THE OLYMPIAD CORNER
No. 204

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We begin this number with a different feature. Arthur Baragar has provided us with a personal account of his experiences as Deputy Leader to the 40th IMO in Romania. He is a professor at the University of Nevada, Las Vegas, and was a member of the Canadian 1981 IMO team, a coordinator in 1995 and leader observer in 1998.

SNAPSHOTS OF THE '99 CANADIAN IMO TEAM
Arthur Baragar, Deputy Leader

* * * July 13th * * *

I'm sitting at a table in a Frankfurt airport waiting room, playing bridge with three high school students. Slung about my waist is a fanny pack stuffed with passports and plane tickets for eight. A cacophony of Greek buzzes about us, as tired travellers wait to board the plane to Athens. Among them is our contingent, easily distinguished by our bright red golf shirts emblazoned with red maple leaves. We are exhausted after our flight from Toronto, and a little envious of the travellers who are boarding. This is our gate, but it isn't our flight. Ours is the next plane, bound for Bucharest, Romania.

My partner at the table is David Arthur, who has just completed grade 12 at Upper Canada College in Toronto. He and the five other high school students in our group have been selected to represent Canada at the 40th International Mathematical Olympiad. He is playing the hand, which doesn't seem too unusual. Somehow, I'm often the dummy. Defending the hand are David Pritchard of Woburn Collegiate Institute in Scarborough, and David Nicholson of Fenelon Falls Secondary School in Fenelon Falls, Ontario.

As the waiting room slowly empties, we begin to realize that not all the travellers are bound for Athens, but that some are young and would also be on our flight. Jessie Lei, from Vincent Massey Secondary School in Windsor, has come prepared to trade, and is soon exchanging coins with the team from Mexico. She discovered the joy of souvenirs last year, when she first represented Canada at the IMO in Taiwan. The other veteran amongst us is our Team Captain, Jimmy Chui, from Earl Haig Secondary School in North York, who is competing in the IMO for the third time. The sixth member of our team is James Lee, from Eric Hamber Secondary School in Vancouver.
We pack in the cards, and I join my associate, Dorette Pronk, a professor from Calvin College in Michigan. She was first invited to escort the Canadian team to the IMO in '98, while she was working at Dalhousie University. She is in a conversation with Michael Albert, the Deputy Leader from New Zealand and a fellow Canadian who graduated from the University of Waterloo. Their team is well rested after three days in Germany recovering from jet lag, but their trip has already run afoul of Murphy's law. Their German stay was spent replacing a lost passport, a reminder of the awkward burden around my waist. They too had exercised the same precautions, and swore the student had been trusted with it for no more than fifteen minutes. My hand went to my waist — our documents were still there.

* * * July 15th * * *

I toss the frisbee back into the crowd on the field. My aim is not too good, and the disc floats to someone other than the intended target. We are working out some of our anxiety with the American team by throwing several frisbees around. We've been joined by a few Australians, as well as a couple of very young Roma, better known as gypsies. Carmen, our Romanian guide, discreetly voices her concern about our uninvited guests, and cautions us to watch our belongings. The fanny pack is still uncomfortably slung about my waist, but we have little else to worry about. I am happier that we can connect on some level with the locals, and that the students are working out some of their pre-exam jitters.

Our arrival in Bucharest was recorded on both print film and video, and played back to the audience at the opening ceremonies earlier today. Our red shirts no doubt attracted the attention of the reporters. Fortunately, we didn't look nearly as bad as our 18 hours of travel made us feel. I'm sure the images brought a sense of pride and longing to our leader, Prof. Ed Barbeau of the University of Toronto. Ed is a veteran coach of IMO teams, and was my coach when I represented Canada at the '81 IMO in Washington. This year, he was with us for the first week of coaching in Waterloo before departing for Romania. Every team sends their leader a few days early to serve on the jury which selects the questions for the competition. Because Ed has already seen the exam, he and the rest of the jury are sequestered. We waved hello to him in his balcony seat, but that is all the contact we will have with him until after the exam.

The first two days in Bucharest were mostly spent recovering from jet lag, getting to know our fellow competitors, and seeing some of the sites. Bucharest might best be described as a city of faded glory. There are many grand buildings from before the war mixed in with the stark architecture of the communist era. But most are in a state of disrepair — broken fences, minimally tended gardens and torn-up streets. A street-car with numerous rusty patches rumbles by our playing field. The weather has been warm and humid, and water is slowly seeping through my shoes as I race to catch a disc in the wet, shin-high grass.
Playing Frisbee with the American team is something of a tradition for us. The Americans are very friendly and seem particularly excited to be with us. They have already spent two weeks in the country training with the Romanian team, and are happy to be amongst native English speakers again (even though we say “zed” instead of “zee”).

* * * July 18th * * *

My real work has begun. I’ve been reunited with Ed, and we are sitting across from the coordinators for question three. The exam was tough this year, and the students were a bit discouraged with their performance. They shouldn’t have been. They were not the only ones who found the exam tough, and I am proud of their performance. Question three was the hard question in the first session, and the first on our grading schedule. No one on our team has a complete solution, but several found a key result. Since we are going for part marks, some of the work is hard to find. After each session, I interviewed the students, taking simple notes on what each student did. During this interview the students were able to point out what they thought was important in their work, now that they knew how to solve the questions. I therefore know where to look for their most promising work. The coordination for the first five students goes better than expected.

