If a domino has a short common edge with this hole it is allowed to shift along itself for one unit and cover this hole (while doing this another hole opens). Prove that using some sequence of those shifts it is possible to move the hole into any corner of the box.

As a second Olympiad set for this number, we give the problems of the Fourth National Mathematical Olympiad of Turkey. My thanks go to Richard Nowakowski, Canadian Team Leader at the IMO in Argentina, for collecting them for us.

FOURTH NATIONAL MATHEMATICAL OLYMPIAD OF TURKEY

Second Round, First Day

December 6, 1996 — Time: 4.5 hours

1. Let $\{A_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ be sequences of positive integers. Assume that, for each positive integer x there exist a unique positive integer N and a unique N-tuple $(x_1, x_2, \ldots, x_N)$ of integers such that

$$x = \sum_{n=1}^{N} x_n A_n, \quad 0 \leq x_n \leq \alpha_n \quad (n = 1, 2, \ldots, N) \quad \text{and} \quad x_N \neq 0.$$

Prove that

(i) $A_{n_0} = 1$ for some n_0;

(ii) if $k \neq j$, then $A_k \neq A_j$,

(iii) if $A_k \leq A_j$, then A_k divides A_j.

2. Given a square $ABCD$ of side length 2, let M and N be points on the edges $[AB]$ and $[CD]$, respectively. The lines CM and BN meet at P, while the lines AN and MD meet at Q. Show that $|PQ| \geq 1$.

3. n integers on the real axis are coloured. Determine for which positive integral values of k there exists a family K of closed intervals satisfying the following conditions:

(i) The union of all closed intervals in K contains all the coloured integers.

(ii) Any two distinct closed intervals in K are disjoint.

(iii) For each $I \in K$, $\frac{b_I}{a_I} = \frac{1}{k}$, where a_I is the number of integers in I, and b_I the number of coloured integers in I.

Second Round, Second Day

December 7, 1996 — Time: 4.5 hours

4. Given a quadrangle $ABCD$, the circle which is tangential to $[AD]$, $[DC]$ and $[CB]$ touches these edges at K, L and M, respectively. Denote the point at which the line which passes through L and is parallel to AD meets $[KM]$ by N, and the point at which $[LN]$ and $[KC]$ meet by P. Prove that

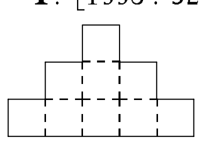
$$|PL| = |PN|.$$

5. Show that $\prod_{k=0}^{n-1}(2^n - 2^k)$ is divisible by $n!$ for each positive integer n.

6. Let $\mathbb{R}$ stand for the set of all real numbers. Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) > f(x)(1 + yf(x))$ for all positive real x, y.

Next, some catching up. Somehow, we misplaced Edward T.H. Wang's solution to problem #1 of the XI Italian Mathematical Olympiad. Here is his solution.

1. [1998 : 323] *XI Italian Mathematical Olympiad 1995.*



Determine for which values of the integer n it is possible to cover up, without overlapping, a square of side n with tiles of the type shown in the picture where each small square of the tile has side 1.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Call a tile of the given shape a "nonatile". We show that a perfect cover of an $n \times n$ board by nonatiles is possible if and only if n is a multiple of 6. First note that a nonatile consists of 9 unit squares. Hence, if a perfect cover exists, then $9 \mid n^2$ and thus, $3 \mid n$. We claim that n must be *even*. Suppose that an $n \times n$ board has been covered up by non-overlapping nonatiles where n is *odd*. We colour all the n^2 unit squares of the board alternatingly black and white, as on an ordinary chessboard. Then clearly: |number of black squares $-$ number of white squares| $= 1$. We now classify all the nonatiles into two kinds: black or white, depending on whether the top (protruded) square is coloured black or white. Note that a black nonatile has 6 black squares and 3 white squares, while a white nonatile has 6 white squares and 3 black squares.

Hence, if there are b black nonatiles and w white nonatiles, then |number of black squares $-$ number of white squares| $= 3|b - w| \neq 1$ and this is a contradiction. Hence, n is even and $6 \mid n$ follows. To finish the proof, it suffices to demonstrate that a 6×6 board admits a perfect cover (using 4 nonatiles) as shown by the diagram below.

1	1	1	1	1	2
3	1	1	1	2	2
3	3	1	2	2	2
3	3	3	4	2	2
3	3	4	4	4	2
3	4	4	4	4	4

Next we give a different generalization and solution of a problem of the Dutch Mathematical Olympiad 1993 [1997 : 197; 1998 : 389].

5. [1997 : 197; 1998 : 389–390] *Dutch Mathematical Olympiad, 1993.*
$P_1, P_2, \ldots, P_{11}$ are eleven distinct points on a line, $P_iP_j \leq 1$ for every pair P_i, P_j. Prove that the sum of all (55) distances P_iP_j, $1 \leq i, j \leq 11$ is smaller than 30.

Solution by Achilleas Sinefakopoulos, student, University of Athens, Greece.

More generally, consider n distinct adjacent points $P_1, P_2, \ldots, P_n$ on a line ($n \geq 2$) such that $P_iP_j \leq 1$ for every pair P_i, P_j. We shall prove by induction on m that if $n = 2m + 1 > 3$, then $\sum_{1 \leq i < j \leq n} P_iP_j < m(m+1)$.

First, note that, if $n = 3$, then $P_1P_2+P_2P_3+P_1P_3 = 2P_1P_3 \le 2$, with equality if and only if $P_1P_3 = 1$. Now, if $n = 5$, then $\sum_{1<i<j<5} P_iP_j < 2$ (since $P_2P_43 < P_1P_5 \le 1$). Therefore,

$$\begin{aligned}\sum_{1\le i<j\le 5} P_iP_j &= \sum_{1<i<j<5} P_iP_j + P_1P_5 + \sum_{1<k<5} (P_1P_k + P_kP_5)\\ &= \sum_{1<i<j<5} P_iP_j + P_1P_5 + \sum_{1<k<5} P_1P_5\\ &= \sum_{1<i<j<5} P_iP_j + 4P_1P_5 < 2+4 \;=\; 6\,.\end{aligned}$$

Suppose that the result is true for some $n = 2m+1 > 3$. We show that it is also true for $n+2$. Indeed, by the inductive hypothesis, we have $\sum_{1<i<j<n+1} P_iP_j < m(m+1)$. Hence,

$$\begin{aligned}\sum_{1\le i<j\le n+2} P_iP_j &= \sum_{1<i<j<n+2} P_iP_j + P_1P_{n+2} + \sum_{1<k<n+2} (P_1P_k + P_kP_{n+2})\\ &= \sum_{1<i<j<n+2} P_iP_j + P_1P_{n+2} + \sum_{1<k<n+2} P_1P_{n+2}\\ &= \sum_{1<i<j<n+2} P_iP_j + (n+1)P_1P_{n+2}\\ &< m(m+1) + (n+1) \;=\; (m+2)(m+1)\,.\end{aligned}$$

The proof is complete.

To round things out with respect to the February 1999 number here is an additional solution.

4. [1999 : 6–7] *Vietnamese Mathematical Olympiad 3 / 1996, Category B.*

Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying simultaneously two conditions:

(i) $f(1995) = 1996$

(ii) for every $n \in \mathbb{Z}$, if $f(n) = m$, then $f(m) = n$ and $f(m+3) = n-3$, ($\mathbb{Z}$ is the set of integers).

Solution by Pierre Bornsztein, Courdimanche, France.

Let $f : \mathbb{Z} \to \mathbb{Z}$ satisfy the two conditions. From (ii), we deduce: $f(f(n)) = n$ for all $n \in \mathbb{Z}$; that is, f is involutory.

The equality $f(m+3) = n-3$ then gives $m+3 = f(n-3)$ or $f(n-3) = f(n)+3$. From this, we easily obtain, by induction, $f(n-3k) = f(n)+3k$ for all positive integers k.

Let m be any integer; m may be written $m = f(n)$ for an n (for $n = f(m)$, actually). Condition (ii) gives $f(m+3) = f(m) - 3$, and by induction: $f(m+3k) = f(m) - 3k$ for all positive integers k. All this can be summed up by $f(n+3k) = f(n) - 3k$ for all $n, k \in \mathbb{Z}$.

It follows that for every $k \in \mathbb{Z}$,

$$\begin{aligned} f(3k) &= f(0) - 3k\,, \\ f(3k+1) &= f(1) - 3k\,, \\ f(3k+2) &= f(2) - 3k\,. \end{aligned}$$

Let $k = 665$. We have $f(1995) = f(0) - 1995 = 1996$ and $f(1996) = 1995 = f(1) - 1995$. Then $f(0) = 3991$ and $f(1) = 3990$.

Thus, for every $k \in \mathbb{Z}$,

$$\begin{aligned} f(3k) &= 3991 - 3k\,, \\ f(3k+1) &= 3990 - 3k\,, \\ f(3k+2) &= f(2) - 3k\,. \end{aligned}$$

We remark that

$$\begin{aligned} f(3\mathbb{Z}) &= 3\mathbb{Z} + 1\,, \\ f(3\mathbb{Z}+1) &= 3\mathbb{Z}\,, \\ \text{and so,}\quad f(3\mathbb{Z}+2) &\subset 3\mathbb{Z} + 2\,. \end{aligned}$$

Moreover f is bijective, so $f(3\mathbb{Z}+2) = 3\mathbb{Z}+2$. Denote $f(2) = 3a + 2$, where $a \in \mathbb{Z}$. Then:

$$\text{if } f \text{ is a solution then } f(m) = \left\{ \begin{array}{ll} 3991 - m & \text{if } m \not\equiv 2 \pmod 3 \\ 3a + 4 - m & \text{if } m \equiv 2 \pmod 3\,. \end{array} \right\}$$

Conversely, a straightforward calculation shows that any such f satisfies the conditions.

Now, we turn to solutions from our readers to problems given in the March 1999 number of the *Corner* with problems of the 19th Austrian-Polish Mathematics Competition 1996 [1999 : 70–71].

1. Let $k \geq 1$ be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties:

(a) The decimal representation of n consists of exactly k digits.

(b) All digits of n are odd.

(c) The number n is divisible by 5.

(d) The number $m = \frac{n}{5}$ has k odd (decimal) digits.

Solution by Jacqueline Freeman and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Call a natural number "ideal" if it satisfies conditions (a)–(d). We show that $n = a_k a_{k-1} \cdots a_1 \in \mathbb{N}$ is ideal if and only if $a_i \in \{5, 7, 9\}$ for all $i = 1, 2, \ldots, k$ and $a_1 = 5$. This is clearly true when $k = 1$, and thus, we assume that $k \geq 2$.

If n is ideal, then (c) implies that $a_1 = 5$. We show that none of the a_i's can equal 1 or 3. Let $n/5 = b_k b_{k-1} \cdots b_1$ and consider the process of long division of n by 5. Clearly, $a_k \neq 1, 3$ for otherwise $n/5$ would have only $k-1$ digits. If $a_i = 1$ or 3 for any i, $2 \leq i \leq k-1$, then clearly $b_i = 0, 2, 4, 6$, or 8, depending on whether the remainder "carried over" when dividing a_{i+1} by 5 is 0, 1, 2, 3, or 4, respectively. This is a contradiction to (d).

Conversely, suppose $n = a_k a_{k-1} \cdots a_1$, where $a_i \in \{5, 7, 9\}$ for all $i = 1, 2, \ldots, k$ and $a_1 = 5$. To show that n is ideal, it clearly suffices to verify (d). Since $a_k \geq 5$, $m = n/5$ has k digits, and we can write $m = b_k b_{k-1} \cdots b_1$. Clearly, b_1 is odd. If b_i is even for some i, $2 \leq i \leq k$, then from $n = 5m$ we see that $a_i = 0, 1, 2, 3$, or 4 depending on whether the "carry" (when b_{i-1} is multiplied by 5) is 0, 1, 2, 3, or 4, respectively. This is a contradiction. Hence, all the b_i's are odd. This completes the proof of our claim.

Finally, since $a_1 = 5$ and each of the a_i's ($i = 2, 3, \ldots, k$) can take on any one of the 3 values 5, 7, or 9, the total number of ideal integers is 3^{k-1}.

Remark. In fact, it is not difficult to show that if n is ideal then all the digits of $n/5$ must be 1, 5, or 9.

2. A convex hexagon $ABCDEF$ satisfies the following conditions:

(a) The opposite sides are parallel (that is, $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$).

(b) The distances between the opposite sides are equal (that is, $d(AB, DE) = d(BC, EF) = d(CD, FA)$, where $d(g, h)$ denotes the distance between lines g and h).

(c) $\angle FAB$ and $\angle CDE$ are right angles.

Show that diagonals BE and CF intersect at an angle of $45°$.

Solution by Toshio Seimiya, Kawasaki, Japan.

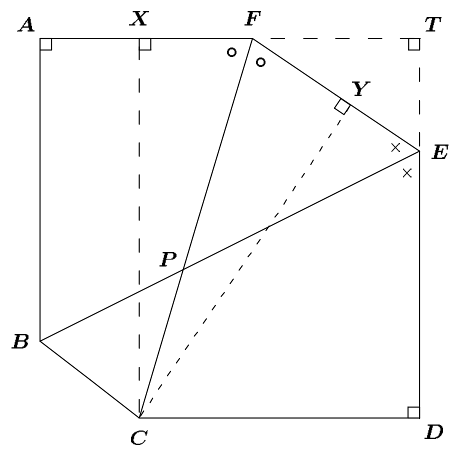
Let X, Y be the feet of the perpendiculars from C to AF, EF, respectively. (See figure.)

Since $AF \parallel CD$, it follows that $CX = d(AF, CD)$.

Since $BC \parallel EF$, it follows that $CY = d(BC, EF)$.

Since $d(AF, CD) = d(BC, EF)$, we have $CX = CY$. Thus, CF bisects $\angle AFE$. Similarly, BE bisects $\angle DEF$. Let T be the intersection of AF and DE. Since $AB \parallel DE$ and $\angle A = 90°$, we get

$$\angle FTE = 90° .$$



Then,

$$\begin{aligned}&\angle AFE + \angle DEF \\ &\quad = (\angle T + \angle TEF) + (\angle T + \angle TFE) \\ &\quad = 2\angle T + (\angle TEF + \angle TFE) \\ &\quad = 90^\circ \times 2 + 90^\circ = 270^\circ .\end{aligned}$$

Thus, we have

$$\angle PFE + \angle PEF = \tfrac{1}{2}(\angle AFE + \angle DEF) = 135^\circ .$$

Hence, we get

$$\begin{aligned}\angle FPE &= 180^\circ - (\angle PFE + \angle PEF) \\ &= 180^\circ - 135^\circ = 45^\circ .\end{aligned}$$

3. The polynomials $P_n(x)$ are defined recursively by $P_0(x) = 0$, $P_1(x) = x$ and

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \quad \text{for} \quad n \geq 2 .$$

For every natural number $n \geq 1$, find all real numbers x satisfying the equation $P_n(x) = 0$.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's write-up.

We will prove that for $n \geq 1$, the only real solution of $P_n(x) = 0$ is $x = 0$.

For $n \geq 2$,

$$P_n(x) - P_{n-1}(x) = (x-1)(P_{n-1}(x) - P_{n-2}(x)) .$$

Then an easy induction leads to

$$P_n(x) - P_{n-1}(x) \;=\; (x-1)^{n-1}(P_1(x) - P_0(x)) \;=\; x(x-1)^{n-1}\,.$$

That is, $P_n(x) = P_{n-1}(x) + x(x-1)^{n-1}$ for $n \geq 2$, and we note that it remains true for $n = 1$.

We deduce that, for $n \geq 1$,

$$\begin{aligned} P_n(x) &= x(x-1)^{n-1} + x(x-1)^{n-2} + \cdots + x + P_0(x) \\ &= x((x-1)^{n-1} + (x-1)^{n-2} + \cdots + 1)\,. \end{aligned}$$

Then, if $x = 2$, $P_n(2) = 2n \neq 0$ and if $x \neq 2$, $P_n(x) = x \cdot \frac{((x-1)^n - 1)}{x-2}$. Thus, $P_n(x) = 0$ if and only if $x = 0$ or $(x-1)^n = 1$ for $x \neq 2$.

If n is even, $(x-1)^n = 1$ if and only if $x = 0$ or $x = 2$. Then $P_n(x) = 0$ if and only if $x = 0$.

If n is odd, $(x-1)^n = 1$ if and only if $x = 2$. Then $P_n(x) = 0$ if and only if $x = 0$.

Thus, for $n \geq 1$, $P_n(x) = 0$ if and only if $x = 0$.

