Here are the hints and answers to the 1999 Bernoulli Trials [1999: 321]. Many thanks to Christopher Small for sending them to us.

**The Bernoulli Trials 1999**

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**Hints and Answers:**

1. TRUE. We have

\[
\frac{1}{1999} = \frac{1}{x} + \frac{1}{y} > \max \left( \frac{1}{x}, \frac{1}{y} \right).
\]

Therefore \( x, y \geq 2000 \). Moreover, if

\[ x > 3998000 = 2000 \times 1999 \]

then \( 1/x < 1/3998000 \), so that

\[
\frac{1}{1999} < \frac{1}{3998000} + \frac{1}{y}.
\]

This reduces to \( y < 2000 \), which is a contradiction.

2. TRUE. The equation implies that

\[ \cos x = \pi/2 \pm \sin x \pmod{2\pi} \]

As \( |\cos x| \leq 1 \) we must have \( \cos x = \pi/2 \pm \sin x \), or

\[ \cos^2 x \pm 2 \cos x \sin x + \sin^2 x = \pi^2/4. \]

Therefore \( |\sin 2x| = \pi^2/4 - 1 > 1 \). But this cannot hold.
3. FALSE. As $n^3 - n = n(n-1)(n+1)$, where $n$ is odd, it follows that the right hand side is divisible by 9 and 16. As the sum of the digits must be 0 (mod 9) it follows that $A + B = 7$ (mod 9). In addition, $B$ must be even. The possibilities for $AB$ are 70, 52, 34, 16 and 88. However, $48AB$ must be divisible by 16.

So the answer is $AB = 16$. In fact, $CD = 76$, although this is not needed.

4. FALSE.

5. TRUE. The colouring of the squares in the first row and first column is arbitrary. There are $2^{15}$ choices for colouring these squares. Once the colours of these squares are determined, the rest of the board can only be coloured in one way to fulfill the condition.

6. TRUE. Consider polynomials. Let $n$ be the degree of $f$. The equation implies that $2n = n^2$. This requires that $f$ be constant or a quadratic. It can be checked that $f(x) = x^2$ works.

7. FALSE. Formally differentiating term by term gives us back the same series. Thus $\phi'(x) = \phi(x)$. This implies that $\phi(x) = Ae^x$ for some constant $A$. We note that $f(x) = A$ works. Since $\phi(0) = 1999$, we find that $\phi(x) = 1999e^x$. Therefore $\phi(-1999) > 0$.

8. TRUE. Write

$$\int_0^1 x^{-x} \, dx = \int_0^1 e^{x \ln (1/x)} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n [\ln (1/x)]^n \, dx$$
Substituting \( x^{n+1} = e^{-y} \) reduces the right-hand side to \( \sum_{n=1}^{\infty} n^{-n} \). The result follows easily from this.

9. FALSE. First note that the event that a triangle is acute is independent of the choice of labels for the vertices.

![ Triangle Diagram ]

Shrink the circle around \( \triangle ABX \) until one of the 3 points hits the boundary. Now relabel the triangle so that the two points in the interior of the new circle are \( A \) and \( B \) in random order, and the point on the boundary of the new circle is called \( X' \).

This shows that with respect to events that are symmetric in the labels, \( \triangle ABX \) and \( \triangle ABX' \) have the same distribution. Thus \( \Pi = \Pi' \).

10. TRUE. Since it takes 4 counters to block the lines in any \( 4 \times 4 \) slice, any solution must have exactly 4 counters in each such horizontal slice. There are two basic patterns, \( A \) and \( B \) as shown, which will block the lines with 4 counters within a slice. Other configurations of four counters are obtained as rotations or reflections of these two basic cases.

![ Counter Patterns ]

Stacking I, II, III, and IV in order can be checked to block all lines parallel to a face or edge.

11. FALSE. Try

Day 1: M1W1 v M2W2 and M3W3 v M4W4
Day 2: M1W3 v M3W1 and M2W4 v M4W2
Day 3: M1W2 v M4W3 and M2W1 v M3W4
We now present the questions of the 1999 Atlantic Provinces Council on the Sciences Annual Mathematics Competition, held this year at Memorial University, St. John's, Newfoundland. The winning team consisted of Ian Caines and Jacky Pak Ki Li from Dalhousie University. The runners-up were the team of Shannon Sullivan and Jerome Terry from Memorial University of Newfoundland. You may view their pictures at

www.math.mun.ca/~apics/picture/shots.html

Send me your nice solutions!

**APICS 1999 Mathematics Competition**

1. Find the volume of the solid formed by one complete revolution about the $x$-axis of the area in common to the circles with equations $x^2 + y^2 - 4y + 3 = 0$ and $x^2 + y^2 = 3$.

2. The Memorial University Philosophers' Jockey Club has just received the bronze busts of the ten members of their hall of fame. Each will be placed in its designated place on a single shelf, above the gold plaque bearing the name of the member. The ten busts are drawn at random from the crate. What is the probability that at no time will there be an empty space between two busts already placed on the shelf?

3. Prove that $\sin^2(x + \alpha) + \sin^2(x + \beta) - 2\cos(\alpha - \beta) \sin(x + \alpha) \sin(x + \beta)$ is a constant function of $x$.

4. In Scottish Dancing, there are three types of dances, two of which are fast rhythms, Jigs and Reels, and one is a slow rhythm, Strathspey.

   A Scottish Dance program always starts with a Jig. The following dances are selected (by type) according to the following rules:

   (i) the next dance is always of a different type from the previous one,

   (ii) no more than two fast dances can be consecutive.

   Find how many different arrangements of Jigs, Reels and Strathspeys are possible in a Scottish Dance list which has (a) seven dances, (b) fifteen dances.

5. Find all differentiable functions $f(x)$ which satisfy the integral equation

$$
(f(x))^{2000} = \int_{1}^{x} (f(t))^{1999} \, dt.
$$
6. Inside a square of side $r$, four quarter circles are drawn, with radius $r$ and centres at the corners of the square.

Find the area of the shaded region.

7. Pat has a method for solving quadratic equations. For example, Pat solves $6x^2 + x - 2 = 0$ as follows:

Step 1. Pat multiplies the leading coefficient by the constant coefficient and solved the simpler equation $x^2 + x - 12 = 0$ to get $(x + 4)(x - 3) = 0$.

Step 2. Pat then replaces each $x$ by $6x$ ($x$ times the leading coefficient) to get $(6x + 4)(6x - 3) = 0$.

Step 3. Pat then simplifies this equation to get $(3x + 2)(2x - 1) = 0$, which solves the original equation.

Prove or disprove that Pat’s method always works.

8. An arbelos consists of three semicircular arcs as shown:

A circle is placed inside the arbelos so that it is tangent to all three semicircles.

Suppose that the radii of the two smaller semicircles are $a$ and $b$, and that the radius of the circle is $r$.

Assuming that $a > b > r$ and that $a$, $b$ and $r$ are in arithmetic progression, calculate $a/b$. 