Let me elaborate by briefly describing the mechanics of the coordination procedure used to award grades. The leaders of each team come before a pair of Romanian judges called coordinators, and present a question for grading. The leaders usually propose the grades, and argue what progress the student has made on the problem to warrant that grade. The coordinators judge the progress with that made by other students and between the four of us, a grade is decided upon. On this question, we got the top of what we thought would be a fair grade for five of the six students. The sixth student was David Arthur. In my interview with him, he gave an outline of how to solve the problem, and described the details of all but the last step. A little while later, he filled in the details of the last step. His verbal description was a little more refined than what he managed on the exam, but much of it was there.

I began by pointing out where each of the steps was, and on a separate page, the missing step. I argued that the student was on the verge of a complete solution. They were not satisfied, and pointed out that even with the last step, only one direction would be established. I realized that they didn’t fully understand this particular proof, so I presented the full proof in my own way — kind of like reading the rule book to an official. We then went back over the paper — a slow-motion review of the play. They still were not happy. David didn’t even have a correct asymptotic result, which many other students found. Such results, by themselves, were not relevant to any proof, and unfortunately, didn’t naturally fall out in this proof until the last step. We soon reached an impasse, and in the end, we had to settle for a low grade. A frustrating, but not unfair result — a little like a puck
ringing off a goal post. Close, but nothing to show for it on the scoreboard.

Coordination for the rest of the questions went well. The Romanian coordinators had a pleasant attitude towards coordinating, and for the most part, seemed to want to give points away. They only demanded from us adequate reasons to award them. On one occasion, they even seemed as eager as us to excavate a student’s work to make sure full credit was awarded. “Excavating” is an appropriate description. Our students are fairly good at expressing their thoughts when they have a complete solution, but are not very good at conveying the ideas which they do not yet know how to use. Some of their work which was worthy of note was hidden among reams of scrap, some of which was inserted with other questions. The interviews provided invaluable help with this archaeological dig.

* * * July 21st * * *

I’m sitting on the front seat of a coach travelling up a windy road which follows a mountain stream. We have a police escort three buses ahead. The car races ahead, lights flashing and siren blaring, waving traffic in both directions out of the way so that our VIP caravan can pass. Without the escort, the trip would take twice as long. I have been impressed by how important the country considers this competition. We have seen IMO posters on the public buses and throughout the city; we have attended receptions at some of the country’s grandest government buildings; but nothing has impressed me more with our importance than this police escort. As we cross into another jurisdiction, the police car ahead pulls over and another takes its place. Our escort for the last hour waves to our entourage as we pass by.

We are on our way to Castle Brun, a medieval fort once used to collect tolls at the entrance of a mountain pass, and rumoured to be the home of Count Dracula. Transylvania, (literally “between hills”) is a very beautiful region, reminiscent of the alpine valleys of Switzerland and Austria. The castle is a treasure, beautifully situated with marvellous views, cosy rooms, secret passageways and a quaint courtyard. The atmosphere invites the imagination to run wild. I could spend a lot of time here, but we do not have much — we have more places to see.

Our escorts whisk us back into Bucharest. Stuffed in my bag are table cloths and souvenirs of our visit. I had recognized an opportunity to get some Christmas shopping done, and even borrowed money from one of my fellow travellers. My fanny pack with most of my money was no longer about my waist, but safely stored in the hotel safe — one of the luxuries enjoyed when I joined the leaders after the exam.

In the city streets which are several lanes wide, cars are able to squeeze in between the buses, and our relation to the police escort becomes less obvious. A car in front of us dutifully stops at a light, bringing the tail end of our convoy to a screeching halt. The occupant of the car is unimpressed with how well our driver can lean on his horn, and our escort soon disappears.
ahead of us. We are left to negotiate city traffic at a rush hour pace. It's not a big deal — we're almost home, and we got almost full value out of the escort.

Tomorrow is a shopping day, followed by a sumptuous dinner and disco. In two days, we leave, proud of our accomplishments and happy to have been here. The Romanians have been wonderful hosts and organized an excellent competition.

There is so much more I haven't described — the tomatoes (fresh, juicy, and a delight with every meal); the cheese (not for me); the desserts; the bonfire; the passing of the Canadian designed IMO flag; tight connections; interviews with the press; our trips to Niagara Falls and Elora during training; the logo; and so on.

*** Coda ***

Results: David Arthur, Jimmy Chui and David Pritchard won Bronze medals. As a team, Canada tied with the Dutch team for 31st place. China and Russia tied for 1st, Vietnam was 3rd, Romania was 4th, and the American team placed 10th.

The Exam: Here are the problems of the 40th IMO Competition.

40th INTERNATIONAL MATHEMATICAL OLYMPIAD
Bucharest
Day I — July 16, 1999

1. (Luxembourg) Determine all finite sets $S$ of at least three points in the plane which satisfy the following condition:

   for any two distinct points $A$ and $B$ in $S$, the perpendicular bisector of the line segment $AB$ is an axis of symmetry for $S$.

2. (India) Let $n$ be a fixed integer, with $n \geq 2$.

   (a) Determine the least constant $C$ such that the inequality

   \[
   \sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left( \sum_{1 \leq i \leq n} x_i \right)^4
   \]

   holds for all real numbers $x_1, \ldots, x_n \geq 0$.

   (b) For this constant $C$, determine when equality holds.

3. (Belarus) Consider an $n \times n$ square board, where $n$ is a fixed even positive integer. The board is divided into $n^2$ unit squares. We say that two different squares on the board are adjacent if they have a common side.