4. The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that $-1 \leq xy + yz + zt + tx \leq 0$.

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's solution.

First, we have:

$$xy + yz + zt + tx \;=\; (x+z)(y+t) \;=\; -(x+z)^2 \;\leq\; 0$$

(since $y + t = -(x+z)$). Then,

$$|xy + yz + zt + tx| \;\leq\; (x^2 + y^2 + z^2 + t^2)^{1/2}(y^2 + z^2 + t^2 + x^2)^{1/2} \;=\; 1$$

(by the Cauchy-Schwarz inequality). The conclusion follows.

Remark. Equality $xy + yz + zt + tx = 0$ holds if and only if $x + z = y + t = 0$. Therefore, inequality $xy + yz + zt + tx \leq 0$ becomes an equality for the quadruplets $(x, y, z, t) = (a, b, -a, -b)$ where a, b are real numbers such that $a^2 + b^2 = \frac{1}{2}$.

If equality $xy + yz + zt + tx = -1$ holds, then we have equality in the Cauchy-Schwarz inequality so that (x, y, z, t) and (y, z, t, x) are proportional. Since at least one of the numbers x, y, z, t must be non-zero, this leads to $x = y = z = t$ or $y = -x$, $z = x$, $t = -x$. The first case is incompatible with the hypotheses, and the second case provides only the quadruplets:

$$\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) \quad \text{and} \quad \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) .$$

Conversely, these two quadruplets satisfy $xy + yz + zt + tx = -1$ and we may conclude: $xy + yz + zt + tx = -1$ holds if and only if $(x, y, z, t) = \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$ or $\left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$.

6. Natural numbers k, n are given such that $1 < k < n$. Solve the system of n equations

$$x_i^3 \cdot (x_i^2 + x_{i+1}^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2 \quad \text{for} \quad 1 \le i \le n ,$$

with n real unknowns x_1, x_2, $\dots$, x_n. Note: $x_0 = x_n$, $x_{n+1} = x_1$, $x_{n+2} = x_2$, and so on.

Solution by Pierre Bornsztein, Courdimanche, France.

We prove that there are two solutions

$$x_1 = x_2 = \cdots = x_n = 0$$

and

$$x_1 = x_2 = \cdots = x_n = \frac{1}{\sqrt[3]{k}} .$$

First, suppose that $(x_1, \dots, x_n)$ is a solution.

First Case. There is $i \in \{1, \dots, n\}$ such that $x_i = 0$.

From the cyclic symmetry, we suppose, without loss of generality, that $x_1 = 0$. Then

$$x_1^3(x_1^2 + \cdots + x_k^2) = 0 = x_n^2 .$$

Thus, $x_n = 0$.

Continuing, we deduce that $x_i = 0$ for all i.

Second Case. $x_i \neq 0$ for all i.

Since $x_i^3(x_i^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2$, we deduce that $x_i > 0$ for all i. From cyclic symmetry, without loss of generality, we have

$$x_1 = \min\{x_i : i = 1, 2, \dots, n\} .$$

Then $x_1^3 \le x_2^3$ and $x_1^2 \le x_{k+1}^2$. Thus,

$$x_1^2 + x_2^2 + \cdots + x_k^2 \le x_2^2 + x_3^2 + \cdots + x_{k+1}^2 .$$

We deduce that

$$\begin{aligned} x_n^2 &= x_1^3(x_1^2 + \cdots + x_k^2) \\ &\le x_2^3(x_2^2 + \cdots + x_{k+1}^2) = x_1^2 . \end{aligned}$$

Thus, $x_n \le x_1$.

What we have actually shown is that if $x_i = \min\{x_1, \ldots, x_n\}$, then $x_{i-1} = x_i$ $(= \min\{x_1, \ldots, x_n\})$. An easy induction leads to $x_1 = x_2 = \cdots = x_n$.

Let a denote the common value. Then $a^3(ka^2) = a^2$; that is, $a = \frac{1}{\sqrt[3]{k}}$.

Conversely, it is easy to verify that $(0, 0, \ldots, 0)$ and $\left(\frac{1}{\sqrt[3]{k}}, \ldots, \frac{1}{\sqrt[3]{k}}\right)$ are solutions.

7. Show that there do not exist non-negative integers k and m such that $k! + 48 = 48(k+1)^m$.

Solution by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Suppose the given equation holds for some non-negative integers k and m. Then $48 \mid k!$. Since $48 = 2^4 \times 3$, we must have $k \geq 6$. If $k = 6$ or 7, the equation becomes $16 = 7^m$ or $106 = 8^m$, respectively. Clearly, neither is possible. Hence, $k \geq 8$ and the given equation can be rewritten as

$$3 \times 5 \times 7 \times 8 \times \cdots \times (k-1) \times k + 1 \;=\; (k+1)^m\,. \tag{1}$$

Suppose $k+1$ is a composite. Then it has a prime divisor q. Since $q \leq k$, we have $q \mid k!$, which implies $q \mid 48$. Since $k \geq 8$, the left side of (1) is odd and thus, q must be odd. Hence, $q = 3$, which is clearly impossible in view of (1), since $3 \not\mid 1$.

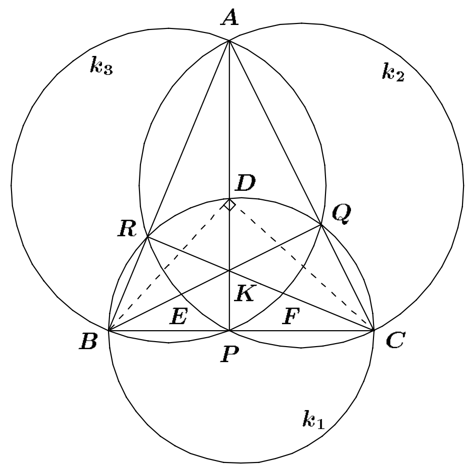
Therefore, $k+1 = p$ is a prime. Then $k! = (p-1)! \equiv -1 \pmod{p}$ by Wilson's Theorem; that is, $p \mid k! + 1$.

Rewriting the given equation as $k! + 1 + 47 = 48p^m$, we infer that $p \mid 47$ and so $p = 47$. Then we have $46! + 48 = 48 \times 47^m$ or $46! = 48(47^m - 1)$. Since the prime divisors of $46!$ include 5, 7, 11 which are all coprime with 48, we have $47^m \equiv 1 \pmod{5, 7, 11}$. Now, straightforward checking reveals that $\mathrm{ord}_5(47) = 4$ (that is, the least positive integer n such that $47^n \equiv 1 \pmod{5}$ is $n = 4$), $\mathrm{ord}_7(47) = 6$ and $\mathrm{ord}_{11}(47) = 5$. Hence, m is divisible by $\mathrm{lcm}\{4, 6, 5\}$; that is, $60 \mid m$. So $m \geq 60$ and we have $48 \times 47^m \geq 48 \times 47^{60}$. Clearly, $48 \times 47^{60} > 46! + 48$ and we have a contradiction.

Next we turn to solutions to problems of the 3rd Turkish Mathematical Olympiad [1999 : 72].

2. For an acute triangle ABC, k_1, k_2, k_3 are the circles with diameters $[BC]$, $[CA]$, $[AB]$, respectively. If K is the radical centre of these circles, $[AK] \cap k_1 = \{D\}$, $[BK] \cap k_2 = \{E\}$, $[CK] \cap k_3 = \{F\}$ and $\mathrm{area}(ABC) = u$, $\mathrm{area}(DBC) = x$, $\mathrm{area}(ECA) = y$, and $\mathrm{area}(FAB) = z$, show that $u^2 = x^2 + y^2 + z^2$.

Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



Let P, Q, and R be the feet of the perpendiculars from A, B, and C, to BC, CA, and AB, respectively.

AP, BQ, and CR are common chords of k_2, k_3; k_3, k_1; and k_1, k_2, respectively. Thus, K is the orthocentre of $\triangle ABC$. Since $\angle BDC = 90°$ and $DP \perp BC$, we get

$$DP^2 = BP \cdot CP \,. \tag{1}$$

Since $BK \perp AC$ and $AP \perp BC$, we have $\angle BKP = \angle ACP$. Further, we have $\triangle BKP \sim \triangle ACP$. It follows that $BP : AP = KP : CP$; that is

$$AP \cdot KP = BP \cdot CP \,. \tag{2}$$

From (1) and (2) we have

$$DP^2 = AP \cdot KP \,.$$

Hence, we have

$$DP^2 \cdot BC^2 = (AP \cdot BC) \times (KP \cdot BC) \,.$$

From this we get

$$x^2 = u \times \text{area}\ (KBC) \,. \tag{3}$$

Similarly, we have

$$y^2 = u \times \text{area}\ (KCA) \,, \tag{4}$$

and

$$z^2 \;=\; u \times \text{area}\;(KAB)\,. \qquad (5)$$

Therefore, we obtain from (3), (4), and (5),

$$\begin{aligned} x^2+y^2+z^2 &= u \times \{\text{area}\;(KBC) + \text{area}\;(KCA) + \text{area}\;(KAB)\} \\ &= u \times \text{area}\;(ABC) \\ &= u^2\,. \end{aligned}$$

3. Let $\mathbb{N}$ denote the set of positive integers. Let A be a real number and $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 1$ and

$$1 < \frac{a_{n+1}}{a_n} \;\leq\; A \quad \text{for all} \quad n \in \mathbb{N}\,.$$

(a) Show that there is a unique non-decreasing surjective function $k : \mathbb{N} \to \mathbb{N}$ such that $1 < \frac{A^{k(n)}}{a_n} \leq A$ for all $n \in \mathbb{N}$.

(b) If k takes every value at most m times, show that there exists a real number $C > 1$ such that $C^n \leq Aa_n$ for all $n \in \mathbb{N}$.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

(a) The condition $1 < \frac{A^{k(n)}}{a_n} \leq A$ is equivalent to $A^{k(n)-1} \leq a_n < A^{k(n)}$ or $k(n) - 1 \leq \frac{\ln(a_n)}{\ln(A)} < k(n)$. Hence, k is necessarily the function given by $k(n) = 1 + \left\lfloor \frac{\ln(a_n)}{\ln(A)} \right\rfloor = 1 + \lfloor \log_A(a_n) \rfloor$ for all $n \in \mathbb{N}$. This shows the unicity of k. Since $\{a_n\}$ is increasing and $a_1 = 1$, we have $a_n \geq 1$. We also note $A > 1$. Hence, $\frac{\ln(a_n)}{\ln(A)}$ is a non-negative real number and $1 + \lfloor \log_A(a_n) \rfloor$ is a positive integer. Thus, we can define a function $k : \mathbb{N} \to \mathbb{N}$ by the formula $k(n) = 1 + \lfloor \log_A(a_n) \rfloor$. For all $n \in \mathbb{N} : a_n < a_{n+1}$, so that $\log_A(a_n) < \log_A(a_{n+1})$ and $k(n) \leq k(n+1)$. Thus, k is non-decreasing. Now, let us remark that, for $s \in \mathbb{N}$, $k(n) = s$ is equivalent to $A^{s-1} \leq a_n < A^s$. Therefore, if $\{a_n\}$ is bounded above, say $a_n \leq M$ for all n, an s satisfying $A^{s-1} > M$ (such an s exists since $A > 1$) cannot be an image under k. Thus, if $\{a_n\}$ satisfies the hypothesis and is convergent (for instance $a_n = \frac{1}{2}\left(1 + \frac{1}{n}\right)^n$), the surjectivity of k cannot be obtained. Consequently, we will henceforth assume that $\{a_n\}$ is not bounded above.

Given $s \in \mathbb{N}$, we prove that the equation $k(n) = s$ has at least one solution. For $s = 1$, $n = 1$ is obviously a solution, so now we suppose that $s \geq 2$. From the supplementary hypothesis, there exists $n \in \mathbb{N}$ such that $a_n \geq A^{s-1}$. Let r be the least of these n's, so that $a_{r-1} < A^{s-1} \leq a_r$. Then $a_r \leq Aa_{r-1} < A^s$ so that $A^{s-1} \leq a_r < A^s$ and r is a solution. The function k fulfils all the demands and (a) follows.

(b) For each $s \in \mathbb{N}$, denote by $N(s)$ the number of n such that $k(n) = s$; that is, such that $A^{s-1} \leq a_n < A^s$. By hypothesis, we have $N(s) \leq m$ for each s.

We will show that (b) holds with $C = A^{1/m} > 1$. Let n be an arbitrary integer. If $k(n) = s$, then

$$n = N(1) + N(2) + \cdots + N(s-1) + j \quad \text{where} \quad j \in \{1, 2, \ldots, N(s)\}$$

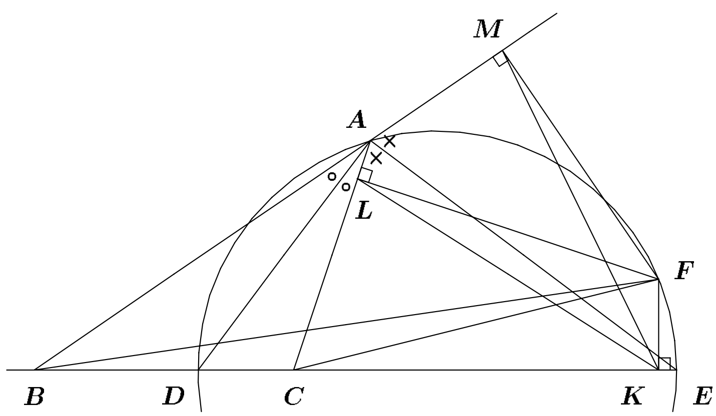
(because there are $N(1) + N(2) + \cdots + N(s-1)$ terms of the sequence $\{a_n\}$ which are $< A^{s-1}$). On the one hand, $a_n A \geq A^{s-1}A = A^s = C^{ms}$, and, on the other hand,

$$\begin{aligned} n &= N(1) + N(2) + \cdots + N(s-1) + j \\ &\leq N(1) + N(2) + \cdots + N(s-1) + N(s) \;\leq\; sm\,. \end{aligned}$$

Hence, $C^n \leq C^{ms}$ and we obtain $C^n \leq a_n A$. So (b) is proved.

4. In a triangle ABC with $|AB| \neq |AC|$, the internal and external bisectors of the angle A intersect the line BC at D and E, respectively. If the feet of the perpendiculars from a point F on the circle with diameter $[DE]$ to the lines BC, CA, AB are K, L, M, respectively, show that $|KL| = |KM|$.

Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.



Since the circle with diameter DE is the Apollonius circle, we have

$$\frac{FB}{FC} = \frac{AB}{AC}\,. \tag{1}$$

By the Law of Sines, we have

$$\frac{AB}{AC} = \frac{\sin C}{\sin B}\,. \tag{2}$$

From (1) and (2), we get
$$\frac{FB}{FC} = \frac{\sin C}{\sin B};$$
that is,
$$FB\sin B = FC\sin C. \tag{3}$$

Since the circle with diameter BF passes through K and M, we have
$$KM = FB\sin B.$$

Similarly, we have
$$KL = FC\sin C.$$

Therefore, we obtain from (3)
$$KL = KM.$$

5. Let $t(A)$ denote the sum of elements of A for a non-empty subset A of integers, and define $t(\phi) = 0$. Find a subset X of the set of positive integers such that for every integer k there is a unique ordered pair of subsets (A_k, B_k) of X with $A_k \cap B_k = \phi$ and $t(A_k) - t(B_k) = k$.

Solution by Pierre Bornsztein, Courdimanche, France.

We will prove that $X = \{3^i : i \in \mathbb{N}\}$ has the desired property.

LEMMA. Every integer n can be written $n = \sum \alpha_i 3^i$ where for all i, $\alpha \in \{-1, 0, 1\}$ and the sum has only a finite number of non-zero terms. Moreover, such a decomposition is unique.

Proof of the Lemma. First we prove the unicity:

If $\sum_{i=0} \alpha_i 3^i = \sum_{j=0} \beta_j 3^j$ with α_i, $\beta_j \in \{-1, 0, 1\}$, we recognize two possibilities.

Case 1. For all i, j, $\alpha_i = \beta_j = 0$, we are done.

Case 2. Otherwise, let i_0 be the smallest subscript such that $\alpha_{i_0} \neq 0$ or $\beta_{i_0} \neq 0$. Then
$$(\alpha_{i_0} - \beta_{i_0})3^{i_0} = -\sum_{i>i_0} \alpha_i 3^i + \sum_{j>i_0} \beta_j 3^j.$$

The right-hand side is divisible by 3^{i_0+1}. We deduce that $\alpha_{i_0} - \beta_{i_0}$ is divisible by 3. But $-2 \leq \alpha_{i_0} - \beta_{i_0} \leq 2$. Thus, $\alpha_{i_0} = \beta_{i_0}$. It follows that, for $0 \leq i \leq i_0$, $\alpha_i = \beta_i$ and $\sum_{i>i_0} \alpha_i 3^i = \sum_{j>i_0} \beta_j 3^j$.

