We present the last of the readers’ solutions to the questions of the 1999 Atlantic Provinces Council on the Sciences Annual Mathematics Competition, which was held last year at Memorial University, St. John’s, Newfoundland [1999: 452].

2. The Memorial University Philosopher’s Jockey Club has just received the bronze busts of the ten members of their hall of fame. Each will be placed in its designated place on a single shelf, above the gold plaque bearing the name of the member. The ten busts are drawn at random from the crate. What is the probability that at no time will there be an empty space between two busts already placed on the shelf?

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Let us index the places on the shelf from left to right with the numbers 1 through 10.

As each bust is pulled from the crate and put in its place on the shelf, the only way that no gap will be left between busts on the shelf is if the new bust belongs in the place to the immediate left, or the immediate right of those already on the shelf as illustrated below.

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  _ _ _ _ _ _ _ _ _ _
  1 2 3 4 5 6 7 8 9 10
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The new bust may go here or here.

Let us call the index of the place belonging to the first object pulled from the crate \( r \). This leaves \( r - 1 \) busts to be placed to the left of the first bust, labeled \( L_{r-1}, L_{r-2}, \ldots, L_1 \), and \( 10 - r \) busts to be placed to the right of the first bust, labelled \( R_{r+1}, R_{r+2}, \ldots, R_{10} \).

To avoid ever leaving gaps between busts already placed on the shelf, the ‘left’ busts must be pulled from the crate in order: \( L_{r-1}, L_{r-2}, \ldots, L_1 \), and the ‘right’ busts must be pulled from the crate in order: \( R_{r+1}, R_{r+2}, \ldots, R_{10} \). Note that this has no effect on whether the next bust pulled from
the crate must be a 'left' object or a 'right' object; it means only that if \( L_x \)

is pulled from the crate, then the next 'left' bust must be \( L_{x-1} \) and if \( R_y \)

is pulled from the crate, then the next 'right' bust must be \( R_{y+1} \).

Since one of the 10 busts has been placed, pulling the remaining busts from the crate and placing them on the shelf will take 9 'moves', where each move consists of placing either the next 'left' bust or the 'right' bust. Hence, we must find the number of permutations of the \( r - 1 \) 'left' and the \( 10 - r \) 'right' moves.

If, for example, the first bust placed was the bust at index 4, then there are \( r - 1 = 4 - 1 = 3 \) 'left' moves and \( 10 - r = 10 - 4 = 6 \) 'right' moves, so some possible permutations of 'left' and 'right' moves are:

\[
\begin{align*}
P1: & \quad L_3 \quad R_5 \quad L_2 \quad R_6 \quad L_1 \quad R_7 \quad R_8 \quad R_9 \quad R_{10} \\
& \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\

P2: & \quad R_5 \quad R_6 \quad R_7 \quad L_3 \quad L_2 \quad R_8 \quad L_1 \quad R_9 \quad R_{10} \\
& \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\

P3: & \quad R_5 \quad L_3 \quad R_6 \quad R_7 \quad R_8 \quad L_2 \quad R_9 \quad L_1 \quad R_{10} \\
& \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9
\end{align*}
\]

Recall that the order of all 'left' moves is fixed \((L_{r-1}, L_{r-2}, \ldots, L_1)\) and the order of all 'right' moves is fixed \((R_{r+1}, R_{r+2}, \ldots, R_{10})\). Hence, once it is known that a move was a 'left' move, the exact 'left' bust placed in that move is determined as it must be the next 'left' bust in the above sequence. The same is true for all 'right' moves. Therefore, the indices of the 'left' and 'right' busts are not necessary. For example, the above permutations can be simplified to the following:

\[
\begin{align*}
P1: & \quad L \quad R \quad L \quad R \quad L \quad R \quad R \quad R \\
& \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\

P2: & \quad R \quad R \quad R \quad L \quad L \quad R \quad L \quad R \\
& \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\

P3: & \quad R \quad L \quad R \quad R \quad R \quad R \quad L \quad L \quad R \\
& \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9
\end{align*}
\]

Thus, we are arranging 9 objects, \( r - 1 \) of one kind (L's) and \( 10 - r \)

of another (R's). Hence, for each of the 10 choices for \( r \), there are
\( P(9; r - 1, 10 - r) \) ways to place the busts on the shelf without ever leaving a space between busts already on the shelf.

Alternatively, we could choose which \( r - 1 \) of the 9 moves will be the 'left' moves, leaving the remaining \( 10 - r \) moves to be the 'right' moves. For each of the 10 choices for \( r \), this would give \( \binom{9}{r-1} \) ways to place the busts on the shelf without ever leaving a space between busts already on the shelf. But

\[
P(9; r - 1, 10 - r) = \frac{9!}{(r - 1)!(10 - r)!}
= \frac{9!}{(r - 1)!(9 - (r - 1))!}
= \binom{9}{r-1}
\]

Hence, the two approaches are equivalent.

It is clear that there are 10! ways in total to place the busts on the shelf (without regard as to whether or not gaps are left). Therefore, the probability that at no time will there be any space between busts already placed is:

\[
\frac{\sum_{r=1}^{10} P(9; r - 1, 10 - r)}{10!} = \frac{\sum_{r=1}^{10} \binom{9}{r-1}}{10!}
= \frac{\sum_{r=0}^{9} \binom{9}{r}}{10!}
= \frac{2^9}{10!} = \frac{2}{14175}
\]

A general solution, for \( n \) busts to be placed in \( n \) specific places on a shelf can be obtained by replacing 10 by \( n \) and following the same reasoning as was given for the case with 10 busts. Hence, a general solution for \( n \) busts is: \( \frac{2^{n-1}}{n!} \).