   $N$ unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

   Determine the smallest possible value of $N$. 
4. (UK) Determine all pairs \((n, p)\) of positive integers such that

\[
p \text{ is a prime,} \\
1 \leq 2p, \quad \text{and} \\
(p - 1)^n + 1 \text{ is divisible by } n^{\nu - 1}.
\]

5. (IMO) Two circles \(G_1\) and \(G_2\) are contained inside the circle \(G\), and are tangent to \(G\) at the distinct points \(M\) and \(N\), respectively. \(G_1\) passes through the centre of \(G_2\). The line passing through the two points of intersection of \(G_1\) and \(G_2\) meets \(G\) at \(A\) and \(B\). The lines \(MA\) and \(MB\) meet \(G_1\) at \(C\) and \(D\), respectively.

Prove that \(CD\) is tangent to \(G_2\).

6. (Bulgaria) Determine all functions \(f : \mathbb{R} \to \mathbb{R}\) such that

\[
f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1
\]

for all real numbers \(x, y\).

Acknowledgments: I should acknowledge the work of Graham Wright, who might be thought of as our team manager. Our sponsors include: Alberta Education, the Bank of Montreal, the Canadian Mathematical Society, the Centre for Education in Mathematics and Computing (University of Waterloo), the Fields Institute for Research in the Mathematical Sciences, Industry Canada, the New Brunswick Department of Education, the Newfoundland and Labrador Department of Education, the Northwest Territories Department of Education, the Ontario Ministry of Education, the Quebec Ministry of Education, the Samuel Beatty Fund, the Saskatchewan Department of Education, the Senator Norman M. Paterson Foundation, the Sun Life Assurance Company of Canada, the University of Calgary, the University of New Brunswick at Fredericton, the University of Ottawa, the University of Toronto and Waterloo Maple Inc.

In memoriam: Jessie Lei was killed in an automobile accident over the New Year holidays. Our sympathy goes out to her family and friends. She will be dearly missed.
As a first Olympiad Problem Set for this number we give the problems of the 10th Mexican Mathematics Olympiad National Contest of November 1996. My thanks go to Professor Richard Nowakowski, Canadian Team Leader to the IMO at Mar del Plata, Argentina, for collecting the problems for us.

10th MEXICAN MATHEMATICS OLYMPIAD NATIONAL CONTEST, November, 1996
First Day

1. Let \(ABCD\) be a quadrilateral and let \(P\) and \(Q\) be the trisecting points of the diagonal \(BD\) (that is, \(P\) and \(Q\) are the points on the line segment \(BD\) for which the lengths \(BP\), \(PQ\) and \(QD\) are all the same). Let \(E\) be the intersection of the straight line through \(A\) and \(P\) with \(BC\) and let \(F\) be the intersection of the straight line through \(A\) and \(Q\) with \(DC\). Prove the following:

(i) If \(ABCD\) is a parallelogram, then \(E\) and \(F\) are the mid-points of \(BC\) and \(CD\), respectively.

(ii) If \(E\) and \(F\) are the mid-points of \(BC\) and \(CD\), respectively, then \(ABCD\) is a parallelogram.

2. There are 64 booths around a circular table and in each one there is a chip. The chips and the booths are numbered 1 to 64 sequentially (each chip has the same number as the booth it is in). At the centre of the table there are 1996 light bulbs turned off. Each minute the chips move simultaneously in a circular way (following the numbering sense) as described: chip \#1 moves one booth, chip \#2 moves two booths, chip \#3 moves three booths, etc., so that more than one chip can be in the same booth at a given minute. For each minute that a chip shares a booth with chip \#1, a bulb is lit (one for each chip sharing position at that moment with chip \#1). Where is chip \#1 on the first minute in which all bulbs are lit?

3. Prove that it is not possible to cover a 6 cm \(\times\) 6 cm square board with eighteen 2 cm \(\times\) 1 cm rectangles, in such a way that each one of the interior 6 cm lines that form the squaring goes through the middle of at least one of the rectangles. Prove also that it is possible to cover a 6 cm \(\times\) 5 cm square board with fifteen 2 cm \(\times\) 1 cm rectangles, in such a way that each one of the interior 6 cm lines that form the squaring and each one of the interior 5 cm lines that form the squaring goes through the middle of at least one of the rectangles.
Second Day

4. For which integers $n \geq 2$ can the numbers 1 to 16 be written each in one square of a squared 4 × 4 paper (no repetitions allowed) such that each of the 8 sums of the numbers in rows and columns is a multiple of $n$, and all of these 8 multiples of $n$ are different from one another?

5. In an $n \times n$ squared paper, the numbers 1 to $n^2$ are written in the usual ordering (from left to right and then down as shown in the picture for the case $n = 3$).

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Any sequence of steps from a square to an adjacent one (sharing a side) starting at square number 1 and ending at square number $n^2$ is called a path. If $\mathcal{C}$ is a path, denote by $\mathcal{L}(\mathcal{C})$ the sum of the numbers through which $\mathcal{C}$ goes:

(i) For a fixed $n$, let $M$ be the largest $\mathcal{L}(\mathcal{C})$ that can be obtained and let $m$ be smallest $\mathcal{L}(\mathcal{C})$ possible. Prove that $M - m$ is a perfect square.

(ii) Prove that there is no $n$ for which one can find a path $\mathcal{C}$ satisfying $\mathcal{L}(\mathcal{C}) = 1996$.