If necessary, we denote by i_1 the smallest subscript such that $i_1 > i_0$ and $\alpha_{i_1} \neq 0$ or $\beta_{i_1} \neq 0$. Then if $0 \leq i < i_1$, $\alpha_i = \beta_i$, and with the same reasoning as above, we get $\alpha_{i_1} = \beta_{i_1}$, and so on. Unicity follows.

Now, let p be a fixed positive integer. The numbers of the form $N = \sum_{i=0}^{p} \alpha_i 3^i$ with $\alpha_i \in \{-1, 0, 1\}$ are all distinct, from above. For each $i \in \{0, \ldots, p\}$, there are 3 possible choices for α_i. Thus, we obtain exactly 3^{p+1} such integers. Moreover

$$-\sum_{i=0}^{p} 3^i \;\leq\; \sum_{i=0}^{p} \alpha_i 3^i \;\leq\; \sum_{i=0}^{p} 3^i\,;$$

that is,

$$-\left(\frac{3^{p+1}-1}{2}\right) \;\leq\; \sum_{i=0}^{p} \alpha_i 3^i \;\leq\; \left(\frac{3^{p+1}-1}{2}\right).$$

But there are exactly 3^{p+1} integers in the interval $\left[-\left(\frac{3^{p+1}-1}{2}\right), \frac{3^{p+1}-1}{2}\right]$. It follows that each integer can be expressed in this form, proving the lemma.

Now, let k be an integer. From the lemma, $k = \sum_{i\geq 0} \alpha_i 3^i$, where $\alpha_i \in \{-1, 0, 1\}$, and $\alpha_i \neq 0$ for at most finitely many values of i. Denote $A_k = \{3^i : \alpha_i = 1\}$ and $B_k = \{3^i : \alpha_i = -1\}$. Then $A_k, B_k \subset X$, $A_k \cap B_k = \emptyset$ and $t(A_k) - t(B_k) = k$. Moreover, suppose that A'_k, B'_k are subsets of X with $A'_k \cap B'_k = \emptyset$ and $t(A'_k) - t(B'_k) = k$. Note that A'_k and B'_k must be finite. Then,

$$\sum_{3^i \in A_k} 3^i - \sum_{3^i \in B_k} 3^i = k = \sum_{3^i \in A'_k} 3^i - \sum_{3^i \in B'_k} 3^i\,.$$

The unicity of the form $\sum \alpha_i 3^i$ gives $A_k = A'_k$ and $B_k = B'_k$.

6. Let $\mathbb{N}$ denote the set of positive integers. Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the condition

$$m \mid n \iff f(m) \mid f(n)$$

for all $m, n \in \mathbb{N}$.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bataille's solution.

For such a function f, we have:

(a) $f(1) = 1$: Since $1 \mid n$, we have $f(1) \mid f(n)$ for all $n \in \mathbb{N}$. But f is surjective, so that $f(n)$ may be any positive integer; hence, $f(1)$ divides all positive integers and $f(1) = 1$.

(b) f is bijective: f is already surjective; moreover, if $f(m) = f(n)$, then $f(m) \mid f(n)$ and $f(n) \mid f(m)$ so that $m \mid n$ and $n \mid m$; that is, $m = n$. Hence, f is also injective.

Obviously, the condition on f also reads: $m \mid n \iff f^{-1}(m) \mid f^{-1}(n)$ for all $m, n \in \mathbb{N}$.

(c) $f(p)$ is prime whenever p is prime: If k divides $f(p)$, $f^{-1}(k)$ divides p, so that $f^{-1}(k) = 1$ or p and $k = 1$ or $f(p)$.

More generally, $f(p^a) = (f(p))^a$ for all positive integers a: Let q be a prime integer dividing $f(p^a)$. Then $f^{-1}(q)$ is a prime integer dividing p^a. It follows that $f^{-1}(q) = p$ and $q = f(p)$. Therefore, $f(p^a)$ is a power of $f(p)$, say $f(p^a) = (f(p))^b$. Now, the $a+1$ distinct divisors of p^a, namely 1, p, $\dots$, p^a, provide $a+1$ distinct divisors of $f(p^a) = (f(p))^b$, namely 1, $f(p)$, $\dots$, $f(p^a)$. As $(f(p))^b$ has $b+1$ divisors, we have $a \le b$. Similarly, using f^{-1}, $b \le a$. Thus, $a = b$ and $f(p^a) = (f(p))^a$.

(d) If u, v are coprime positive integers, then $f(u)$, $f(v)$ are also coprime and $f(uv) = f(u)f(v)$. That $f(u)$, $f(v)$ are coprime is immediate since any prime integer p dividing $f(u)$ and $f(v)$ would provide a prime integer, namely $f^{-1}(p)$, dividing u and v.

Since $u \mid uv$ and $v \mid uv$, $f(u) \mid f(uv)$ and $f(v) \mid f(uv)$. Taking into account the result we have just shown, we get $f(u) \cdot f(v) \mid f(uv)$. With f^{-1}, $f(u)$, $f(v)$ instead of f, u, v, we obtain $u \cdot v \mid f^{-1}(f(u) \cdot f(v))$ and consequently $f(uv) \mid f(u) \cdot f(v)$. Hence, $f(uv) = f(u)f(v)$. Let P be the set of prime integers. We have obtained the following concerning f: The restriction of f to P is a bijection from P onto P and, combining the last two results, if $n = \prod_{p \in P} p^{v_p(n)}$, then $f(n) = \prod_{p \in P} (f(p))^{v_p(n)}$. [Here, $v_p(n)$ denotes the exponent of p in the standard factorization of n into prime powers: of course, we have $v_p(n) = 0$ for all but a finite number of p and the sequence $(v_p(n))_{p \in P}$ is uniquely determined by n.]

Conversely, let $\phi : P \to P$ be any bijection from P onto P and let $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n) = \prod_{p \in P} (\phi(p))^{v_p(n)} \quad \text{when} \quad n = \prod_{p \in P} p^{v_p(n)} . \qquad (1)$$

Clearly, f is surjective. Moreover,

$$\begin{aligned} m \mid n &\iff v_p(m) \le v_p(n) && \text{for all } p \in P \\ &\iff v_{\phi(p)}(f(m)) \le v_{\phi(p)}(f(n)) && \text{for all } p \in P \\ &\iff v_q(f(m)) \le v_q(f(n)) && \text{for all } q \in P \\ &\iff f(m) \mid f(n) . \end{aligned}$$

In conclusion, the solutions are the functions extending to $\mathbb{N}$ the bijections $\phi : P \to P$ by means of the formula (1).

That completes the *Olympiad Corner* for this issue. Send me your Olympiad contest materials and your nice solutions!

BOOK REVIEWS

ALAN LAW

Calculus Mysteries and Thrillers, by R. Grant Woods
published by the Mathematical Association of America, 1998,
ISBN 0-88385-711-1, softcover, 131+ pages, $24.95 (US).
Reviewed by ***G.J. Griffith****, University of Saskatchewan, Saskatoon, Saskatchewan.*

This book consists of eleven single variable calculus problems together with their solutions. The problems are posed in story form. In each case, the student is supposedly a client who is to provide a complete solution to a judge, a government agency or some other "math freak", which implies that slip-shod solutions are unacceptable.

My favourite problem is the first, which involves the mathematics of banking pool balls off a parabolic rail. I like it because it is relatively easy and should be accessible to some of my better freshmen students (and perhaps it reflects my mis-spent youth). I also like the *Sunken Treasure* problem, which involves lowering a parabolic hull onto the ocean floor (classified as difficult) and the somewhat standard *Designing Dipsticks* problem, which requires the student to calculate volumes of solids of revolution.

I dislike *The Case of the Swivelling Spotlight*, *Saving Lunar Station Alpha* and *The Case of the Alien Agent*, since I consider these to be far too complicated even for top rate freshmen.

Dr. Woods has assigned seven of the eleven problems to groups of students in his calculus classes and claims that "this experience has convinced [him] that these projects are feasible for reasonably bright freshmen working in groups." I can only comment that from my experience, this implies that Manitoba freshmen are significantly ahead of their counterparts who reside in the neighbouring province to the West.

The Math Chat Book, by Frank Morgan,
published by the Mathematical Association of America
(Spectrum Series), 2000.
ISBN 0-88385-530-5, softcover, 113 pages, $19.95 (U.S.).
Reviewed by ***Sandy Graham****, University of Waterloo, Waterloo, Ontario.*

This book is a compilation of questions and answers inspired by problems posed to the author during a live call-in TV show also called Math Chat. The book/show does for mathematics, what the CBC program Quirks and Quarks does for science. The author tries to pose interesting problems that will make everyone enjoy math regardless of their past experiences. He has divided the book into five sections called stories: Time, Probabilities and Possibilities, Prime Numbers and Computing, Geometry, and Physics and the World. Each story is divided into several short episodes of related questions. There is even a puzzle at the beginning of the book with a $1000 prize for the best solution submitted to the Mathematical Association of America's website.

The problems range from modular arithmetic relating to leap years, to some game theory analysis of tic-tac-toe, to classic statistics fallacies when flipping a coin, to discussing the largest prime number computed to date. Some of the problems posed, however, seem to have little to do with mathematics, such as "Which makes the water level in a bucket rise more, adding a pound of salt or a pound of sand?" In a few cases, the author poses a problem but does not give a solution. In most cases, the solutions go into very little detail of the underlying mathematics.

The book is quick to read for anyone interested in mathematics, and it may spark some interest for those who have shied away from this discipline in the past. The problems could provide interesting subject matter for discussion in a high school math club. With its wide variety of topics, teachers could use some of the Math Chat scenarios as starting points for more detailed analysis in almost any mathematics class, from high school to university. Teachers may even be inspired to generate more quirky mathematical problems relating to the topics in their courses.

Although the mathematical content is sometimes questionable, overall the book is worth reading. My favourite excerpt comes from Episode 7 — Predicting the Random. It is typical of many of the "problems" of the book — a little math mixed with a little fun.

Mathematical Trick

1. What is $2 + 2$?
2. What is $4 + 4$?
3. What is $8 + 8$?
4. What is $16 + 16$?
5. Quick, pick a number between 12 and 5.

Math Chat predicts that your answer is 7. Try it out on your friends; it works. Why not start your next discussion about it?

A Nice COMC Problem

Daryl Tingley

Question B12 of the 1999 Canadian Open Mathematics Challenge was:

Q: *Triangle ABC is any one of the set of triangles having base BC equal to a and height from A to BC equal to h, with $h < \frac{\sqrt{3}}{2}a$. P is a point inside the triangle such that the value of*

$$\measuredangle PAB = \measuredangle PBA = \measuredangle PCB = \alpha\,.$$

Show that the measure of α is the same for every triangle in the set.

The problem is one implication of the following locus problem:

L: *Let BC be parallel to the line ℓ such that the distance from BC to ℓ is less than $\frac{\sqrt{3}}{2}$ of the length of BC. Let A be a variable point on ℓ. Find the locus of all points P inside $\triangle ABC$ such that*

$$\measuredangle PAB = \measuredangle PBA = \measuredangle PCB. \qquad (*)$$

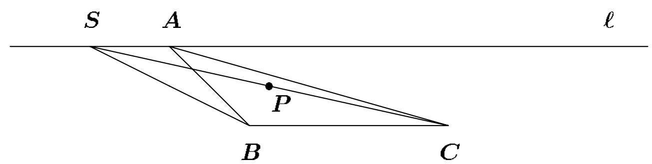
The two constraints: "the distance from BC to ℓ is less than $\frac{\sqrt{3}}{2}$ of the length of BC", and "P inside $\triangle ABC$" restrict the more general question:

G: *Let BC be parallel to the line ℓ. Let A be a variable point of ℓ. Find the locus of all points P that satisfy* $(*)$.

These are nice problems for computer investigation. The locus for **L** is hard to guess. When **G** is considered, the locus consists of several branches, some of which are unfamiliar and difficult curves. A computer geometry package, such as Geometer's Sketchpad, easily allows one to find solutions P associated with each A. (Note that P must lie on the perpendicular bisector of AB.) Such an investigation should reveal the answer to **L**, and indicate that the answer to **G** is more difficult.

To use the geometry package to produce a trace of the locus, a compass and straight edge type construction for P must be found. That is, given A, a compass and straight edge construction must be found for the various P that satisfy $(*)$. Such constructions are (of course) tied closely to a synthetic proof of the problem. Finally, a computer package such as **MAPLE** can be used to find an equation for the unfamiliar branches of the trace. (After eliminating the radicals, an 8^{th} degree polynomial, with highest order term x^4y^4, is the result.)

In this article, we outline a solution of **G**, leaving much of the work to the reader. To avoid tiresome details, special cases are ignored: cases such as when P lies on the lines that form the edges of the triangle. The reader can verify (easily) that there is nothing exciting about them.



Theorem 1. Let BC be parallel to ℓ. Let A be on ℓ and P inside $\triangle ABC$ with $\measuredangle PAB = \measuredangle PBA = \measuredangle PCB = \alpha$. If the line PC meets ℓ at S, then the points B, P, A and S lie on a circle. As a consequence, $SB = BC$ and $\measuredangle ASC = \measuredangle BCS = \measuredangle BSC$.

Proof. $\measuredangle BCS = \measuredangle ASC = \alpha$ (interior opposite angles), showing that $\measuredangle ASC = \measuredangle ASP = \measuredangle ABP$, and that $ASBP$ is concyclic. Hence, $\measuredangle BSP = \measuredangle BAP = \alpha$ (angles on the same chord), so that $SB = BC$.

Remarks:

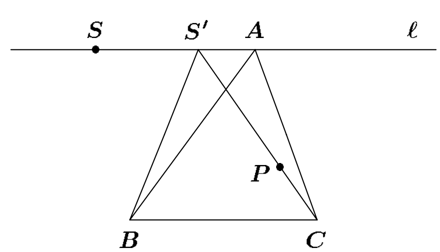
1. If $h > a$, it follows from Theorem 1 that there are no points P inside $\triangle ABC$ satisfying $(*)$.

2. If $h < a$ (we leave the case $h = a$ to the reader), there are two points (independent of A), call them S and S', on ℓ, with $SB = S'B = BC$ (we assume that S and S' are labelled so that $\measuredangle SBC > 90°$ and $\measuredangle S'BC < 90°$). Theorem 1 shows that the locus of P (inside $\triangle ABC$) lies on $SC \cup S'C$.

3. The point P is either at the intersection of the line SC, the circle SAB and the perpendicular bisector of AB, or at the intersection of the line $S'C$, the circle $S'AB$ and the perpendicular bisector of AB. Thus, it is easily constructed with compass and straight edge.

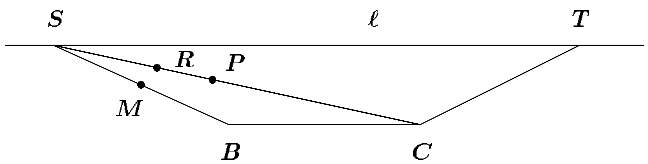
4. With the orientation of the above diagram, both S and S' must be to the left of A, since the line CP must pass though the segment AB if P is inside $\triangle ABC$.

To answer **Q**, note that if $h < \frac{\sqrt{3}}{2}a$, then P cannot be inside $\triangle ABC$ and lie on $S'C$. For then we would have $\measuredangle S'BC < 60°$, giving that $\measuredangle BCA > \measuredangle BCS' = \measuredangle BS'C > 60°$ (remember that S' must be to the left of A) and that the sum of the angles of $\triangle ABC$ is greater than $\measuredangle PCB + \measuredangle PAB + \measuredangle PBA = 3\measuredangle BCS' > 180°$.



To complete the discussion of **L**, we must decide which points on SC can be the point P associated with some choice of A. As in the proof of Theorem 1, $SAPB$ is concyclic. Thus, given A on ℓ, we can find P on SC, and *vice versa*, using that circle.

Let M be the mid-point of SB, and let R be the point on SC such that $RM \perp SB$. Since $\measuredangle SBC > 90^\circ$, the point R exists.



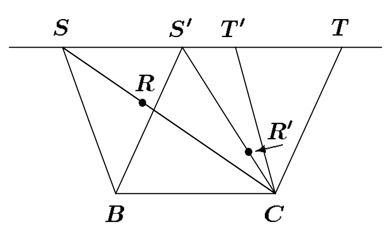
Theorem 2. The locus of question **L** is RC.

