THE SKOLIAD CORNER
No. 48
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This issue we give the preliminary round of the Senior High School Mathematics Contest of the British Columbia Colleges written March 8, 2000. My thanks go to Jim Totten, The University College of the Cariboo, for sending them for use in the Corner.

BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Preliminary Round — March 8, 2000

1. Antonino sets out on a bike ride of 40 km. After he has covered half the distance he finds that he has averaged 15 km/hr. He decides to speed up. The rate at which he must travel the rest of the trip in order to average 20 km/hr for the whole journey is:
   (a) 25 km/hr    (b) 30 km/hr    (c) 35 km/hr    (d) 36 km/hr    (e) 40 km/hr

2. A circle with centre at (3, 2) intersects the x-axis at the origin, O, and at the point B. The tangents to the circle at O and B intersect at the point P. The y-coordinate of P is:
   (a) $-3\frac{1}{2}$    (b) -4    (c) $-4\frac{1}{2}$    (d) -5    (e) none of these

3. From five students whose ages are 6, 7, 8, 9, and 10, two are randomly chosen. The probability that the difference in their ages will be at least 2 years is:
   (a) $\frac{1}{2}$    (b) $\frac{2}{5}$    (c) $\frac{3}{5}$    (d) $\frac{7}{10}$    (e) $\frac{3}{4}$
4. The centres of three circles of radius 2 units are located at the points 
(0, 0), (12, 0) and (0, 5). If the circles represent pulleys, what is the length 
of the belt which goes around all 3 pulleys as shown in the diagram?

(a) $30 + \pi$ (b) $30 + 4\pi$ (c) $36 + \pi$ (d) $60 - 4\pi$  
(e) none of these

5. If Mark gets 71 on his next quiz, his average will be 83. If he gets 
99, his average will be 87. How many quizzes has Mark already taken?

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

6. While 10 pin bowling (see diagram) 
Sam left 3 pins standing which formed 
the vertices of an equilateral triangle. 
How many such equilateral triangles 
are possible?

(a) 15 (b) 14 (c) 12 (d) 10 (e) none of these

7. If I place a $6 \text{ cm} \times 6 \text{ cm}$ square on a triangle, I can cover up to 60% 
of the triangle. If I place the triangle on the square, I can cover up to $\frac{2}{3}$ 
of the square. What is the area, in cm$^2$, of the triangle?

(a) $22\frac{1}{2}$ (b) 24 (c) 36 (d) 40 (e) 60

8. Two circles, each with radius 10 cm, are placed so they are tangent 
to each other and a straight line. A smaller circle is nestled between them 
so that it is tangent to the larger circles and the line. What is the radius, in 
centimetres, of the smaller circle?

(a) $\sqrt{10}$ (b) 2.5 (c) $\sqrt{2}$ (d) 1 (e) none of these
9. Arrange the following in ascending order:

\[2^{5555}, 3^{3333}, 6^{2222}\]

(a) \(2^{5555}, 3^{3333}, 6^{2222}\)  (b) \(2^{5555}, 6^{2222}, 3^{3333}\)
(c) \(6^{2222}, 3^{3333}, 2^{5555}\)  (d) \(3^{3333}, 6^{2222}, 2^{5555}\)
(e) \(3^{3333}, 2^{5555}, 6^{2222}\)

[Editor's note: Astute readers will notice that this is the same question as Question 10 from the Junior Contest given last issue. The solution appears in this issue.]

10. Given that \(0 < x < y < 20\), the number of integer solutions \((x, y)\) to the equation \(2x + 3y = 50\) is:

(a) 25  (b) 16  (c) 8  (d) 5  (e) 3

11. Suppose \(A, B,\) and \(C\) are positive integers such that

\[
\frac{24}{5} = A + \frac{1}{B + \frac{1}{C + 1}}.
\]

The value of \(A + 2B + 3C\) equals:

(a) 9  (b) 12  (c) 15  (d) 16  (e) 20

12. A box contains \(m\) white balls and \(n\) black balls. Two balls are removed randomly without replacement. The probability one ball of each colour is chosen is:

(a) \(\frac{mn}{(m+n)(m+n-1)}\)  (b) \(\frac{mn}{(m+n)^2}\)  (c) \(\frac{2mn}{(m+n-1)(m+n-1)}\)
(d) \(\frac{2mn}{(m+n)(m+n-1)}\)  (e) \(\frac{m(m-1)}{(m+n)(m+n-1)}\)

13. If it takes \(x\) builders \(y\) days to build \(z\) houses, how many days would it take \(q\) builders to build \(r\) houses? Assume these builders work at the same rate as the others.