6. The picture below shows a triangle $\triangle ABC$ in which the length $AB$ is smaller than that of $BC$, and the length of $BC$ is smaller than that of $AC$. The points $A'$, $B'$ and $C'$ are such that $AA'$ is perpendicular to $BC$ and the length of $AA'$ equals that of $BC$; $BB'$ is perpendicular to $AC$ and the length of $BB'$ equals that of $AC$; $CC'$ is perpendicular to $AB$ and the length of $CC'$ equals that of $AB$. Moreover $\angle AC'B$ is a 90° angle. Prove that $A'$, $B'$ and $C'$ are collinear.
The final contest set we give this issue is the Bi-National Israel-Hungary Competition, 1996. Thanks go to J.P. Grossman for collecting these problems and forwarding them to me when he was Canadian Team Leader to the IMO at Mumbai.

THE BI-NATIONAL ISRAEL-HUNGARY COMPETITION, 1996

Technion IIT, Israel
March 27, 1996 (Time: 4 hours)

Each problem is worth 7 points.

1. Find all sequences of integers \(x_1, x_2, \ldots, x_{1997}\) such that

\[
\sum_{k=1}^{1997} 2^{k-1}(x_k)_{1997} = 1996 \prod_{k=1}^{1997} x_k.
\]

2. Let \(n > 2\) be an integer, and suppose that \(n^2\) can be represented as the difference of the cubes of two consecutive positive integers. Prove that \(n\) is the sum of two squares. Prove that such an \(n\) really exists.

3. A given convex polyhedron has no vertex which is incident with exactly 3 edges. Prove that the number of faces of the polyhedron which are triangles, is at least 8.

4. Let \(a_1, a_2, \ldots, a_n\) be arbitrary real numbers and \(b_1, b_2, \ldots, b_n\) real numbers satisfying the condition \(1 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq 0\). Prove that there is a positive integer \(k \leq n\) for which the inequality \(|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq |a_1 + a_2 + \cdots + a_k|\) holds.

We next turn to readers' solutions to problems of the 4th Mathematical Olympiad of the Republic of China (Taiwan) [1998: 322–323].

1. Let \(P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n\) be a polynomial with complex coefficients. Suppose the roots of \(P(x)\) are \(\alpha_1, \alpha_2, \ldots, \alpha_n\) with \(|\alpha_1| > 1, |\alpha_2| > 1, \ldots, |\alpha_j| > 1, \text{and } |\alpha_{j+1}| \leq 1, \ldots, |\alpha_n| \leq 1\). Prove:

\[
\prod_{i=1}^{j} |\alpha_i| \leq \sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}.
\]

Solution by Mohammed Aassila, Strasbourg, France.

Let

\[
Q(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m,
\]

\[
R(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n,
\]

\[
Q(x)R(x) = c_0x^{n+m} + c_1x^{n+m-1} + \cdots + c_{n+m}
\]

and

\[
Q(x)R\left(\frac{1}{x}\right) = d_{-m}x^m + \cdots + d_nx^n
\]
with \( a_0 = b_0 = 1 \).

We claim that

\[
\sum_{i=0}^{m+n} c_i^2 = \sum_{k=-m}^{n} d_k^2.
\]  

Indeed, \( d_k \) is the sum of \( a_\alpha b_\beta \) with \( \alpha - \delta = k \), so that \( d_k^2 \) is the sum of \( a_\alpha b_\beta a_\gamma b_\delta \) with \( \alpha - \delta = \gamma - \beta = k \), and summing over \( k \) means we take all \( \alpha, \beta, \gamma, \delta \) with \( \alpha - \delta = \gamma - \beta \). Hence

\[
\sum_{i=0}^{m+n} c_i^2 = \sum_{\alpha + \beta = \gamma + \delta} a_\alpha b_\beta a_\gamma b_\delta
\]

\[
= \sum_{\alpha - \delta = \beta - \gamma} a_\alpha b_\beta a_\gamma b_\delta = \sum_{k=-m}^{n} d_k^2.
\]

Hence, we set

\[
Q(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_j),
\]

\[
R(x) = (x - \alpha_{j+1})(x - \alpha_{j+2}) \cdots (x - \alpha_n).
\]

Because of (1), we know that \( |a_0|^2 + \cdots + |a_n|^2 \) is equal to the sum of the squares of the absolute values of the coefficients of \( Q(x)R(\frac{1}{x}) \), and in particular we get \( |\alpha_1 \cdots \alpha_j|^2 \leq |a_0|^2 + \cdots + |a_n|^2 \).

2. Given a sequence of integers: \( x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \), one constructs a second sequence: \( |x_2 - x_1|, |x_3 - x_2|, |x_4 - x_3|, |x_5 - x_4|, |x_6 - x_5|, |x_7 - x_6|, |x_8 - x_7|, |x_1 - x_8| \). Such a process is called a single operation. Find all the 8-term integral sequences having the following property: after finitely many applications of the single operation, the sequence becomes an integral sequence with all terms equal.

Comment by Mohammed Aassila, Strasbourg, France.

This problem is a special case of problem 3 by Wai Ling Yee, [1998: 34].

3. Suppose \( n \) persons meet in a meeting. Every one among them is familiar with exactly eight other participants of that meeting. Furthermore suppose that each pair of two participants who are familiar with each other has four acquaintances in common at that meeting, and each pair of two participants who are not familiar with each other has only two acquaintances in common. What are the possible values of \( n \)?

Solution by Pierre Bornszttein, Courdimanche, France.

Such a situation cannot occur.

Suppose, for a contradiction, that such a meeting is possible. Let \( x \) be a participant, and let \( x \) be familiar with \( x_1, x_2, \ldots, x_8 \).