Proof. Let T be the point on ℓ with $\measuredangle SBC = \measuredangle BCT$. Suppose that P is on RC. Let A be the point (other than S) where circle SBP meets ℓ. (Consider the case that circle SBP is tangent to ℓ separately. Then, $A = S$, $P = R$.) If A is to the left of S, then $\measuredangle ASP + \measuredangle ABP = 180^\circ$ (opposite angles of a cyclic quadrilateral). However, $\measuredangle TSP = \measuredangle SBR < \measuredangle SBP < \measuredangle ABP$, so that $\measuredangle ASP + \measuredangle ABP > \measuredangle ASP + \measuredangle TSP = 180^\circ$, a contradiction. Thus, A must be to the right of S. Since $ASBP$ is concyclic, it follows that P is inside $\triangle ABC$.

We have already shown that if A and P satisfy the conditions of problem **L**, then P is on SC, $SAPB$ is concyclic, and A is to the right of S. We leave it to the reader to complete this theorem by showing that P must be on the segment RC, and also to show that A is on the segment ST.

We now turn to problem **G**. First, consider the case in which P is (again) inside $\triangle ABC$.

If $\frac{\sqrt{3}}{2}a < h < a$, the above ideas can be used to show that there is a segment $R'C$ on $S'C$ such that $RC \cup R'C$ is the locus of P. The point R' is the intersection of the perpendicular bisector of $S'B$ with $S'C$. For there to be a solution P on $R'C$, the point A must lie on the segment $S'T'$, and S' must be to the left of T'. In this case, there are two solutions for P inside $\triangle ABC$: one on RC and one on $R'C$. Note that S' is to the left of T' if and only if $h > \frac{\sqrt{3}}{2}a$.



For a given A, allowing P to be either inside or outside $\triangle ABC$, results in more solutions. Indeed, as A moves to the left of S, and P is the intersection of the circle ASB with the line SC, the point P moves on the line SC past R, then past S. It can be shown that any point P on the ray $\overrightarrow{CS}$ satisfies (*). Likewise, P can be any point on the ray $\overrightarrow{CS'}$ (even if there are no solutions on CS' that lie inside $\triangle ABC$). Thus, the locus of **G** includes the rays $\overrightarrow{CS}$ and $\overrightarrow{CS'}$.

If A is to the right of T, the circle ASB intersects SC at a point P below BC. Then, $\measuredangle PCB$ is supplementary to the equal angles $\measuredangle PAB$ and $\measuredangle PBA$. Calling such a P a supplementary solution, we have that any point P on lines SC or $S'C$ is either a solution or a supplementary solution to **G**; that is, there is an A on ℓ with $\measuredangle PAB = \measuredangle PBA = \measuredangle PCB$, or $\measuredangle PAB = \measuredangle PBA = 180° - \measuredangle PCB$.

We have by no means solved **G**. We have seen that if P is required to be inside $\triangle ABC$, then the locus is $RC \cup R'C$ ($\frac{\sqrt{3}}{2}a < h < a$), or RC ($0 < h < \frac{\sqrt{3}}{2}a$). If P is also allowed to be outside $\triangle ABC$, then the locus includes $\overrightarrow{CS} \cup \overrightarrow{CS'}$.

For each A, the lines AB, BC and CA divide the plane into seven regions: the inside of $\triangle ABC$, the infinite regions adjacent to the vertices A, B, C (each bounded by two rays), and the infinite regions adjacent to the sides AB, BC and CA (each bounded by two rays and a line segment).

Suppose that P is in the infinite region adjacent to A. Then $\measuredangle PAB > \measuredangle PCB$. (Extend PA to meet BC, and use the result that an exterior angle of a triangle is greater than either of the two interior opposite angles.) Thus, P cannot be a solution to **G**. Similarly, there is no solution P to **G** inside the infinite regions adjacent to B or C.

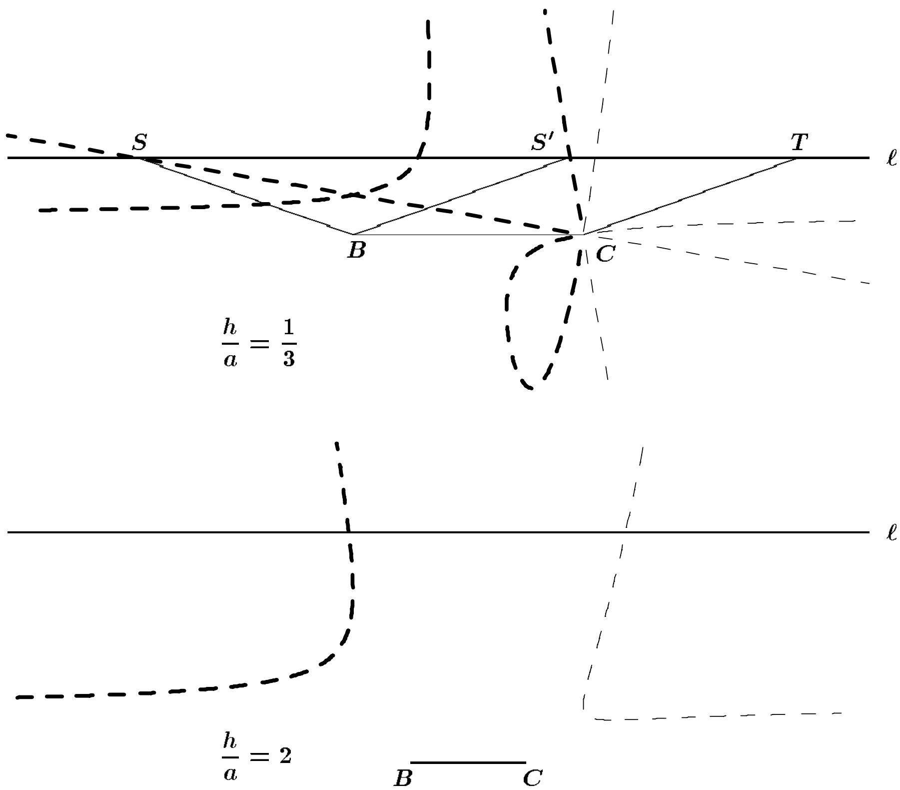
For any A on ℓ, the point P where the perpendicular bisector of AB meets the circle ABC to the left of AB is a solution to **G**. (Use the circle theorems!) Conversely, if P is a solution to **G** inside the infinite region adjacent to AB, then $PACB$ is concyclic, since $\measuredangle PCB = \measuredangle PAB$. Thus, P is at the intersection of the perpendicular bisector of AB and the circle ABC, the intersection to the left of AB.

If P is a solution to **G** in the infinite region adjacent to BC, then P is again a point of intersection of the circle ABC and the perpendicular bisector of AB — this time, the intersection to the right of AB. However, if $h > a$, then there is no choice of A that leads to a solution P in this region.

Finally, we must consider P in the infinite region adjacent to AC . We have already seen some solutions in this region: if A is to the left of S (S'), then there is a solution P on $\overrightarrow{RS}$ ($\overrightarrow{R'S'}$) which will be in the region adjacent to AC. We leave it to the reader to prove that these are the only solutions in this region.

For some points A, there are supplementary solutions for **G** in the region adjacent to AC, namely, the point of intersection of the perpendicular bisector of AB and the circle ABC that lies to the right of AB.

Theorem 3. The locus of **G** is indicated in the figures below for $h < a$ and $h > a$. The solutions are indicated with thick dashed lines, while the supplementary solutions are indicated with thin dashed lines. Together, these loci consist of the lines SC and SC', and of those points where, for each A on ℓ, the perpendicular bisector of AB intersects the circle through A, B and C.



Daryl Tingley
Department of Mathematics and Statistics
University of New Brunswick
Fredericton, New Brunswick
Canada E3B 5A3
daryl@math.unb.ca

THE SKOLIAD CORNER

No. 49

R.E. Woodrow

This issue we continue the theme with the problems of the Final Round of the Junior High School Mathematics Contest of the British Columbia Colleges. On the basis of the Preliminary Round, students are invited to write this contest as part of a visit and tour of a college campus. My thanks go to Jim Totten, The University College of the Cariboo, one of the organizers, for forwarding the materials to us.

BRITISH COLUMBIA COLLEGES

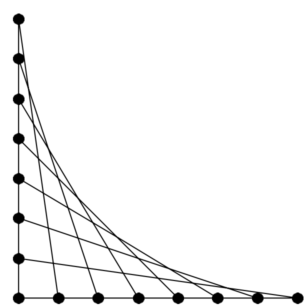
Junior High School Mathematics Contest

Part A — Final Round — May 5, 2000

1. The last (ones) digit of a perfect square cannot be:

(a) 1 (b) 4 (c) 5 (d) 6 (e) 8

2. Suppose a string art design is constructed by connecting nails on a vertical axis and on a horizontal axis by line segments as follows: The nail furthest from the origin on the vertical axis is connected to the nail nearest the origin on the horizontal axis. Then proceed toward the origin on the vertical axis and away on the horizontal axis as shown in the diagram. If this were done on a project with 10 nails on each axis, the number of points of intersection of line segments would be:



(a) 45 (b) 46 (c) 47 (d) 48 (e) none of these

3. Assume there is an unlimited supply of pennies, nickels, dimes, and quarters. An amount (in cents) which cannot be made using exactly 6 of these coins is:

(a) 91 (b) 87 (c) 78 (d) 51 (e) 49

4. Given $x^2 + y^2 = 28$ and $xy = 14$, the value of $x^2 - y^2$ equals:

(a) -14 (b) 0 (c) 14 (d) 28 (e) 42

5. Given that $0 < x < y < 20$, the number of integer solutions (x, y) to the equation $2x + 3y = 50$ is:

(a) 16 (b) 9 (c) 8 (d) 5 (e) 3

6. The numbers $1, 3, 6, 10, 15\ldots$ are known as triangular numbers. Each triangular number can be expressed as $\frac{n(n+1)}{2}$ where n is a natural number. The largest triangular number less than 500 is:

(a) 494 (b) 495 (c) 496 (d) 497 (e) 498

7. An 80 m rope is suspended at its two ends from the tops of two 50 m flagpoles. If the lowest point to which the mid-point of the rope can be pulled is 36 m from the ground, then the distance, in metres, between the flagpoles is:

(a) $6\sqrt{39}$ (b) $36\sqrt{13}$ (c) $12\sqrt{39}$ (d) $18\sqrt{13}$ (e) $12\sqrt{26}$

8. At a certain party, the first time the door bell rang one guest arrived. On each succeeding ring two more guests arrived than on the previous ring. After 20 rings, the number of guests at the party was:

(a) 39 (b) 58 (c) 210 (d) 361 (e) 400

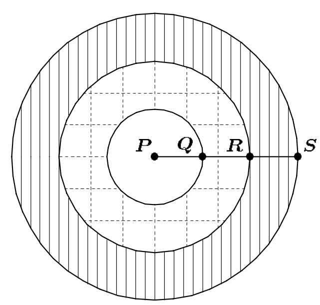
9. An operation $*$ is defined such that

$$A * B = A^B - B^A .$$

The value of $2 * (-1)$ is:

(a) -3 (b) -1 (c) $-\frac{1}{2}$ (d) 0 (e) $\frac{3}{2}$

10. Three circles with a common centre P are drawn as shown with $PQ = QR = RS$. The ratio of the area of the region between the inner and middle circles (shaded with squares) to the area of the region between the middle and outer circles (shaded with lines) is:



(a) $\frac{1}{3}$ (b) $\frac{4}{9}$ (c) $\frac{1}{2}$ (d) $\frac{3}{5}$ (e) $\frac{2}{3}$

Part B — Final Round — May 5, 2000

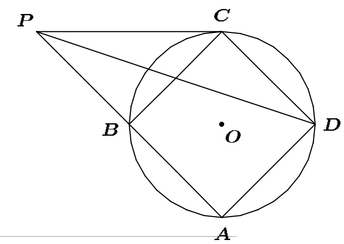
1. (a) How many three-digit numbers can be formed using only the digits 1, 2, and 3 if both of the following conditions hold:

(i) repetition is allowed;

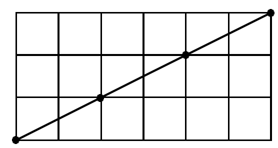
(ii) no digit can have a larger digit to its left.

(b) Repeat for a four-digit number using the digits 1, 2, 3, and 4.

2. The square $ABCD$ is inscribed in a circle of radius one unit. ABP is a straight line, PC is tangent to the circle. Find the length of PD. Make sure you explain thoroughly how you got **all** the things you used to find your solution!



3. If a diagonal is drawn in a 3×6 rectangle, it passes through four vertices of smaller squares. How many vertices does the diagonal of a 45×30 rectangle pass through?



4. Let a and b be any real numbers. Then $(a - b)$ is also a real number, and consequently $(a - b)^2 \geq 0$. Expanding gives $a^2 - 2ab + b^2 \geq 0$. If we add $2ab$ to both sides of the inequality, we get $a^2 + b^2 \geq 2ab$. Thus, for any real numbers a and b, we have $a^2 + b^2 \geq 2ab$.

Prove that for any real numbers a, b, c, d

(a) $2abcd \leq b^2c^2 + a^2d^2$.

(b) $6abcd \leq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2$.

5. A circular coin is placed on a table. Then identical coins are placed around it so that each coin touches the first coin and its other two neighbours.

(a) If the outer coins have the same radius as the inner coin, show that there will be exactly 6 coins around the outside.

(b) If the radius of all 7 coins is 1, find the total area of the spaces between the inner coin and the 6 outer coins.

Last issue we gave the problems of the Preliminary Round of the British Columbia Colleges Senior High School Contest for 2000. Here, thanks to Jim Totten, The University College of the Cariboo, one of the organizers, are some of the "official" solutions. Look for the rest in the next issue.

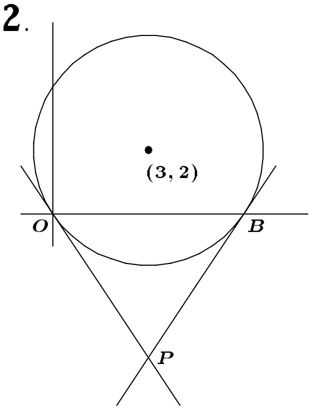
BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Preliminary Round — March 8, 2000

1. Antonino sets out on a bike ride of 40 km. After he has covered half the distance he finds that he has averaged 15 km/hr. He decides to speed up. The rate at which he must travel the rest of the trip in order to average 20 km/hr for the whole journey is:

(a) 25 km/hr **(b)** 30 km/hr **(c)** 35 km/hr **(d)** 36 km/hr **(e)** 40 km/hr

Solution. The answer is **(b)**. Antonino averages 15 km/hr for the first 20 km. This means it takes him $20/15 = 4/3$ hours to cover the first 20 km. In order to average 20 km/hr for a 40 km distance, he must cover the distance in 2 hours. He only has $2/3$ hours remaining in which to cover the last 20 km. His speed over this last 20 km then must be (on average) $20/(2/3) = 30$ km/hr.

2.



A circle with centre at $(3, 2)$ intersects the x–axis at the origin, O, and at the point B. The tangents to the circle at O and B intersect at the point P. The y–coordinate of P is:

(a) $-3\frac{1}{2}$ **(b)** -4 **(c)** $-4\frac{1}{2}$ **(d)** -5 **(e)** none of these

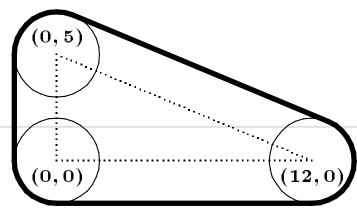
Solution. The answer is **(c)**. Let C be the centre of the circle. Since the points O and B are equidistant from the centre of the circle and also equidistant from the point P, and since both O and B lie on the x–axis, we see that P has coordinates $(3, y)$ with $y < 0$. The slope of OC is $2/3$. Since PO is the tangent line to the circle at O, we know that $PO \perp OC$. Therefore the slope of PO is $-3/2$. However, the slope of PO is computed to be $(y - 0)/(3 - 0) = y/3$. Together these imply that $y = -9/2$.