(a) \(\frac{wr}{xz}\)  (b) \(\frac{r^2}{qz}\)  (c) \(\frac{qz}{rxy}\)  (d) \(\frac{wx}{qz}\)  (e) \(\frac{r}{qxy}\)

14. If \(x^2 + xy + x = 14\) and \(y^2 + xy + y = 28\), then a possible value for the sum of \(x + y\) is:

(a) \(-7\)  (b) \(-6\)  (c) \(0\)  (d) \(1\)  (e) \(3\)
15. Two congruent rectangles each measuring 3 cm × 7 cm are placed as in the figure. The area of overlap (shaded), in cm², is:

\[ \text{Area} = \frac{27}{7} \] (a) \[ \frac{27}{7} \] (b) \[ \frac{20}{7} \] (c) \[ \frac{20}{7} \] (d) \[ \frac{21}{7} \] (e) none of these

The problems given last issue were those of the preliminary round of the Junior High School Contest of the British Columbia Colleges. My thanks for these “official solutions” to Jim Totten, The University College of the Cariboo.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Preliminary Round — March 8, 2000

1. After 15 litres of gasoline was added to a partially filled fuel tank, the tank was 75% full. If the tank’s capacity is 28 litres, then the number of litres in the tank before adding the gas was:

Answer. (d). Let \( x \) be the number of litres of gasoline in the tank prior to filling. Then \( x + 15 = \frac{3}{4} \times 28 \), or \( x = 6 \).

2. The following figures are made from matchsticks.

If you had 500 matchsticks, the number of squares in the largest such figure you could build would be:

Answer. (b). The first figure is composed of one square of side 1 (consisting of 4 matchsticks) plus 2 squares of side 1 each missing 1 matchstick, for a total of \( 4 + 2 \cdot 3 = 10 \) matchsticks. Each subsequent figure consists of the previous figure plus 2 squares of side 1 each missing 1 matchstick. Thus, the 11th figure in the sequence contains \( 4 + 2 \cdot 3 \cdot n = 6n + 4 \) matchsticks. The largest value \( n \) for which 500 matchsticks is sufficient is thus 82 (which uses up \( 6 \cdot 82 + 4 = 496 \) matchsticks). Now the number of squares in the
first figure is 3 and each subsequent figure contains 2 more squares than the previous one. Therefore the number of squares in the \( n \)th figure is \( 2n + 1 \). For \( n = 82 \) this means that 165 squares would be in the largest figure made with 500 matchsticks.

3. The perimeter of a rectangle is 56 metres. The ratio of its length to width is 4 : 3. The length, in metres, of a diagonal of the rectangle is:

**Answer.** (b). Let \( l \) and \( w \) be the length and width (in metres) of the rectangle in question. Since the perimeter is 56 metres, we have \( 2l + 2w = 56 \), or \( l + w = 28 \). We are also told that \( l : w = 4 : 3 \), or \( l = \frac{4}{3}w \). Using this in the first equation we get

\[
\frac{4}{3}w + w = 28
\]

\[
\frac{7}{3}w = 28
\]

\[
w = 12
\]

which implies that \( l = 16 \). By the Theorem of Pythagoras the length of the diagonal is

\[
\sqrt{12^2 + 16^2} = \sqrt{400} = 20.
\]

4. If April 23 falls on Tuesday, then March 23 of the same year was a:

**Answer.** (a). Since there are 31 days in March, there are 31 days between March 23 and April 23. That is, the period in question is 4 weeks and 3 days. Since April 23 is a Tuesday, we must have March 23 a Saturday, namely 3 days earlier in the week.