As \( x \) is familiar with \( x_8 \) they have four acquaintances in common, among \( x_1, \ldots, x_7 \), say \( x_1, x_2, x_3, x_4 \). Then \( x_7 \) and \( x_8 \) are not familiar.
Now $x$ is familiar with $x_7$, so they have four acquaintances in common among $x_1, x_2, \ldots, x_6$. So at least two must be among $x_1, x_2, x_3, x_4$, say $x_1$ and $x_2$.

But then $x_7$ and $x_8$, who are not familiar, have $x_1, x_2$ and $x$ in common, and that is too many, a contradiction.

4. Given $n$ distinct integers $m_1, m_2, \ldots, m_n$, prove that there exists a polynomial $f(x)$ of degree $n$ and with integral coefficients which satisfies the following conditions:

(1) $f(m_i) = -1$, for all $i$, $1 \leq i \leq n$.

(2) $f(x)$ cannot be factorized into a product of two non-constant polynomials with integral coefficients.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. The solutions were similar. We give Bornsztein’s solution.

It suffices to show that $f(x) = (x - m_1) \ldots (x - m_n) - 1$ satisfies condition (2).

Suppose, for a contradiction, that $f = QR$, with $Q, R \in \mathbb{Z}[x]$, and $Q, R$, with degree at most $n - 1$.

For all $i \in \{1, \ldots, n\}$, since $f(m_i) = -1 = Q(m_i)R(m_i)$, with $Q(m_i), R(m_i)$ integers, then

$$Q(m_i) = 1 \quad \text{and} \quad R(m_i) = -1, \quad \text{or} \quad Q(m_i) = -1 \quad \text{and} \quad R(m_i) = 1.$$ 

So, $Q + R$ is a polynomial with degree $\leq n - 1$ and with $n$ distinct roots, then $Q + R \equiv 0$; that is, $Q \equiv -R$. Thus $f = -Q^2$, which is impossible since the leading coefficient of $f$ is 1, a contradiction. Thus $f$ is irreducible in $\mathbb{Z}[X]$.

5. Let $P$ be a point on the circumscribed circle of $\triangle A_1A_2A_3$. Let $H$ be the orthocentre of $\triangle A_1A_2A_3$. Let $B_1$ ($B_2$, $B_3$ respectively) be the point of intersection of the perpendicular from $P$ to $A_2A_3$ ($A_1A_2$, $A_2A_3$ respectively). It is known that the three points $B_1$, $B_2$, $B_3$ are collinear. Prove that the line $B_1B_2B_3$ passes through the mid-point of the line segment $PH$.

Solution by Pierre Bornsztein, Courdimanche, France.

The line $B_1B_2B_3$ is known as the Simson line.

Let $C_1, C_2, C_3$ be such that $B_1$, $B_2$, $B_3$ are the mid-points of $PC_1$, $PC_2$, $PC_3$, respectively.
It is known that $C_1, C_2, C_3$ are collinear. The line $C_1C_2C_3$ is the Steiner line, $L$. (The Steiner line is the image of the Simson line by the homothetic transformation with centre $P$ and ratio 2.)

It suffices to show that $H$ is on the Steiner line.

If $P$ is one of $A_1, A_2, A_3$, then $L$ is an altitude in $\triangle A_1A_2A_3$ and $H \in L$.

If $P \not\in \{A_1, A_2, A_3\}$, let $K$ be symmetric with $H$ with respect to $A_2A_3$. It is known that $K$ is on the circle $L$.

If $H$ is among $C_1, C_2, C_3$, dearly $H \in L$. Otherwise, with angles evaluated modulo $\pi$: [Ed. $(AB; CD)$, means the angle from the line segment $AB$ to the line segment $CD$.]

\[(C_1H; C_2H) = (C_1H; C_1P) + (C_1P; C_2P) + (C_2P; C_2H)\]
\[(C_1H; C_1P) = (KH; KP) \text{ symmetry in } A_2A_3\]
\[= (KA_1; KP)\]
\[= (A_2A_1; A_2P) \text{ concyclic.}\]

Similarly \[(C_2P; C_2H) = (A_1P; A_1A_2).\]

Thus \[(C_1H; C_2H) = (A_2A_1; A_2P) + (C_1P; C_2P) + (A_1P; A_1A_2)\]
\[= (A_1P; A_2A_1) + (A_2A_1; A_2P) + (C_1P; C_2P)\]
\[= (PA_1; PA_2) + (C_1P; C_2P).\]

Moreover \[(C_1P; C_2P) = (B_1P; B_2P)\]
\[= (B_1P; PA_2) + (PA_2; PA_1) + (PA_1; B_2P).\]
Then
\[(C_1 H; C_2 H) = (B_1 P; PA_2) + (PA_1; B_2 P)\]
\[= \left( B_1 P; B_2 A_2 \right) + \left( B_1 A_2; PA_2 \right)_{(\pi/2)}\]
\[+ (PA_1; A_1 B_2) + \left( A_1 B_2; B_2 P \right)_{(\pi/2)}\]
\[= (A_3 P; PA_2) + (PA_1; A_1 A_3)\]
\[= (A_2 A_3; A_3 P) + (A_3 P; A_2 A_1) \quad \text{(concyclic)}\]
\[= 0 \pmod{\pi}.
\]
Then \(C_1, C_2, H\) are collinear and in all cases \(H \in \mathcal{L}\).