3. From five students whose ages are 6, 7, 8, 9, and 10, two are randomly chosen. The probability that the difference in their ages will be at least 2 years is:

(a) $\frac{1}{2}$ **(b)** $\frac{2}{5}$ **(c)** $\frac{3}{5}$ **(d)** $\frac{7}{10}$ **(e)** $\frac{3}{4}$

Solution. The answer is **(c)**. First of all the number of possible ways to choose a pair of distinct students from a set of five is $\binom{5}{2} = \frac{5!}{2!3!} = 10$. From this we need only eliminate those whose age difference is 1. Clearly there are exactly 4 such, namely $(6, 7)$, $(7, 8)$, $(8, 9)$, and $(9, 10)$. Thus, our probability of success is 6 out of 10, or $3/5$.

4. The centres of three circles of radius 2 units are located at the points $(0, 0)$, $(12, 0)$ and $(0, 5)$. If the circles represent pulleys, what is the length of the belt which goes around all 3 pulleys as shown in the diagram?



(a) $30 + \pi$ **(b)** $30 + 4\pi$ **(c)** $36 + \pi$ **(d)** $60 - 4\pi$ **(e)** none of these

Solution. The answer is **(b)**. The straight sections of the belt are tangent to all 3 pulleys and thus perpendicular to the radius of each pulley at the point of contact. Thus the straight sections of the belt are the same lengths as the distances between the centres of the pulleys, which are 5, 12, and $\sqrt{5^2 + 12^2} = 13$. Thus, the straight sections of belt add up to 30 units. The curved sections of belt, when taken together, make up one complete pulley, or a circle of radius 2. Thus the curved sections add up to $2\pi(2) = 4\pi$. Thus, the full length of the belt is $30 + 4\pi$ units.

5. If Mark gets 71 on his next quiz, his average will be 83. If he gets 99, his average will be 87. How many quizzes has Mark already taken?

(a) 4 **(b)** 5 **(c)** 6 **(d)** 7 **(e)** 8

Solution. The answer is **(c)**. Let n be the number of quizzes Mark has already taken. Let x be his total score on all n quizzes. Then we have the following:

$$\frac{x + 71}{n + 1} = 83 \implies x = 83(n + 1) - 71\,,$$
$$\frac{x + 99}{x + 1} = 87 \implies x = 87(n + 1) - 99\,.$$

Solving this system yields $n = 6$.

That completes the *Skoliad Corner* for this issue. The solutions to the problems of the British Columbia Colleges Senior High School Contest for 2000 will be completed in the next issue.

Send me suitable contest materials and suggestions for the future of the *Corner*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M65 3G3. The electronic address is

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University)

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 8 of 2001.

High School Problems

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H277.

(a) Find all right triangles with integer sides with perimeter 60.

(b) Find all right triangles with integer sides with area 600.

H278. Consider the time as seen on a digital clock in 24 hour mode. (24 hour mode is representing the time relative to 12 midnight. For example, $6{:}25$ am is $06{:}25$, but $6{:}25$ pm is $18{:}25$. Also, $12{:}45$ am counts as $00{:}45$.) Let n be the number we get when we remove the colon from the time T as seen on a digital clock in 24 hour mode. Find all times T such that:

(i) n is a palindrome, [Ed. reads the same backwards as forwards.]

(ii) m, the number of minutes that T is after midnight, is a palindrome, and

(iii) $n = m$.

H279. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Let a and b be integers such that $a \equiv b \pmod 3$. Prove that

$$\tfrac{2}{3}\,\left(a^2+ab+b^2\right)$$

can be expressed as a sum of three non-negative squares.

H280. *Proposed by Fotifo Casablanca, Bogotá, Colombia.*

In the spirit of the Olympics: There are 9 regions inside the rings of the Olympics. Put a different positive whole number in each so that the five products of the numbers in each ring form a set of five consecutive integers.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A253. *Proposed by Mohammed Aassila, Strasbourg, France.*

Does there exist a polynomial $f(x,y,z)$ with real coefficients, such that $f(x,y,z) > 0$ if and only if there exists a non-degenerate triangle with side lengths $|x|$, $|y|$, and $|z|$?

A254. In the acute triangle ABC, the bisectors of $\angle A$, $\angle B$, and $\angle C$ intersect the circumcircle again at A_1, B_1 and C_1, respectively. Let M be the point of intersection of AB and B_1C_1, and let N be the point of intersection of BC and A_1B_1. Prove that MN passes through the incentre of triangle ABC.

(1997 Baltic Way)

A255. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Define $A = (\sum_{i=1}^{n} a_i)/n$, $G = \sqrt[n]{\prod_{i=1}^{n} a_i}$, and $H = n/(\sum_{i=1}^{n} 1/a_i)$ for positive real numbers a_1, a_2, $\ldots$, a_n. It is known that $A \geq G \geq H$, from which it follows that $0 \geq \log(G/A)$ and $0 \geq 1 - A/H$. Prove that $0 \geq \log(G/A) \geq 1 - A/H$, and determine when equality holds.

A256. ~~*Proposed by Mohammed Aassila, Strasbourg, France.*~~

Prove that for any positive integer n, there exist $n + 1$ points M_1, M_2, $\ldots$, M_{n+1} in $\mathbb{R}^n$ such that for any integers i and j for which $1 \leq i < j \leq n + 1$, the Euclidean distance between M_i and M_j is 1.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C97. Given a positive integer n, let $\overline{0}$, $\overline{1}$, $\ldots$, $\overline{n-1}$ denote the integers modulo n (so that $\overline{a}$ is the reduction of a modulo n). Find all positive integers n with the property that the set

$$\{\overline{a} \mid 0 < a < n/2 \text{ with } a \text{ and } n \text{ relatively prime}\}$$

is a group under multiplication.

C98. Find all pairs of integers (x, y) which satisfy the equation

$$x^2 - 34y^2 = -1 .$$

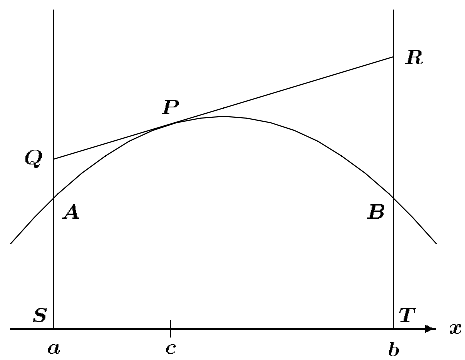
Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. The graph of the function $y = f(x)$ is concave downward over the interval $a \leq x \leq b$. A and B are the points $(a, f(a))$, $(b, f(b))$, respectively. The tangent to $y = f(x)$ at any point $P(x, f(x))$, $a \leq x \leq b$, meets the line $x = a$ at Q and the line $x = b$ at R. If the area bounded by the curve, the tangent at P, the line $x = a$, and the line $x = b$, is minimized, prove that the x–coordinate of the point P is independent of the function $f(x)$.

(1997 Descartes, Problem 12)

Solution.



Let P be at $(c, f(c))$. Let S and T be the points $(a, 0)$ and $(b, 0)$, respectively. Without loss of generality, we can assume that the points A, B, P, Q, and R, lie above the x-axis. (If they do not, we can merely translate the points upward until they do lie above the x-axis, and this translation will not affect the area in question.)

Now, the area under the curve between $x = a$ and $x = b$ is fixed. Hence, we want to find the c such that the area of trapezoid $QRTS$ is a minimum.

The area of the trapezoid is $\frac{1}{2}(b - a)(Q_y + R_y)$, where Q_y and R_y are the y–coordinates of Q and R, respectively. But since $\frac{1}{2}(b - a)$ is constant, that only means we must find the c that yields the minimum value of $Q_y + R_y$. This is equivalent to finding the c that yields the minimum value of $\frac{1}{2}(Q_y + R_y)$; that is, the average value of the y–coordinates of Q and R.

Consider the tangent line $y = g(x)$. This tangent line varies with c. Because $f(x)$ is concave down, the line $g(x)$ lies entirely above $f(x)$ except at the contact point. In other words, $g(x) \geq f(x)$ with equality if and only if $x = c$.

Now $\frac{1}{2}(Q_y + R_y) = \frac{1}{2}(g(a) + g(b)) = g(\frac{a+b}{2})$, with the last equality holding since $g(x)$ is a linear function. But $g(\frac{a+b}{2}) \geq f(\frac{a+b}{2})$, with equality if and only if $c = \frac{a+b}{2}$. In turn, this means that the minimum value of $\frac{1}{2}(Q_y + R_y)$ is $f(\frac{a+b}{2})$, and this occurs only when $c = \frac{a+b}{2}$.

Hence, the area in question is minimized when $c = \frac{a+b}{2}$, which is independent of $f(x)$, QED.

Remark. This problem can also be solved using calculus.

J.I.R. McKnight Problems Contest 1986 — Solution

4. (b) Prove that in any acute triangle ABC,

$$\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4K},$$

where K is the area of triangle ABC.

Solution by Vedula N. Murty, Dover, PA, USA.

We use the following formulae:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$\sin A = a/(2R)$, where R is the circumradius of the triangle ABC, and $K = abc/(4R)$.

From these, we have

$$\cot A = \frac{R(b^2 + c^2 - a^2)}{abc},$$

Using this equation and similar equations for $\cot B$ and $\cot C$, we have

$$\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4K}.$$

This appears to be a little simpler than the one given in [1999 : 296–297]. Now, why do we need the triangle to be acute?

Another Do-It-Yourself Proof of the $n = 3$ case of Fermat's Last Theorem

Andy Liu

Part Zero.

Everyone knows Fermat's Last Theorem, which states that the Diophantine equation $x^n + y^n = z^n$ has no solution in non-zero integers for all $n \geq 3$. This article offers a proof of the case $n = 3$ different from the one presented in [2]. The idea of the present proof is taken from [1], with two changes. First, the concept of quadratic residues, though very useful in general, would take us too far afield. It was discovered that the reference to quadratic residues in the argument in [1] is actually redundant, and is accordingly excised. Second, the main lemma in the argument in [1] is rather unmotivated. It is now presented in a manner which makes its discovery plausible.

Part One.

We wish to prove that the Diophantine equation $x^3 + y^3 = z^3$ has no solutions in non-zero integers. Our approach is indirect. Suppose that such a solution exists.

Problem 1.

Prove that no two of x, y and z are equal.

By the Well Ordering Principle, there exists a solution such that $|xyz|$ is minimal.

Problem 2.

Prove that x, y and z are pairwise relatively prime to one another, and that exactly one of them is even.

We will attempt to construct another solution for which $|xyz|$ is smaller. This will yield the desired contradiction. We may assume that z is even. Then $x + y$ and $x - y$ are both even. Hence, there exist integers u and w such that $x + y = 2u$ and $x - y = 2w$. It follows that $x = u + w$ and $y = u - w$. We then have $z^3 = (u + w)^3 + (u - w)^3 = 2u(u^2 + 3w^2)$.

Problem 3.

Prove that u and w are relatively prime to each other, and that exactly one of them is odd.

Case 1.

Suppose u is not divisible by 3. Then u and $3w$ are relatively prime to each other.

Problem 4.

Prove that $2u$ and $u^2 + 3w^2$ are relatively prime to each other.

It follows that there exist integers r and s such that $2u = r^3$ and $u^2 + 3w^2 = s^3$. We shall continue the analysis in Part Four.

Case 2.

Suppose u is divisible by 3. Then $u = 3v$ for some integer v, and $z^3 = 18v(3v^2 + w^2)$.

Problem 5.

Prove that $3v$ and w are relatively prime to each other, as are $18v$ and $3v^2 + w^2$.

It follows that there exist integers r and s such that $18v = r^3$ and $3v^2 + w^2 = s^3$. We shall continue the analysis in Part Four.

Part Two.

Both cases in Part One lead us to consider the Diophantine equation $a^2 + 3b^2 = s^3$ where $(a, 3b) = 1$ and $a + b \equiv 1 \pmod 2$. We first explore the case where s is a prime.

Problem 6.

Prove that $s > 3$ and that neither a nor b is divisible by s.

Let b^{-1} be the inverse of b modulo s and let $g = ab^{-1}$. From $a^2 + 3b^2 \equiv 0 \pmod s$, we have $g^2 + 3 \equiv 0 \pmod s$. Let $q = \lfloor \sqrt{s} \rfloor$. Then $q < \sqrt{s} < q + 1$.

Problem 7.

Prove that $g(i' - i'') \equiv j' - j'' \pmod s$ for some i', j', i'' and j'', each an integer between 0 and q inclusive, such that $(i', j') \neq (i'', j'')$.

Define $i = |i' - i''|$ and $j = |j' - j''|$. Then either $gi + j \equiv 0 \pmod s$ or $gi - j \equiv 0 \pmod s$. In any case, $g^2i^2 - j^2 \equiv 0 \pmod s$.

Problem 8.

Prove that $0 < i < \sqrt{s}$ and $0 < j < \sqrt{s}$.

From $g^2 + 3 \equiv 0 \pmod s$, we have $g^2i^2 + 3i^2 \equiv 0 \pmod s$. Hence, $3i^2 + j^2 \equiv 0 \pmod s$, so that $3i^2 + j^2 = hs$ for some integer h.

Problem 9.

Prove that $h = 1$ or 3.

If $h = 1$, then we have $s = 3i^2 + j^2$. If $h = 3$, then 3 divides j also, so that $j = 3k$ for some integer k. We then have $s = i^2 + 3k^2$. In summary, we have $s = m^2 + 3n^2$ for some integers m and n.

Problem 10.

Prove that $(m, 3n) = 1$ and $m + n \equiv 1 \pmod 2$.

We have $a^2 + 3b^2 = s^3 = (m^2 + 3n^2)^3 = m^6 + 9m^4n^2 + 27m^2n^4 + 27n^6$. We may take $a = m^3 - Amn^2$ and $b = Bm^2n - 3n^3$, so that $a^2 + 3b^2 = m^6 + (3B^2 - 2A)m^4n^2 + (A^2 - 18B)m^2n^4 + 27n^6$.

Problem 11.

Prove that the system of equations $3B^2 - 2A = 9$ and $A^2 - 18B = 27$ has two solutions, namely, $(A, B) = (-3, -1)$ and $(9, 3)$.

Since in Part One we attempt to find a solution smaller than the minimal solution, we take $a = m^3 - 9mn^2$ and $b = 3m^2n - 3n^3$.

Part Three.

From our exploration in Part Two, we claim that $a^2 + 3b^2 = s^3$, with $(a, 3b) = 1$ and $a + b \equiv 1 \pmod 2$, has a solution given by $s = m^2 + 3n^2$, $a = m^3 - 9mn^2$ and $b = 3m^2n - 3n^3$ for some integers m and n such that $(m, 3n) = 1$ and $m + n \equiv 1 \pmod 2$. We shall justify the claim using induction on the number ℓ of prime factors of s.

Problem 12.

Prove the claim for the case $\ell = 0$.

We could also have used the case $\ell = 1$ as the basis of our induction. Consider now the case where s has $\ell + 1$ prime factors. Let p be one of them. Then $s = tp$ where t has ℓ prime factors.

We have $a^2 + 3b^2 = s^3$, so that $a^2 + 3b^2 \equiv 0 \pmod p$. From Part Two, there exist integers c and d such that $p^3 = c^2 + 3d^2$. Moreover, $p = m_1^2 + 3n_1^2$, $c = m_1^3 - 9m_1n_1^2$ and $d = 3m_1^2n_1 - 3n_1^3$ where m_1 and n_1 are integers such that $(m_1, 3n_1) = 1$ and $m_1 + n_1 \equiv 1 \pmod 2$. Now,

$$\begin{aligned} t^3p^6 &= (a^2 + 3b^2)(c^2 + 3d^2) = (ac + 3bd)^2 + 3(ad - bc)^2 \\ &= (ac - 3bd)^2 + 3(ad + bc)^2 . \end{aligned}$$

Problem 13.

Prove that $(ad+bc)(ad-bc) = p^3(t^3d^2 - b^2)$ and deduce that p divides exactly one of $ad + bc$ and $ad - bc$.

We may assume that p divides $ad - bc$ but not $ad + bc$, since the other case can be handled in an analogous manner.

Problem 14.

Prove that p^3 divides $ad - bc$ as well as $ac + 3bd$.

Let $e = (ac + 3bd)/p^3$ and $f = (ad - bc)/p^3$.

Problem 15.

Prove that $e^2 + 3f^2 = t^3$, $a = ce + 3df$ and $b = de - cf$, and deduce that $(e, 3f) = 1$ and $e + f \equiv 1 \pmod 2$.

By the induction hypothesis, there exist integers m_2 and n_2 such that $(m_2, 3n_2) = 1$, $m_2 + n_2 \equiv 1 \pmod 2$, $t = m_2^2 + 3n_2^2$, $e = m_2^3 - 9m_2n_2^2$ and $f = 3m_2^2n_2 - 3n_2^3$. Let $m = m_1m_2 + 3n_1n_2$ and $n = m_2n_1 - n_2m_1$.