5. Consider the dart board shown in the diagram. If a dart may hit any point on the board with equal probability, the probability it will land in the shaded area is:
Answer. (d). The total area of the board is $25x^2$ square units. The area of the shaded region is $x \cdot 4x + x \cdot 3x = 7x^2$ square units. Therefore, the probability of hitting the shaded area is

$$\frac{7x^2}{25x^2} = \frac{7}{25} = 0.28.$$ 

6. The proper divisors of a number are those numbers that are factors of the number other than the number itself. For example, the proper divisors of 12 are 1, 2, 3, 4 and 6. An abundant number is defined as a number for which the sum of its proper divisors is greater than the number itself. For example, 12 is an abundant number since $1 + 2 + 3 + 4 + 6 > 12$. Another example of an abundant number is:

Answer. (c). Let us compute the sum of the proper divisors of each of the 5 possible answers in the list:

<table>
<thead>
<tr>
<th>Number</th>
<th>Sum of Proper Divisors</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>1 &lt; 13</td>
</tr>
<tr>
<td>16</td>
<td>1 + 2 + 4 + 8 = 15 &lt; 16</td>
</tr>
<tr>
<td>30</td>
<td>1 + 2 + 3 + 5 + 6 + 10 + 15 = 42 &gt; 30</td>
</tr>
<tr>
<td>44</td>
<td>1 + 2 + 4 + 11 + 22 = 40 &lt; 44</td>
</tr>
<tr>
<td>50</td>
<td>1 + 2 + 5 + 10 + 25 = 43 &lt; 50</td>
</tr>
</tbody>
</table>

The only one of these which qualifies as an abundant number is 30.

7. The figure below is a right trapezoid with side lengths 4 cm, 4 cm, and 6 cm as labelled. The circle has radius 2 cm. The area, in cm$^2$, of the shaded region is:

![Diagram of a right trapezoid with a circle inside, shaded region is the area between the trapezoid and the circle.]  

Answer. (d). The area in question is the area of a trapezoid less the area of a semicircle. The area of the semicircle is obviously $\frac{1}{2}\pi 2^2 = 2\pi$ cm$^2$. The area of the trapezoid is $\frac{1}{2}(4 + 6) = 20$ cm$^2$. Thus, the shaded area is $20 - 2\pi$ cm$^2$. 
8. Three vertices of parallelogram \( PQRS \) were \( P(-3,-2) \), \( Q(1,-5) \), and \( R(9,1) \) with \( P \) and \( R \) diagonally opposite. The sum of the coordinates of vertex \( S \) is:

Answer. (e). Let the coordinates of the point \( S \) be \((x, y)\). Since \( PS \parallel QR \), they must have the same slope:

\[
\frac{y + 2}{x + 3} = \frac{-5 - 1}{1 - 9} = \frac{3}{4}
\]

or \( 4y - 3x = 1 \).

Since \( RS \parallel QP \), we also have (by the same argument):

\[
\frac{y - 1}{x - 9} = \frac{-5 + 2}{1 + 3} = -\frac{3}{4}
\]

or \( 4y + 3x = 31 \).

From these two equations in 2 unknowns we easily solve for \( x = 5 \) and \( y = 4 \). Thus, \( x + y = 9 \).

9. Which shape cannot be filled, without any overlapping, using copies of the tile shown on the right?

Answer. (b). The diagram below shows how figures (a), (c), (d), and (e) can be filled with copies of the "T" tile. No matter how one tries figure (b) cannot be filled with copies of it.

!(a)(c)(d)(e)!

10. Arrange the following in ascending order:

\[ 2^{5555} \quad 3^{3333} \quad 6^{2222} \]

Answer. (e). Note first that

\[ 2^{5555} = (2^5)^{1111} = 32^{1111} \]
\[ 3^{3333} = (3^3)^{1111} = 27^{1111} \]
\[ 6^{2222} = (6^2)^{1111} = 36^{1111} \]

Since \( 27 < 32 < 36 \), we have \( 27^{1111} < 32^{1111} < 36^{1111} \), which means

\[ 3^{3333} < 2^{5555} < 6^{2222} \].
11. 2000 days, 2000 hours, 2000 minutes, and 2000 seconds would be equivalent to \( N \) million seconds. Of the choices offered, the closest approximation of \( N \) is:

**Answer.** (d). Let us first compute the number of seconds in 1 day, 1 hour, 1 minute, and 1 second, and then multiply by 2000. Now 1 day plus 1 hour is clearly 25 hours. Then 1 day, 1 hour, plus 1 minute is \( 25 \times 60 + 1 = 1501 \) minutes. Expressed in seconds this is \( 1501 \times 60 = 90060 \) seconds. Thus, 1 day, 1 hour, 1 minute, and 1 second is 90,061 seconds. The answer to the problem is this figure multiplied by 2000; that is, 180,122,000, which to the nearest million is 180,000,000.

12. A three-digit decimal number \( abc \) may be expressed as \( 100a + 10b + c \) where each of the digits is multiplied by its respective place value and subsequently summed. If \( a = b = c \) and \( a > 0 \), which of the following numbers must be a factor of the three-digit number \( abc \)?

**Answer.** (e). If \( a = b = c \), then \( 100a + 10b + c = 100a + 10a + a = 111a \). Since \( a \) can be any digit, in order for a number to be a factor of the three-digit number, it must be a factor of 111. The factors of 111 are 1, 3, 37, and 111. The only one of these appearing in the list is 37.

13. If \((x + y)^2 - (x - y)^2 > 0\), then

**Answer.** (a).

\[
(x + y)^2 - (x - y)^2 > 0 \iff x^2 + 2xy + y^2 - x^2 + 2xy - y^2 > 0
\]

\[
\iff 4xy > 0
\]

\[
\iff xy > 0.
\]

The last condition clearly holds if and only if \( x \) and \( y \) have the same sign; that is, both are positive or both are negative.

14. Consider all non-congruent triangles with all sides having whole number lengths and a perimeter of 12 units. The following statements correspond to these triangles.

(i) There are only three such triangles.
(ii) The number of equilateral triangles equals the number of scalene triangles.
(iii) None of these triangles are right angled.
(iv) None of these triangles have a side of length 1 unit.

Of the four statements made, the number of true statements is:

**Answer.** (d). Let \( a, b, c \) be the three sides of the triangle. Let us assume that \( a \leq b \leq c \). Since the perimeter is 12, we have \( a + b + c = 12 \). Let us now list all possible sets of integers \((a, b, c)\) satisfying the above conditions:

\[
(1, 1, 10), \quad (1, 2, 9), \quad (1, 3, 8), \quad (1, 4, 7), \quad (1, 5, 6), \quad (2, 2, 8),
\]

\[
(2, 3, 7), \quad (2, 4, 6), \quad (2, 5, 5), \quad (3, 3, 6), \quad (3, 4, 5), \quad (4, 4, 4).
\]
However, it is clear that some of these "triangles" do not actually exist, since in any triangle the sum of the lengths of the two shorter sides must be greater than the length of the longest side. With this additional condition we have only the following triangles \((a, b, c)\):

\[(2, 5, 5), \quad (3, 4, 5), \quad (4, 4, 4).\]

We can now examine the four statements and conclude that (i), (ii) and (iv) are clearly true. As for (iii), we see that triangle \((3, 4, 5)\) above is right-angled; hence, (iii) is false.

15. An altitude, \(h\), of a triangle is increased by a length \(m\). How much must be taken from the corresponding base, \(b\), so that the area of the new triangle is one-half that of the original?

**Answer.** (e). The area of the original triangle is \(\frac{1}{2}bh\). The new triangle has altitude \(h + m\) and base \(b - x\). We need to find \(x\) such that the area of the new triangle is \(\frac{1}{2}(h + m)(b - x)\). Thus,

\[
\frac{\frac{1}{2}bh}{2(h + m)} = b - x
\]

\[
x = b - \frac{\frac{1}{2}bh}{2(h + m)} = \frac{2bh + 2bm - bh}{2(h + m)} = \frac{b(2m + h)}{2(h + m)}
\]

That completes the *Skoliad Corner* for this issue. Send me your contest materials and any communications about the *Corner*. 