6. Let \(a, b, c, d\) be integers such that \(ad - bc = k > 0\), \((a, b) = 1\), and \((c, d) = 1\). Prove that there are exactly \(k\) ordered pairs of real numbers \((x_1, x_2)\) satisfying \(0 \leq x_1, x_2 < 1\) and for which both \(ax_1 + bx_2\) and \(cx_1 + dx_2\) are integers.

\[\text{Solution by Mohammed Aassila, Strasbourg, France.}\]

Because of Pick's Theorem, we know that the area of \(P = \{(ax_1 + bx_2, cx_1 + dx_2), 0 \leq x_1, x_2 \leq 1\}\) is equal to the sum of the number of interior lattice points and half the number of boundary points. Since \((x_1, x_2)\) is a lattice point if and only if \((1 - x_1, 1 - x_2)\) is as well, we conclude that the number of lattice points in \(P\) is exactly the area, which is \(ad - bc = k\).

We next turn to solutions to problems of the XI Italian Mathematical Olympiad 1995, [1998 : 323-324].

2. In a class of 20 students no two of them have the same ordered pair (written and oral examinations) of scores in mathematics. We say that student \(A\) is better than \(B\) if his two scores are greater than or equal to the corresponding scores of \(B\). The scores are integers between 1 and 10.

(a) Show that there exist three students \(A, B, C\) such that \(A\) is better than \(B\) and \(B\) is better than \(C\).

(b) Would the same be true for a class of 19 students?

\[\text{Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.}\]

(a) Let the ordered pairs be denoted by \((a_i, b_i), i = 1, 2, \ldots, 20\), where \(a_i\) and \(b_i\) denote, respectively, the scores of the \(i^{th}\) student. For convenience of notation, write \((a_i, b_i) < (a_j, b_j)\) if the \(j^{th}\) student is better than the \(i^{th}\) student. Thus \((a_i, b_i) < (a_j, b_j)\) if and only if \(a_i \leq a_j\) and \(b_i \leq b_j\). Since there are 20 \(a_i's\) which are all in \(\{1, 2, \ldots, 10\}\) we must have either

(1): For some \(m \in \{1, 2, \ldots, 10\}\), \(a_i = a_j = a_k = m\) where \(i \neq j \neq k \neq i\), or
(2): Every \( m \in \{1, 2, \ldots, 10\} \) appears exactly twice as the first component in the ordered pairs.

Similarly, either

(3): For some \( n \in \{1, 2, \ldots, 10\} \), \( b_i = b_j = b_k = n \) where \( i \neq j \neq k \neq i \), or

(4): Every \( n \in \{1, 2, \ldots, 10\} \) appears exactly twice as the second component in the ordered pairs.

In case (1), we would have three ordered pairs of the form \((m, b_i), (m, b_j), \) and \((m, b_k)\). Since \( b_i, b_j, \) and \( b_k \) must all be distinct, we may assume, without loss of generality, that \( b_i < b_j < b_k \), and then \((m, b_i) < (m, b_j) < (m, b_k)\). Similarly, in case (3), we would have three ordered pairs of the form \((a_i, n), (a_j, n), \) and \((a_k, n)\). Assuming \( a_i < a_j < a_k \), we then have \((a_i, n) < (a_j, n) < (a_k, n)\). If neither (1) nor (3) holds, then both (2) and (4) must hold and so every \( l \in \{1, 2, \ldots, 10\} \) appears exactly twice as the first component and exactly twice as the second component in the ordered pairs. In particular, there must be ordered pairs of the form \((1, b_i) \) and \((1, b_j)\) for some \( b_i < b_j \). Since 10 must appear twice as the second component and \( b_i \neq b_j \), there must be at least one ordered pair of the form \((a_k, 10)\) for some \( a_k \neq 1 \). Then we have \((1, b_i) < (1, b_j) < (a_k, 10)\) and the conclusion follows.

(b) No. For example, if the 19 ordered pairs are \((1, 10), (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2), (10, 1)\) and \((2, 10), (3, 9), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4), (9, 3), (10, 2)\), then no two ordered pairs from the first ten are comparable and no two ordered pairs from the other nine are comparable. Hence the required "chain" of three students does not exist.

3. In a town there are 4 pubs, \( A, B, C \) and \( D \), connected as shown in the picture.

A drunkard wanders about the pubs starting with \( A \) and, after having a drink, goes to any of the pubs directly connected with equal probability.

(a) What is the probability that the drunkard is at pub \( C \) at his fifth drink?

(b) Where is the drunkard more likely to be after \( n \) drinks? \((n > 5)\)

Solution by Pierre Bornstein, Courdimanche, France.

Let \( a_n, b_n, c_n, d_n \) be the probabilities for the drunkard to be in pub \( A, B, C, D \) respectively for drink number \( n \) \((n \geq 1)\). Let \( (a_{n+1}; b_n) \) (etc.)
be the probability that the drunkard is in \( A \) for the \((n + 1)\)th drink knowing that he was in \( B \) for the \( n\)th drink.

We have \( a_1 = b_1 = c_1 = d_1 = 0 \) and \( a_2 = d_2 = 0 \), \( b_2 = c_2 = 1/2 \).

For \( n \geq 1 \) we have

\[
a_{n+1} = a_n \times (a_{n+1}; a_n) + b_n \times (a_{n+1}; b_n) + c_n \times (a_{n+1}; c_n) + d_n \times (a_{n+1}; d_n)
\]

with \((a_{n+1}; a_n) = 0 = (a_{n+1}; d_n)\) and \((a_{n+1}; b_n) = (a_{n+1}; c_n) = 1/3\).

Then \( a_{n+1} = \frac{1}{3} b_n + \frac{1}{3} c_n \).