Problem 16.

Prove that $(m, 3n) = 1$, $m + n \equiv 1 \pmod 2$, $s = m^2 + 3n^2$, $a = m^3 - 9mn^2$ and $b = 3m^2n - 3n^3$.

This completes the induction argument and justifies the claim.

Part Four.

We now complete the analysis begun in Part One.

Case 1.

The equation $u^2 + 3w^2 = s^3$ has a solution in which $u = m^3 - 9mn^2$ for some integers m and n such that $(m, n) = 1$ and $m + n \equiv 1 \pmod 2$. Then $r^3 = 2u = 2m(m - 3n)(m + 3n)$.

Problem 17.

Prove that $2m$, $m - 3n$ and $m + 3n$ are pairwise relatively prime to one another.

It follows that $2m = \alpha^3$, $m - 3n = \beta^3$ and $m + 3n = \gamma^3$ for some integers α, β and γ.

Problem 18.

Prove that $\alpha^3 = \beta^3 + \gamma^3$ and $0 < |\alpha\beta\gamma| < |xyz|$.

This is the desired contradiction.

Case 2.

The equation $3v^2 + w^2 = s^3$ has a solution in which $v = m^2 n - 3n^3$ for some integers m and n such that $(m, n) = 1$ and $m + n \equiv 1 \pmod 2$. Then $r^3 = 18v = 3^3(2n)(m - n)(m + n)$.

Problem 19.

Prove that $2n$, $m - n$ and $m + n$ are pairwise relatively prime to one another.

It follows that $2n = \alpha^3$, $m - n = \beta^3$ and $m + n = \gamma^3$ for some integers α, β and γ.

Problem 20.

Prove that $\alpha^3 = \beta^3 + \gamma^3$ and $0 < |\alpha\beta\gamma| < |xyz|$.

This is the desired contradiction.

Bibliography.

[1] W. Sierpinski, *Elementary Theory of Numbers*, North-Holland, Amsterdam (1988) pp. 30 and 415–418.

[2] R. Vakil, *A Do-It-Yourself Proof of the $n = 3$ case of Fermat's Last Theorem*, CRUX with MAYHEM **26** (2000) pp. 36–44.

Andy Liu
Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
aliu@vega.math.ualberta.ca

An Interesting Application of the Sophie Germain Identity

Carl Johan Ragnarsson

Abstract

The readers of this article should be familiar with modular arithmetic, and may also recall Sophie Germain's identity. This article deals with an application of the identity in solving the equation $3^x + 4^y = 5^z$.

A word about the identity

As the name says, Sophie Germain's identity was first discovered by Sophie Germain. It reads

$$\begin{aligned} a^4 + 4b^4 &= a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 = (a^2 + 2b^2)^2 - 4a^2b^2 \\ &= (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2) . \end{aligned}$$

What is interesting about this identity is that sums of even powers do not generally factor. Further, such sums factor only when the term we complete the square with is, in itself, a perfect square. Its main application in contest problem solving has, so far, often been trivial because of everyone knowing the identity. When starting to read the chapter about number theory in [1], I found that the identity was used in solving some simple problems related to factoring integers, and the result was an immediate consequence of it. At the time, I was quite sure that whenever I saw another problem involving the identity, I would solve it immediately. I was totally wrong! We present here an interesting application of the identity which is by no means obvious!

The problem

We propose the following problem, which was on the IMO Short List as late as 1991, and which also appears with this source in [2].

Problem. Solve $3^x + 4^y = 5^z$ in non-negative integers.

It is natural to try to prove that the only solutions are the well-known triple $(2, 2, 2)$ and the trivial $(0, 1, 1)$. After starting to try to prove this, one may always change one's course if an obstacle is found.

Counting modulo 4, we have $(-1)^x \equiv 1 \pmod 4$, so that x is even. Let $x = 2n$, so that $9^n + 4^y = 5^z$. Now, counting modulo 5, we get $(-1)^n + (-1)^y \equiv 0 \pmod 5$, since $z \geq 1$, showing that n and y have opposite parities. We split into two cases:

Case 1: n even and y odd. Let $n = 2m$ and $y = 2t + 1$, giving that $9^{2m} + 4^{2t+1} = 5^z$, or, by application of Sophie Germain's identity,

$$\begin{aligned} &(3^m)^4 + 4 \times (2^t)^4 \\ &= \left[(3^m)^2 + 2 \times 3^m 2^t + 2 \times (2^t)^2\right]\left[(3^m)^2 - 2 \times 3^m 2^t + 2 \times (2^t)^2\right] \\ &= 5^z . \end{aligned}$$

Noting that the difference between the two factors is $3^m 2^{t+2}$, which is not divisible by 5, we conclude that both brackets are not multiples of 5. This implies

$$(3^m)^2 - 2 \times 3^m 2^t + 2 \times (2^t)^2 \;=\; (3^m - 2^t)^2 + (2^t)^2 \;=\; 1 ,$$

and since the two squares sum up to 1, they are 0 and 1 respectively, with $m = 0$ and $t = 0$ as the only solution. Tracing back, we easily see that this yields the triple $(0, 1, 1)$ in the original equation.

Case 2: n odd and y even. Now, similarly to Case 1, letting $y = 2t$, the equation becomes $9^n + 16^t = 5^z$. Now, counting modulo 8, we get $1 \equiv 5^z \pmod 8$, and thus, z is even. Thus, letting $z = 2w$, we obtain that $9^n + 16^t = 25^w$, or equivalently

$$(5^w)^2 - (4^t)^2 \;=\; (5^w + 4^t)(5^w - 4^t) \;=\; 3^{2n} ,$$

and since the difference between the brackets is 2^{2t+1}, which is not divisible by 3, we conclude that $5^w - 4^t = 1$, or $5^w = 4^t + 1$. Again, counting modulo 5, we have $0 \equiv (-1)^t + 1 \pmod 5$, and thus, t is odd. Now, letting $t = 2s + 1$, we get, by a second application of Sophie Germain's identity,

$$\begin{aligned} 1 + 4^{2s+1} &= 1 + 4 \times (2^s)^4 \\ &= \left[1 + 2 \times 2^s + 2 \times (2^s)^2\right]\left[1 - 2 \times 2^s + 2 \times (2^s)^2\right] \\ &= 5^w . \end{aligned}$$

Finally, since the difference between the two brackets in the expression above, 2^{s+2}, is not divisible by 5, we conclude that

$$1 - 2 \times 2^s + 2 \times (2^s)^2 \;=\; (1 - 2^s)^2 + (2^s)^2 \;=\; 1 ,$$

and further, that $s = 0$. Tracing back, we easily see that this case leads back to a unique triple as well, namely $(2, 2, 2)$.

Summing up, we conclude that the equation has in all, only the two solutions $(0, 1, 1)$ and $(2, 2, 2)$.

It is interesting how much we have learned from solving this problem. Not least, it uses only the least bit of elementary number theory. It is interesting to note that once again, the triple $(3, 4, 5)$, which has appeared so many times, appears here again, and allows for some interesting problem solving methods. This also allows for brushing up some classics using the same concepts, as seen above.

Further investigation

1. Find another solution of the proposed problem!
2. For which positive integers m, n can we factor $ma^k + nb^k$?
3. In the reals, solve the system

$$\begin{aligned} 3^x + 4^y &= 5^z\,,\\ 3^y + 4^z &= 5^x\,,\\ 3^z + 4^x &= 5^y\,. \end{aligned}$$

4. Prove that for $n \geq 2$, $n^4 + 4^n$ is composite.

References

[1] Arthur Engel, *Problem Solving Strategies*, New York: Springer, 1998.

[2] Naoki Sato, *Number Theory* (unpublished).

Carl Johan Ragnarsson
Paalsjoevaegen 16 SE–223 63 Lund
Sweden
cjr2000@telia.com

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2576. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Characterize the numbers n such that $n!$ finishes (in base 2 notation) with exactly $n-1$ zeros.

2577. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Let a_1, a_2, ..., a_n ($n \geq 2$) be positive integers. Determine the values of n and k ($2 \leq k \leq n$) for which the following identity holds:

$$\underset{1 \leq i_1 < \ldots < i_k \leq n}{\gcd} \left(\operatorname{lcm}\{a_{i_1}, \ldots, a_{i_k}\}\right) = \underset{1 \leq i_1 < \ldots < i_k \leq n}{\operatorname{lcm}} \left(\gcd\{a_{i_1}, \ldots, a_{i_k}\}\right)$$

2578. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For each integer n, determine the hundreds and the units digits of the number $\dfrac{1+5^{2n+1}}{6}$.

2579. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

The excircle on the side BC of triangle ABC touches AC and AB, respectively at Y_A and Z_A. Likewise, the one on CA touches BC and BA at X_B and Z_B, and the one on AB touches CA and CB at Y_C and X_C. Let A' be the intersection of Z_BX_B and X_CY_C, B' be that of X_CY_C and Y_AZ_A, and C' be that of Y_AZ_A and Z_BX_B. Show that AA', BB' and CC' are concurrent. What is the point of intersection of these three lines?

2580. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Suppose that a, b and c are positive real numbers. Prove that

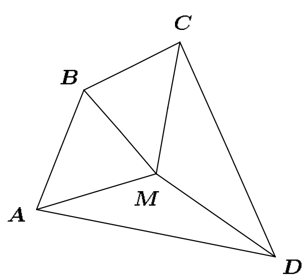
$$\frac{b+c}{a^2+bc}+\frac{c+a}{b^2+ac}+\frac{a+b}{c^2+ab} \;\le\; \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\,.$$

2581. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{ab+c^2}{a+b}+\frac{bc+a^2}{b+c}+\frac{ca+b^2}{c+a} \;\ge\; a+b+c\,.$$

2583. *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*



Given a point M inside the convex quadrangle (see diagram), such that $\angle AMB = \angle MAD + \angle MCD$, $\angle CMD = \angle MCB + \angle MAB$ and $MA = MC$.

Prove that $AB \cdot CM = BC \cdot MD$.

2584. *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

You are given that X, Y, Z and T are points on the chord AB of the circle Γ. Circles Γ_1 and Γ_2 pass through the points X and Y, and touch the circle Γ at points P and S, respectively, while the circles Γ_3 and Γ_4 pass through the points Z and T, respectively, and touch the circle Γ at points Q and R, respectively. Also, Q belongs to the arc APB and the segments XY and ZT do not have common points. Prove that the segments PR, QS and AB intersect at the same points.

2585. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Prove that, for $0 < \theta < \pi/2$,

$$\tan\theta + \sin\theta \;>\; 2\theta\,.$$

2586. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Find all (real or complex) solutions of the system

$$\begin{aligned} 3x + x^3 &= y(1+3x^2)\,, \\ 3y + y^3 &= z(1+3y^2)\,, \\ 3z + z^3 &= w(1+3z^2)\,, \\ 3w + w^3 &= x(1+3w^2)\,. \end{aligned}$$

2587. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

In the half plane $Z = \{(x,y) : y \geq 0\}$, let f be the union of the set of all semicircles lying in Z with diameters on the x–axis, with the set of all lines in Z perpendicular to the x–axis.

Denote by F_{XY} the unique member of f that goes through any two points X and Y in Z. For any three points A, B and C in Z, denote by $\triangle ABC$ the curvilinear triangle formed by the arcs f_{AB}, f_{BC} and f_{CA}.

Let A, B and C be any three points on the x–axis. Let P be any point in the interior of $\triangle ABC$. Let $A' = f_{AP} \cap f_{BC}$, $B' = f_{BP} \cap f_{CA}$ and $C' = f_{CP} \cap f_{AB}$. Let α be the angle at A', interior to $\triangle CAA'$, let β be the angle at B' interior to $\triangle ABB'$, and let γ be the angle at C' interior to $\triangle BCC'$.

Prove that $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right) = 1$.

2588. *Proposed by Niels Bejlegaard, Stavanger, Norway.*

Each positive whole integer a_k $(1 \leq k \leq n)$ is less than a given positive integer N. The least common multiple of any two of the numbers a_k is greater than N.

(a) Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k} < 2$.

(b)* Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k} < \frac{6}{5}$.

(c)* Find the smallest real number γ such that $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k} < \gamma$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

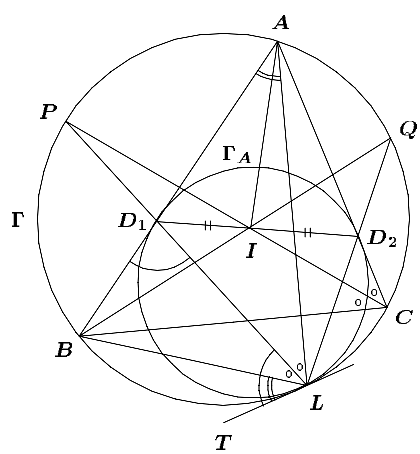
2464. [1999 : 366] *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Given triangle ABC with circumcircle Γ, the circle Γ_A touches AB and AC at D_1 and D_2, and touches Γ internally at L. Define E_1, E_2, M, and F_1, F_2, N in a corresponding way. Prove that

(a) AL, BM, CN are concurrent;

(b) D_1D_2, E_1E_2, F_1F_2 are concurrent, and that the point of concurrency is the incentre of $\triangle ABC$.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*



Let P and Q be the second points of intersection of LD_1 and LD_2 with Γ, respectively. As shown in the figure above, let LT be the common tangent to Γ and Γ_A. Then

$$\angle BD_1L = \angle TLD_1 \qquad \text{and} \qquad \angle TLB = \angle BAL\,.$$

Hence, $\angle BLP = \angle TLD_1 - \angle TLB = \angle BD_1L - \angle BAL = \angle ALP$ so that $\angle ACP = \angle ALP = \angle BLP = \angle BCP$. Therefore, CP is the bisector of $\angle ACB$. Similarly, BQ is the bisector of $\angle ABC$.

Let I be the intersection of CP and BQ. Then I is the incentre of $\triangle ABC$. Since hexagon $ABQLPC$ is inscribed in Γ, by Pascal's Theorem D_1, I and D_2 are collinear. Thus, D_1D_2 passes through the incentre I of $\triangle ABC$. Similarly, E_1E_2 and F_1F_2 pass through I. Therefore, D_1D_2, E_1E_2 and F_1F_2 are concurrent at the incentre of $\triangle ABC$, and part (b) is proved.

Since I is the incentre of $\triangle ABC$, we have $\angle D_1AI = \angle D_2AI$. Since AD_1 and AD_2 are tangent to Γ_A, we have $AD_1 = AD_2$. Thus, $D_1I = D_2I$ and $AI \perp D_1D_2$. Since LD_1 and LD_2 are bisectors of $\angle ALB$ and $\angle ALC$, respectively, we have

$$\frac{BL}{BD_1} = \frac{AL}{AD_1} = \frac{AL}{AD_2} = \frac{CL}{CD_2}.$$

Thus, we get

$$\frac{BL}{CL} = \frac{BD_1}{CD_2}. \tag{1}$$

Since $\frac{1}{2}\angle BAC + \frac{1}{2}\angle ABC + \frac{1}{2}\angle ACB = 90°$, we have $\angle D_1AI + \angle D_1BI + \angle D_2CI = 90°$, so that

$$\angle D_1IB = \angle AD_1I - \angle D_1BI = (90° - \angle D_1AI) - \angle D_1BI = \angle D_2CI.$$

Similarly, we have $\angle D_1BI = \angle D_2IC$. Hence, we have $\triangle BD_1I \sim \triangle ID_2C$. Thus,

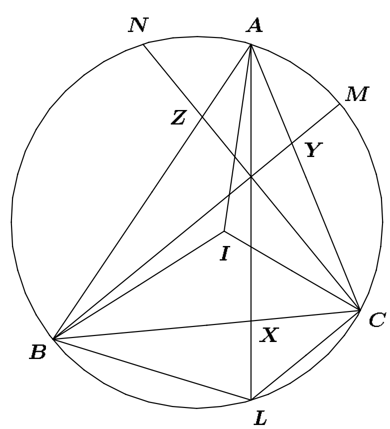
$$\frac{BD_1}{ID_2} = \frac{ID_1}{CD_2} = \frac{BI}{CI}.$$

Thus, we have

$$\frac{BD_1}{CD_2} = \frac{BD_1}{ID_2} \cdot \frac{ID_1}{CD_2} = \frac{BI^2}{CI^2}. \tag{2}$$

From (1) and (2) we have

$$\frac{BL}{CL} = \frac{BI^2}{CI^2}. \tag{3}$$



Now let X be the intersection of AL with BC (see diagram above). Since $\angle ABL + \angle ACL = 180°$, we have from (3)

$$\begin{aligned} \frac{BX}{XC} &= \frac{[ABL]}{[ACL]} = \frac{\frac{1}{2}AB \cdot BL \cdot \sin\angle ABL}{\frac{1}{2}AC \cdot CL \cdot \sin\angle ACL} \\ &= \frac{AB}{AC} \cdot \frac{BL}{CL} = \frac{AB}{AC} \cdot \frac{BI^2}{CI^2}, \end{aligned}$$

where $[PQR]$ denotes the area of triangle PQR. Let Y and Z be the points of intersection of BM and CN with AC and AB, respectively. Then we can similarly show:

$$\frac{CY}{YA} = \frac{BC}{BA} \cdot \frac{CI^2}{AI^2} \qquad \text{and} \qquad \frac{AZ}{ZB} = \frac{CA}{CB} \cdot \frac{AI^2}{BI^2}.$$

Therefore,

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{AB}{AC} \cdot \frac{BC}{BA} \cdot \frac{CA}{CB} \cdot \frac{BI^2}{CI^2} \cdot \frac{CI^2}{AI^2} \cdot \frac{AI^2}{BI^2} = 1.$$

By Ceva's Theorem AX, BY and CZ are concurrent. This implies that AL, BM and CN are concurrent.

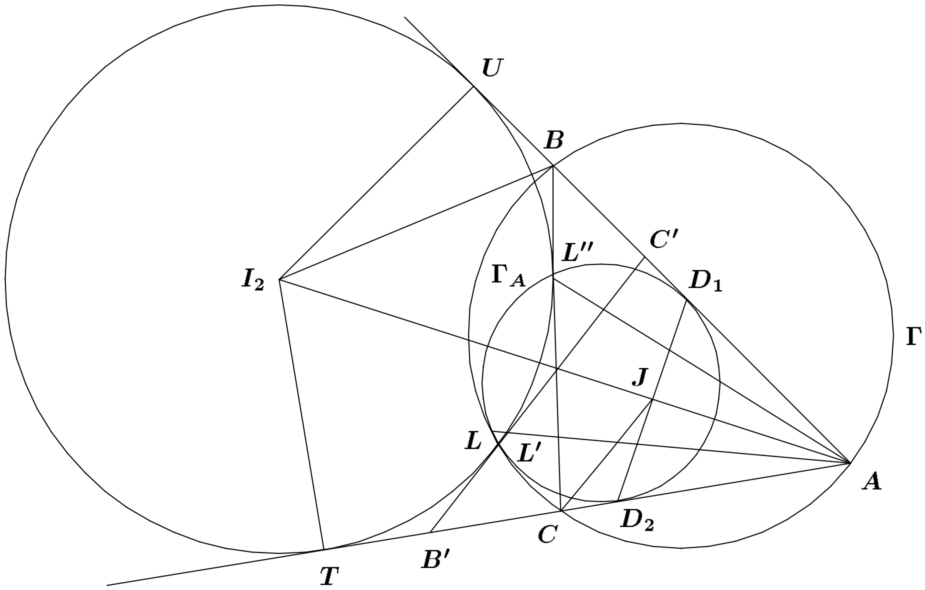
II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let $AB'C'$ be the symmetric triangle of triangle ABC relative to the bisector of angle A. Then both triangles have the same incircle and excircle in the angle A. Let I_a be the centre of the excircle which touches AB at U, BC at L'', CA at T, and $B'C'$ at L'.

We apply inversion with pole A and power $AB \cdot AC$. The inverse point of B is C' and the inverse point of C is B'. The inverse of the circle Γ is therefore the line $B'C'$ and hence, since the circle Γ_A touches AB, AC and Γ, then its inverse is a circle that touches the sides of triangle $AB'C'$. (See figure on page 435.)

If Γ_A were outside of Γ, then its inverse would be the incircle of $\triangle ABC$, but now the inverse of Γ_A is the excircle in the angle A. Thus, the inverse of D_1 is the point U, the inverse of D_2 is the point T, and the inverse of L is the point L'. Since L' is the symmetric point of L'', it follows that AL is the isogonal line of AL''. But it is known (F.G.-M. 1242, 1242a Gergonne-like theorems) that AL'' (and BM'' and CN'', which correspond to the other angles of $\triangle ABC$) passes through Nagel's point. Thus, the isogonals AL, BM and CN are also concurrent.

Now, AI_a is the bisector of angle A and intersects D_1D_2 at J. The inverse of the line D_1D_2 is a circle which passes through A, U and T, and since the points A, U, I_a, T are concyclic, then the inverse of J is the point I_a.



Hence,

$$\begin{aligned} AJ \cdot AI_a = AB \cdot AC &\implies \frac{AJ}{AC} = \frac{AB}{AI_a} \\ &\implies \triangle AJC \sim \triangle ABI_a \\ &\implies \angle ACJ = \angle AI_aB = \frac{C}{2}. \end{aligned}$$

Thus, the point J is the incentre of $\triangle ABC$, and D_1D_2, E_1E_2, F_1F_2 are concurrent.

Other consequences:

1. It is obvious that if Γ_A were outside of Γ, then the results are the same, but the point of concurrency would be the excentre instead of the incentre.

2. The inverse of the circumcircles of $\triangle ABD_2$ and $\triangle ACD_1$ are the lines $C'T$ and $B'U$, respectively (which are concurrent with AL', F.G.-M. 1242), and hence, these circumcircles intersect AL at a point that is the inverse of the symmetric of the adjoint of Gergonne's point of $\triangle ABC$ (F.G.-M. 1242).

3. If r, r_1, r_2, r_3 are the radii of the incircle and the excircles of $\triangle ABC$, s is the semiperimeter of $\triangle ABC$ and R_1, R_2, R_3 are the radii of Γ_A, Γ_B, Γ_C, respectively, then since the power of A to the excircle is $AT^2 = s^2$, then

$$R_1 = \frac{bc}{s^2} \cdot r_1 = \frac{bc}{s^2} \cdot \frac{sr}{s-a} = \frac{r}{\cos^2(A/2)},$$

and hence,

$$\begin{aligned} R_1 + R_2 + R_3 &= r\left(\frac{1}{\cos^2(A/2)} + \frac{1}{\cos^2(B/2)} + \frac{1}{\cos^2(C/2)}\right) \\ &\geq \frac{3r}{\cos^2((A+B+C)/6)} \;=\; 4r\,, \end{aligned}$$

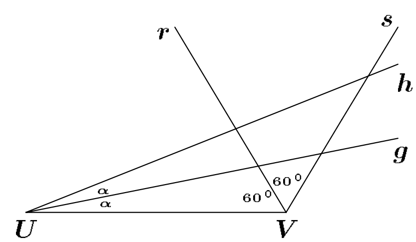
by Jensen's inequality, since the function $f(x) = 1/\cos^2 x$ is convex.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Janous observes that Bankoff (Mixtilinear Adventure, CRUX 9 (1983), pp. 2-7) *has shown that the incentre I is even the mid-point of all three line segments; he further mentions a forthcoming note by Paul Yiu (to appear in the* American Mathematical Monthly), *entitled* Mixtilinear Incircles, *in which he shows that AL, BM and CN are concurrent at the external centre of similitude of the circumcircle and the incircle of $\triangle ABC$.*

2467. [1999 : 367] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given is a line segment UV and two rays, r and s, emanating from V such that $\angle(UV, r) = \angle(r, s) = 60°$, and two lines, g and h, on U such that $\angle(UV, g) = \angle(g, h) = \alpha$, where $0 < \alpha < 60°$.



The quadrilateral $ABCD$ is determined by g, h, r and s. Let P be the point of intersection of AB and CD.

Determine the locus of P as α varies in $(0, 60°)$.

(Editor's note: As was pointed out by several solvers, P should be the intersection of AC and BD.)

Solution by Nikolaos Dergiades, Thessaloniki, Greece and Michael Lambrou, University of Crete, Crete, Greece (amalgamated and adapted by the editors)

We solve a more general problem by replacing the condition:

"$\angle(UV, r) = \angle(r, s) = 60°$"

by

"UV is the external bisector of $\angle(r, s)$",

and replacing the condition

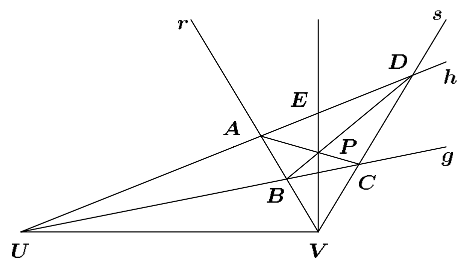
"$\angle(UV, g) = \angle(g, h) = \alpha$"

by

"the lines g and h each meet the rays r and s".

In this case the locus is (part of) the internal bisector of $\angle(r,s)$. Thus, for the problem as given originally, the locus is (part of) the perpendicular to UV at V.

Let $A = h \cap r$, $B = g \cap r$, $C = g \cap s$, $D = h \cap s$, and let $P = AC \cap BD$. Let VP meet the line h at E.



In triangle VAD we have from Ceva's Theorem:

$$\frac{\overline{EA}}{\overline{ED}} \cdot \frac{\overline{BV}}{\overline{BA}} \cdot \frac{\overline{CD}}{\overline{CV}} = -1\,, \tag{1}$$

and from Menelaus' Theorem (relative to U, B, C):

$$\frac{\overline{UA}}{\overline{UD}} \cdot \frac{\overline{BV}}{\overline{BA}} \cdot \frac{\overline{CD}}{\overline{CV}} = 1\,. \tag{2}$$

From (1) and (2) we have

$$\frac{\overline{UA}}{\overline{UD}} = -\frac{\overline{EA}}{\overline{ED}}\,,$$

which means that the points U, E are harmonic conjugates to A, D. Since VU is the external bisector of $\angle AVD$, (that is, $\angle(r,s)$), we conclude that VE is the internal bisector.

Finally, as g turns so that $\angle(UV,g)$ runs from 0 until g is parallel to s and as h turns so that $\angle(UV,h)$ runs from 0 until h is parallel to r (as in the original problem), excepting the case when h is parallel to s, the point P sweeps the said bisector.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

2469. [1999 : 367] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a triangle ABC, consider the altitude and the angle bisector at each vertex. Let P_A be the intersection of the altitude from B and the bisector at C, and Q_A the intersection of the bisector at B and the altitude at C.

These determine a line P_AQ_A. The lines P_BQ_B and P_CQ_C are analogously defined. Show that these three lines are concurrent at a point on the line joining the circumcentre and the incentre of triangle ABC. Characterize this point more precisely.

[Ed. the solution is combined with that of the next problem.]

2470. [1999 : 368] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a triangle ABC, consider the median and the angle bisector at each vertex. Let P_A be the intersection of the median from B and the bisector at C, and Q_A the intersection of the bisector at B and the median at C. These determine a line P_AQ_A. The lines P_BQ_B and P_CQ_C are analogously defined. Show that these three lines are concurrent. Characterize this intersection more precisely.

Combined generalization of **2469** *and* **2470** *devised independently by Nikolaos Dergiades, Thessaloniki, Greece and by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Given $\triangle ABC$, its incentre I, and a point X not on the sides of the triangle, define

$$\begin{array}{lll} P_A = BX\bigcap CI & P_B = CX\bigcap AI & P_C = AX\bigcap BI \\ Q_a = BI\bigcap CX & Q_B = CI\bigcap AX & Q_C = AI\bigcap BX, \end{array}$$

and prove that

(i) P_AQ_A, P_BQ_B, P_CQ_C are concurrent at a point S, and

(ii) S lies on the line joining I to the isogonal conjugate of X.

Solution to (i) by Peter Y. Woo, Biola University, La Mirada, CA, USA.

The concurrency is an immediate consequence of Pappus' theorem: Q_A, P_B, X are three points on the line CX while Q_B, P_A, I are three points on CI. The cross-joins intersect at $S = P_AQ_A\bigcap Q_BP_B$, $P_C = IQ_A\bigcap Q_BX$, and $Q_C = P_AX\bigcap IP_B$, whose collinearity implies that S is on the axis P_CQ_C, which proves (i).

Editor's comment. Note the appropriateness of an alternative statement of Pappus' theorem — "If two triangles are doubly perspective, they are triply perspective." Here we are given that the triangles of P-points and of Q-points are perspective from both I and X, so our conclusion is that they are perspective from a third point S.

Solution to (ii) by Michel Bataille, Rouen, France (whose separate solutions to **2469** and **2470** have been combined by the editor).

We shall use trilinear coordinates relative to $\triangle ABC$ with $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ and $I(1,1,1)$. Let the coordinates of the given point X be $(1/u, 1/v, 1/w)$ so that its isogonal conjugate is the point $X'(u,v,w)$. Let P_A have the unknown coordinates (x,y,z) that we want to evaluate in terms of the given u, v and w.

P_A is on BX so that $\det\begin{pmatrix} x & 0 & \frac{1}{u} \\ y & 1 & \frac{1}{v} \\ z & 0 & \frac{1}{w} \end{pmatrix} = 0$, which gives $\dfrac{x}{w} = \dfrac{z}{u}$.

P_A is on CI so that $\det\begin{pmatrix} x & 0 & 1 \\ y & 0 & 1 \\ z & 1 & 1 \end{pmatrix} = 0$, which gives $y = x$.

From this we get $P_A(w, w, u)$ and, by suitable permutations, $Q_A(v, u, v)$, $P_B(v, u, u)$, $Q_B(w, w, v)$, $P_C(v, w, v)$ and $Q_C(w, u, u)$. By adding P_A to Q_A, and so forth, we see that P_AQ_A, P_BQ_B, P_CQ_C concur at the point S that satisfies

$$S = (v + w, w + u, u + v).$$

By adding the coordinates of S to $X'(u, v, w)$ we see that S, X' and the incentre $I(1, 1, 1)$ are collinear, as desired.

In **2469** we are given that X is the orthocentre $H(\sec A, \sec B, \sec C)$; its isogonal conjugate is the circumcentre $X'(u, v, w) = O(\cos A, \cos B, \cos C)$. Here S is the point on IO with coordinates $(\cos B + \cos C, \cos C + \cos A, \cos A + \cos B)$; this point is X_{65} in Kimberling's list [Ed. Clark Kimberling, Central Points and Central Lines in the Plane of a Triangle, *Math. Mag.* **67**:3 (June 1994) 163-187. Alternatively, one can consult his web page: http://cedar.evansville.edu/~ck6/encyclopedia/].

Editor's comments. Janous and Yiu both show that I lies between O and S and divides that segment in the ratio $R : r$. Most solvers noted that S is the isogonal conjugate of the Schiffler point (X_{21} in Kimberling's list), which Kimberling named for the proposer of problem **1018** [1986: 150-152].

In **2470** we are given $X = G(1/a, 1/b, 1/c)$ whose isogonal conjugate is the Lemoine point $L(a, b, c)$. Here $S = (b + c, c + a, a + b)$, which is "the simplest unnamed centre" X_{37} in Kimberling's list.

[*Comment.* Janous and Yiu show instead that the centroid G lies between S and the isotomic conjugate of the incentre, and it divides that segment in the ratio $1 : 2$. This fact also appears in Kimberling's table 3.]

Both problems were also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

2471. [1999 : 368] *Proposed by Vedula N. Murty, Dover, PA, USA.*

For all integers $n \geq 1$, determine the value of $\displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}k}{k+1}\binom{n+1}{k}$.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

Let S_n, $n \geq 1$, be the sum in question. It is straightforward to check that the equality

$$\frac{k}{k+1}\binom{n+1}{k} = \binom{n+1}{k} - \frac{1}{n+2}\binom{n+2}{k+1}$$

holds for $k = 1, 2, \ldots, n$. Applying the above equality and the Binomial Theorem, we find

$$\begin{aligned} S_n &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k} - \frac{1}{n+2}\sum_{k=1}^{n}(-1)^{k-1}\binom{n+2}{k+1} \\ &= -(1-1)^{n+1} + 1 - (-1)^n \\ &\qquad -\frac{1}{n+2}\left((1-1)^{n+2} - 1 + (n+2) - (-1)^n\right) \\ &= \frac{1-(-1)^n(n+1)}{n+2}. \end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOSÉ LUIS DIAZ, Universitat Politècnica de Catalunya, Terrassa, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HO-JOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; DIGBY SMITH, Mount Royal College, Calgary, Alberta; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

2472. [1999 : 368] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

If A, B, C are the angles of a triangle, prove that

$$\begin{aligned} &\cos^2\left(\frac{A-B}{2}\right)\cos^2\left(\frac{B-C}{2}\right)\cos^2\left(\frac{C-A}{2}\right) \\ &\qquad\qquad \geq \left(8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)^3. \end{aligned}$$

I. *Editorial comment.*

As some solvers pointed out, this follows immediately from problem **2382** [1999 : 440]. In fact, in [1999 : 441], it is given that

$$\cos^2\left(\frac{B-C}{2}\right) \;\geq\; 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \quad (=\;\frac{2r}{R}),$$

and, by symmetry, the same is true for $\cos^2[(A-B)/2]$ and $\cos^2[(C-A)/2]$, and so we are done.

II. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

In ***CRUX*** **585** [1981 : 303] it was shown that

$$\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right) \;\geq\; 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\,. \qquad (1)$$

By squaring (1) and multiplying the right hand side of the resulting inequality by $8\sin(A/2)\sin(B/2)\sin(C/2)$, which is ≤ 1 (see O. Bottema et al, *Geometric Inequalities*, item 2.12), the desired inequality follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HO-JOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

*Dergiades and Lambrou also derive the stronger inequality (1), though without referring to **CRUX** **585** to do it. Janous finds the even stronger inequality*

$$\prod\cos\left(\frac{B-C}{2}\right) \;\geq\; \left(9-8\prod\sin\frac{A}{2}\right)\prod\sin\frac{A}{2} \;\geq\; 8\prod\sin\frac{A}{2}\,.$$

In the other direction, Janous also proves

$$\prod\cos\left(\frac{B-C}{2}\right) \;\leq\; \frac{1}{2}+3\prod\sin\frac{A}{2}+8\left(\prod\sin\frac{A}{2}\right)^2\,.$$

Lambrou further connected this problem to another earlier CRUX problem, as follows. Put $A=\pi-2A_1$, etc.; then A_1, B_1, C_1 are the angles of an acute triangle, and (1) becomes

$$\prod\cos(B_1-A_1) \;\geq\; 8\prod\cos A_1\,.$$

Rewriting A_1 as A, etc., we get that (1), over all triangles, is equivalent to the inequality

$$\prod\cos(B-A)\geq 8\prod\cos A \qquad (2)$$

over all acute triangles. Now let O be the circumcentre of triangle ABC, let AO meet the circle BOC again at A', and define B' and C' similarly. Then the radius R_1 of circle BOC satisfies

$$R_1=\frac{R}{2\cos A}\,.$$

Letting O_1 be the centre of circle BOC, we find that $\angle O_1OA'=B-C$, so

$$OA'=2R_1\cos(B-C)=\frac{R\cos(B-C)}{\cos A}\,,$$

and cyclically for OB' and OC'. Thus (2) is equivalent to

$$OA'\cdot OB'\cdot OC'\geq 8R^3,$$

which is unused problem 13 from the 1996 IMO; see [1999 : 8] for a solution.

2473. [1999 : 368] *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

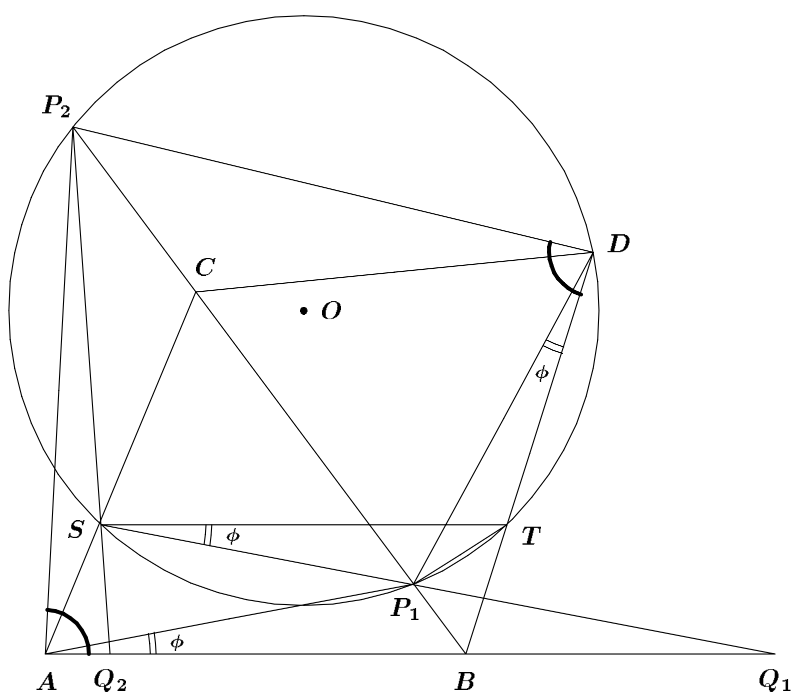
Given a point S on the side AC of triangle ABC, construct a line through S which cuts lines BC and AB at P and Q, respectively, such that $PQ = PA$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Let D be the reflection of A in BC, so that $\triangle DBC$ is congruent to $\triangle ABC$.

Let the line through S parallel to AB intersect BD at T. Let the circumcircle of $\triangle STD$ intersect BC at P_1 and P_2. Let SP_1 and SP_2 intersect AB and Q_1 and Q_2, respectively.

Then $P_1Q_1 = P_1A$ and $P_2Q_2 = P_2A$.



Proof. Note that BC is an axis of symmetry of quadrilaterals

$$ABCD\,,\quad ABDP_1\,,\quad \text{and}\quad ABDP_2\,. \tag{1}$$

Since quadrilateral SP_1TD is cyclic, we have

$$\angle P_1ST \;=\; \angle P_1DT \;=\; \phi\,, \text{ say.} \tag{2}$$

Now, (1) and (2) imply that $\angle P_1AB = \angle P_1DB = \phi$. Since $ST \| AB$, we have

$$\angle P_1Q_1A \;=\; \angle P_1ST \;=\; \phi\,. \tag{3}$$

Now, (2) and (3) imply that $P_1Q_1 = P_1A$. ■

From (1), we obtain that

$$\angle P_2AB \;=\; \angle P_2DB\,. \qquad (4)$$

Since quadrilateral P_2STD is cyclic, we have

$$\angle P_2ST \;=\; \angle P_2Q_2B \;=\; 180^\circ - \angle P_2AB\,. \qquad (5)$$

Now, (4) and (5) imply that $\angle P_2AQ_2 = \angle P_2Q_2A$, so that $P_2Q_2 = P_2A$. ■

Also solved by MICHEL BATAILLE, Rouen, France; NIELS BEJLEGAARD, Stavanger, Norway; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Smeenk commented that he found it to be a very interesting problem on which he spent much time, but found only a quadratic equation with unknown $\tan\phi$, *which was quite unsatisfactory. He then asked a friend, who asked another friend, who found this elegant solution in an old issue of* Euclides, *a Dutch mathematical periodical.*

Bataille, Bejlegaard and Lambrou all made reference to part of this problem being a well-known classical problem. Bataille refered to H. Dörrie, *100 Great Problems of Elementary Geometry, Dover, 1965, and Lambrou, to* B. Bold, *Famous Problems of Geometry, also published by Dover.*

2474*. [1999 : 368] *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing continuous function satisfying, for all x, $y \in \mathbb{R}^+$:

$$f(x+y) + f\left(f(x) + f(y)\right) \;=\; f\Big(f\left(x + f(y)\right) + f\left(y + f(x)\right)\Big)\,.$$

Obviously $f(x) = c/x$ $(c > 0)$ is a solution. Determine all other solutions.

Editor's remark: This problem should have been starred when posed, since it was proposed without a solution.

Comment by proposer: One can prove that if f satisfies the functional equation, then $f(f(x)) = x$ for all $x \in \mathbb{R}^+$ (this was problem 5 of the 1997 Iranian Mathematical Olympiad), but determining all the solutions seems to be a very challenging problem.

This problem remains open.

2475. [1999 : 368] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Prove that

$$\sum_{j=0}^{n}\sum_{k=1}^{n}(-1)^{j+k}\binom{2n}{2j}\binom{2n}{2k-1} \;=\; 0\,.$$

I. *Solution by David Doster, Choate Rosemary Hall, Wallingford, CT, USA.*

Let $A_n = \sum_{j=0}^{n}(-1)^j\binom{2n}{2j}$ and $B_n = \sum_{k=1}^{n}(-1)^k\binom{2n}{2k-1}$.

Then $\sum_{j=0}^{n}\sum_{k=1}^{n}(-1)^{j+k}\binom{2n}{2j}\binom{2n}{2k-1} = A_nB_n$ and $(1+i)^{2n} = A_n - iB_n$.

Thus, $(-4)^n = (2i)^{2n} = \left((1+i)^2\right)^{2n} = (1+i)^{4n} = (A_n^2 - B_n^2) - 2iA_nB_n$, from which we infer that $A_n^2 - B_n^2 = (-4)^n$ and $A_nB_n = 0$.

II. *Solution by Gerry Leversha, St. Paul's School, London, England.*

It is sufficient to note that the given expression factorizes:

$$\sum_{j=0}^{n}\sum_{k=1}^{n}(-1)^{j+k}\binom{2n}{2j}\binom{2n}{2k-1} = \left(\sum_{j=0}^{n}(-1)^j\binom{2n}{2j}\right)\left(\sum_{k=1}^{n}(-1)^k\binom{2n}{2k-1}\right)$$

and then to note that, by the symmetry of Pascal's triangle, the first bracket on the right side is zero for odd values of n and the second bracket is zero for even values of n.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; CHOONGYUP SUNG, Pusan Science High School, Pusan, Korea; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

All the submitted solutions are minor variations of either I or II above, with the only exception of that by the proposer, which uses De Moivre's formula.

2476*. [1999 : 429] *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let n be a positive integer and consider the set $\{1, 2, 3, \ldots, 2n\}$. Give a **combinatorial** proof that the number of subsets A such that

1. A has exactly n elements, and
2. the sum of all elements in A is divisible by n,

is equal to

$$\frac{1}{n}\sum_{d|n}(-1)^{n+d}\binom{2d}{d}\phi\left(\tfrac{n}{d}\right) ,$$

where ϕ is the Euler function. [Ed. Note the correction in the line above.]

Note: When n is prime, proving the formula is problem 6 of the 1995 IMO. A non-combinatorial proof of the formula is due to Roberto Dvornicich and Nikolay Nikolov.

Editor's remark: This problem should have been starred when posed, since it was proposed without a solution.

This problem remains open.

2478. [1999 : 429] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n \in \mathbb{N}$, evaluate $\displaystyle\sum_{k=0}^{n}\frac{n-k}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k}$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (adapted by the editor).

[Ed: This solution assumes familiarity with Newton's generalized binomial coefficients together with some identities thereof.]

Let S_n denote the summation to be evaluated.

The following formulas regarding generalized binomial coefficients are known and easy to verify:

$$(-4)^k\binom{-\frac{1}{2}}{k} = \binom{2k}{k}, \tag{1}$$

$$\binom{-1}{n} = (-1)^n, \tag{2}$$

$$\binom{m}{k+1} = \frac{m}{k+1}\binom{m-1}{k}, \tag{3}$$

$$\sum_{k=0}^{n}\binom{l}{k}\binom{m}{n-k} = \binom{l+m}{n}. \tag{4}$$

[Ed: In all formulas, l and m denote arbitrary integers and n and k, non-negative integers. Formula (4) is usually referred to as the Vandermonde's Convolution Formula.]

Using (3) with $m = \frac{1}{2}$ and $-\frac{1}{2}$, respectively, we obtain

$$\binom{\frac{1}{2}}{k+1} = \frac{1}{2(k+1)}\binom{-\frac{1}{2}}{k} \qquad (5)$$

and

$$\binom{-\frac{1}{2}}{n-k} = \frac{-1}{2(n-k)}\binom{-\frac{3}{2}}{n-k-1}. \qquad (6)$$

Therefore, we have

$$\begin{aligned}
S_n &= \sum_{k=0}^{n-1} \frac{n-k}{k+1}(-4)^k\binom{-\frac{1}{2}}{k}(-4)^{n-k}\binom{-\frac{1}{2}}{n-k} && \text{(by (1))}\\
&= (-4)^n \sum_{k=0}^{n-1} 2\binom{\frac{1}{2}}{k+1}(-\frac{1}{2})\binom{-\frac{3}{2}}{n-k-1} && \text{(by (5), (6))}\\
&= -(-4)^n \sum_{k=0}^{n-1} \binom{\frac{1}{2}}{k+1}\binom{-\frac{3}{2}}{n-k-1}\\
&= -(-4)^n \left(\binom{-1}{n} - \binom{\frac{1}{2}}{0}\binom{-\frac{3}{2}}{n}\right) && \text{(by (4))}\\
&= -(-4)^n \left((-1)^n + 2(n+1)\binom{-\frac{1}{2}}{n+1}\right) && \text{(by (2), (6))}\\
&= -4^n + \frac{n+1}{2}\binom{2n+2}{n+1} && \text{(by (1))}\\
&= (2n+1)\binom{2n}{n} - 4^n\,.
\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and the proposer.

Most of the other submitted solutions involve differentiation and integration of the power series expansion of the function $f(x) = (1-4x)^{-\frac{1}{2}}$ *for* $|\,x\,| < \frac{1}{4}$.

2479. [1999 : 430] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Writing $\tau(n)$ for the number of divisors of n, and $\omega(n)$ for the number of distinct prime factors of n, prove that

$$\sum_{k=1}^{n}(\tau(k))^2 = \sum_{k=1}^{n} 2^{\omega(k)} \sum_{j=1}^{\lfloor n/k\rfloor} \left\lfloor \frac{\lfloor n/k\rfloor}{j}\right\rfloor .$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany. If $n, k \in \mathbb{N}$, then there exist $m \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, k-1\}$ such that $n-1 = km + r$. Then

$$\left\lfloor \frac{n}{k}\right\rfloor = m + \left\lfloor \frac{r+1}{k}\right\rfloor \quad \text{and} \quad \left\lfloor \frac{n-1}{k}\right\rfloor = m .$$

Hence,

$$\left\lfloor \frac{n}{k}\right\rfloor - \left\lfloor \frac{n-1}{k}\right\rfloor = \begin{cases} 1 & \text{if } k|n. \\ 0 & \text{otherwise.}\end{cases}$$

Let S_n, $n \in \mathbb{N}_0$, denote the sum on the right hand side of the desired equation (empty sums are understood to be 0). If $n \in \mathbb{N}$, then from the above result we have

$$S_n - S_{n-1} = \sum_{k|n} 2^{\omega(k)} \sum_{j=1}^{\lfloor n/k\rfloor} \left(\left\lfloor \frac{\lfloor n/k\rfloor}{j}\right\rfloor - \left\lfloor \frac{\lfloor (n-1)/k\rfloor}{j}\right\rfloor \right) ,$$

or, by applying the result once again,

$$S_n - S_{n-1} = \sum_{k|n} 2^{\omega(k)} \tau\left(\frac{n}{k}\right) .$$

With the notation of Dirichlet products (see, for example, T.M. Apostol, Introduction to Analytic Number Theory, 2nd Ed., Springer, 1984, 29-39), the latter equation may be written as $S_n - S_{n-1} = (f * \tau)(n)$, $n \in \mathbb{N}$, where the arithmetical function f is defined by $f(n) = 2^{\omega(n)}$, $n \in \mathbb{N}$. We claim that $f * \tau = \tau^2$ (where τ^2 means ordinary pointwise multiplication). Since f, τ, and τ^2 are all multiplicative, it suffices to show that $(f * \tau)(p^e) = (\tau(p^e))^2$ when p is a prime and $e \in \mathbb{N}$. We have

$$\begin{aligned}(f * \tau)(p^e) &= (\tau * f)(p^e) = \sum_{j=0}^{e} \tau(p^j) f(p^{e-j}) \\ &= \tau(p^e) + 2\sum_{j=0}^{e-1} \tau(p^j) = (e+1) + 2\sum_{j=0}^{e-1}(j+1) \\ &= (e+1)^2 = (\tau(p^e))^2 .\end{aligned}$$

It follows that $S_n - S_{n-1} = (\tau(n))^2$ for all $n \in \mathbb{N}$. Replacing n by k and summing over $k = 1, 2, \ldots, n$, $n \in \mathbb{N}$, gives the requested equation.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and the proposer.

Janous observes that the identity

$$(\tau(n))^2 = \sum_{k|n} 2^{\omega(k)} \tau\left(\frac{n}{k}\right)$$

(implied in the above proof, but not explicitly stated) may be new and is worth noting in its own right, and that applying the Möbius inversion formula to it yields another identity which may be new:

$$\sum_{k|n} 2^{\omega(k)} = \sum_{k|n} \mu(k)\tau^2\left(\frac{n}{k}\right)$$