Doing the same for \( b_{n+1}, c_{n+1}, \) and \( d_{n+1}, \) we obtain

\[
a_{n+1} = \frac{1}{3} b_n + \frac{1}{3} c_n = d_{n+1},
\]

\[
b_{n+1} = \frac{1}{2} a_n + \frac{1}{3} c_n + \frac{1}{2} d_n,
\]

\[
c_{n+1} = \frac{1}{2} a_n + \frac{1}{3} b_n + \frac{1}{2} d_n.
\]

Because \( b_1 = c_1 \) and \( b_2 = c_2 \), we deduce by induction that \( b_n = c_n \) for every \( n \geq 1 \).

Moreover \( a_n + b_n + c_n + d_n = 1 \).

Thus, for \( n \geq 2 \), \( a_n + b_n = \frac{1}{2} \), and, for \( n \geq 1 \), \( a_{n+1} = \frac{2}{3} b_n \).

Thus, for \( n \geq 2 \), \( a_{n+1} = \frac{1}{3} - \frac{2}{3} a_n \) \((*)\).

Let \( u_n = a_n - \frac{1}{5} \). Using (*) we obtain \( u_{n+1} = -\frac{2}{5} u_n \), for \( n \geq 2 \).

Then, for \( n \geq 2 \), \( u_n = \left( -\frac{2}{5} \right)^{n-2} u_2 \) with \( u_2 = a_2 - \frac{1}{5} = -\frac{1}{5} \). Thus,

\[
u_n = \frac{1}{15} \left( \frac{2}{3} \right)^n \text{ for } n \geq 2,
\]

and

\[
a_n = \frac{1}{5} - \frac{2}{5} \left( \frac{2}{3} \right)^n d_n.
\]

Thus \( b_n = c_n = \frac{1}{2} - a_n = \frac{3}{10} + \frac{8}{5} \left( -\frac{2}{3} \right)^{n-2}d_n \).

(a) We deduce that \( c_5 = \frac{3}{10} + \frac{8}{5} \times \frac{13}{27} = \frac{13}{54} \).

(b) For \( n > 5 \), \( a_n < b_n \) \(\iff\) \(-1 < 4\left( -\frac{2}{3} \right)^{n-2} \).

This is true for \( n \) even.

For odd \( n \), \( 4\left( -\frac{2}{3} \right)^{n-2} \) forms an increasing sequence with limit 0, and for \( n = 7 \) we have \( 4\left( -\frac{2}{3} \right)^7 - 2 > -1 \). Thus \( a_n = a_n < b_n = c_n \) for \( n > 5 \).

Thus for \( n > 5 \), the drunkard is more likely to be in \( B \) than \( C \).

4. An acute-angled triangle \( ABC \) is inscribed in a circle with centre \( O \). Let \( D \) be the intersection of the bisector of \( A \) with \( BC \) and suppose that the perpendicular to \( AO \) through \( D \) meets the line \( AC \) in a point \( P \) interior to the segment \( AC \). Show that \( AB = AP \).
Solution by Pierre Bornsztein, Courdimanche, France.

Since $ABC$ is acute, $O$ is interior to $ABC$, $P \in AC$, so $O$ is interior to $ADC$. We have $\angle AOC = 2\beta$ and $OA = OC$, therefore $\angle CAO = \frac{\pi}{2} - 2\beta = \frac{\pi}{2} - 2\beta$. Thus

$$\angle OAD = \frac{\alpha}{2} - \angle CAO = \frac{\alpha}{2} + \beta - \frac{\pi}{2}.$$ 

Let $I = PD \cap AO$. We have that $\triangle AID$ is right-angled at $I$.

Then

$$\angle PDA = \angle IDA = \frac{\pi}{2} - \angle OAD = \pi - \frac{\alpha}{2} - \beta = \angle BDA.$$ 

We deduce that triangles $ABD$ and $APD$ are similar with $AD$ in common. Therefore triangles $ABD$ and $APD$ are isometric, so that $AP = AB$.

6. Find all pairs of positive integers $x, y$ such that

$$x^2 + 615 = 2^y.$$ 

Solutions by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

The only solution is $x = 59$ and $y = 12$. Note first that for all non-negative integers $k$

$$2^{2k+1} \equiv 4^k \cdot 2 \equiv (-1)^k \cdot 2 \equiv 2 \text{ or } 3 \pmod{5}.$$ 

Since $x^2 \equiv 0, 1 \text{ or } 4 \pmod{5}$, $y$ must be even. Letting $y = 2z$, then $x^2 + 615 = 2^y$ becomes $(2^z - x)(2^z + x) = 615$. Since $615 = 3 \times 5 \times 41$ there are only 4 cases:
We next turn to the Yugoslav Federal Competition 1995, Third and Fourth Grade [1998: 325].

1. Let $p$ be a prime number. Prove that the number

$$11 \cdots 122 \cdots 2 \cdots 99 \cdots 9 - 123456789$$

is divisible by $p$, where dots indicate that the corresponding digit appears $p$ times consecutively.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang’s solution.

Let $n$ denote the given number and let $c = 123456789$. First note that the conclusion clearly holds if $p = 3$, since in this case both terms of $n$ are divisible by 3. Assume that $p \neq 3$. By assumption, we have

$$n = \sum_{k=0}^{p-1} 10^{6p+k} + 2 \sum_{k=0}^{p-1} 10^{7p+k} + \cdots + 9 \sum_{k=0}^{p-1} 10^k - c. \quad (1)$$

For $m = 0, 1, 2, \ldots, 8$ we get from (1) that

$$n = \frac{1}{9} (10^p - 1)(10^{6p} + 2 \times 10^{7p} + \cdots + 8 \times 10^p + 9 \times 1) - c$$

$$= \frac{1}{9} (10^{6p} + 10^{8p} + \cdots + 10^p - 9) - c. \quad (2)$$

Since $p \mid n$ if and only if $9p \mid 9n$, it suffices to show that [Ed. because $p \neq 3$]

$$9p \mid 10^{6p} + 10^{8p} + \cdots + 10^p - 9 - 9c. \quad (3)$$

Since $9 + 9c = 1,111,111,110 = 10^9 + 10^8 + \cdots + 10$, (3) is equivalent to

$$9p \mid (10^{6p} + 10^{8p} + \cdots + 10^p) - (10^9 + 10^8 + \cdots + 10). \quad (4)$$

By Fermat’s Little Theorem, we have for all $m = 1, 2, \ldots, 9, 10^{mp} \equiv (10^m)^p \equiv 10^m \pmod{p}$. Furthermore, as $10 \equiv 1 \pmod{9}$, we
have $10^{mp} \equiv 10^m \pmod{9}$ for all $m = 1, 2, \ldots, 9$. Since $p \neq 3$, $(p, 9) = 1$. Thus $10^{mp} \equiv 10^m \pmod{9p}$ and (4) follows.

This completes the proof.

Comment: Very interesting problem, indeed!

2. A polynomial $P(x)$ with integer coefficients is said to be divisible by a positive integer $m$ if and only if the number $P(k)$ is divisible by $m$ for all $k \in \mathbb{Z}$. If the polynomial

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

is divisible by $m$, prove that $a_n n!$ is divisible by $m$.

Solution by Pierre Bornsztein, Courdimanche, France. Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang’s comment.

Assuming there is no error in the statement, then this problem seems trivial, since if $m \mid p(k)$ for all $k \in \mathbb{Z}$, then $m \mid p(0)$; that is, $m \mid a_n$, and so $m \mid a_n n!$.

3. A chord $AB$ and a diameter $CD \perp AB$ of a circle $k$ intersect at a point $M$. Let $P$ lie on the arc $ACB$ and let $a \not\in \{A, B, C\}$. Line $PM$ intersects the circle $k$ at $P$ and $Q \neq P$, and line $PD$ intersects chord $AB$ at $R$. Prove that $RD > MQ$.

Comment by Pierre Bornsztein, Courdimanche, France.

This result is due to P. Erdős. See a solution in Ross Honsberger’s “Mathematical Morsels”.

Finally we discuss reader submissions about the Yugoslav Federal Competition 1995, Selection of the IMO Team [1998: 325].

1. Find all the triples $(x, y, z)$ of positive rational numbers such that $x \leq y \leq z$ and

$$x + y + z, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz \in \mathbb{Z}.$$

Solution by Pierre Bornsztein, Courdimanche, France. Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang’s comment.

This problem was the same as problem #1 of the 45th Mathematical Olympiad in Poland (Final Round) and has appeared in [1997: 323]. A solution by Murray S. Klamkin appeared in [1998: 394–395].

2. Let $n$ be a positive integer having exactly 1995 1’s in its binary representation. Prove that $2^{n-1995}$ divides $n!$. 
Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

We will prove that if \( n \in \mathbb{N}^* \) and \( \lambda(n) \) is the number of 1's in the binary representation of \( n \), then \( 2^{n-\lambda(n)} \) divides \( n \).

**Lemma.** Let \( p \in \mathbb{N}^* \). The greatest power of 2 which divides \( (2^p)! \) is \( 2^N \) with \( N = 2^p - 1 \).

**Proof of the lemma.** It is well known that, with the notation of the lemma

\[
N = \left\lfloor \frac{2^p}{2} \right\rfloor + \left\lfloor \frac{2^p}{2^2} \right\rfloor + \left\lfloor \frac{2^p}{2^3} \right\rfloor + \cdots
\]

(where \( \lfloor \cdot \rfloor \) is the integer part).

Then \( N = 2^{p-1} + 2^{p-2} + \cdots = 2^p - 1 \).

We prove the main result by induction on \( \lambda(n) = k \in \mathbb{N}^* \).

For \( k = 1 \), then \( n \) is a power of 2, \( n = 2^p \), \( p \in \mathbb{N} \) and we conclude with the lemma.

Suppose the result for a fixed \( k \geq 1 \).

Then \( n = 2^p + m \) where \( m \in \mathbb{N}^*, m < 2^p \) and \( \lambda(m) = k \), and we have

\[
n! = (2^p)! (2^p + 1) (2^p + 2) \cdots (2^p + m).
\]

But, by the lemma, we have \( (2^p)! \equiv 0 \text{ (mod } 2^{2^p - 1}) \). Moreover \( (2^p + 1)(2^p + 2) \cdots (2^p + m) \) is the product of \( m \) consecutive positive integers, so that \( (2^p + 1)(2^p + 2) \cdots (2^p + m) \) is divisible by \( m! \) and, by the induction hypothesis, \( m! \equiv 0 \text{ (mod } 2^{m-k}) \).

Then \( (2^p + 1) \cdots (2^p + m) \equiv 0 \text{ (mod } 2^{m-k}) \). And so \( n! \) is divisible by \( 2^{2^p - 1} \times 2^{m-k} = 2^{n-m-(k+1)} = 2^{n-\lambda(n)} \).

**Question:** Is \( 2^{n-\lambda(n)} \) the greatest power of 2 which divides \( n! \)?

That completes the Corner for this issue. Please send me your Olympiad Contests and nice solutions.