SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2329*. [1998: 176, 301] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that p and $t > 0$ are real numbers. Define

$$\lambda_p(t) := t^p + t^{-p} + 2^p \quad \text{and} \quad \kappa_p(t) := \left(t + t^{-1}\right)^p + 2\,.$$

(a) Show that $\lambda_p(t) \le \kappa_p(t)$ for $p \ge 2$.

(b) Determine the sets of p: A, B and C, such that

1. $\lambda_p(t) \le \kappa_p(t)$,
2. $\lambda_p(t) = \kappa_p(t)$,
3. $\lambda_p(t) \ge \kappa_p(t)$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

(a) This part of the problem is included in (b), so we pass on to the latter.

(b) We show that

$$A := \{p|\lambda_p(t) \le \kappa_p(t) \quad \text{for all } t > 0\} \;=\; [0,1] \cup [2,\infty)\,,$$

$$B := \{p|\lambda_p(t) = \kappa_p(t) \quad \text{for all } t > 0\} \;=\; \{0,\,1,\,2\}\,,$$

and

$$C := \{p|\lambda_p(t) \ge \kappa_p(t) \quad \text{for all } t > 0\} \;=\; (-\infty,0] \cup [1,2]\,.$$

The case for B will be settled once we observe $B = A \cap C$, so we determine A and C.

We shall make repeated use of the inequalities: for any $x > 0$,

$$x^q - 1 \;\ge\; q(x-1) \qquad \text{if } q \le 0 \;\text{ or }\; q \ge 1\,, \tag{1}$$
$$x^q - 1 \;\le\; q(x-1) \qquad \text{if } 0 \le q \le 1 \tag{2}$$

(see for example Hardy, Littlewood and Pólya, *Inequalities*, Theorem 42, page 40).

Set $f_p(t) = \kappa_p(t) - \lambda_p(t)$. First suppose $p \ge 2$. For $0 < t < 1$, and by using (1) twice with $q = p - 1 \ge 1$ [and with first $x = t^2 + 1$ and then

$x = t^2$], we have (since $1 - t^2 \geq 0$)

$$\begin{aligned}
\frac{df_p(t)}{dt} &= p(t+t^{-1})^{p-1}\left(1-\frac{1}{t^2}\right) - pt^{p-1} + pt^{-p-1} \\
&= \frac{-p}{t^{p+1}}\left[(1-t^2)(t^2+1)^{p-1} + t^2(t^2)^{p-1} - 1\right] \\
&\leq \frac{-p}{t^{p+1}}\left[(1-t^2)\Big(1+(p-1)t^2\Big) + t^2\Big(1+(p-1)(t^2-1)\Big) - 1\right] \\
&= 0\,. \qquad (3)
\end{aligned}$$

Thus $f_p(t)$ is decreasing in $(0,1]$. Note that $f_p(1/t) = f_p(t)$ so $f_p(t)$ is increasing in $[1,\infty)$, so $t = 1$ gives an absolute minimum. Thus $f_p(t) \geq f_p(1) = 0$ for all $t > 0$, showing that $[2,\infty) \subseteq A$.

If now $1 \leq p \leq 2$ then $q = p - 1 \in [0,1]$, so we use (2), showing that inequality (3) is reversed and hence that $[1,2] \subseteq C$. If $0 \leq p \leq 1$ then $q = p - 1 \leq 0$ so we use (1) again to show $[0,1] \subseteq A$. Finally, if $p \leq 0$ then $q = p - 1 \leq 0$ and $-p \geq 0$, so by use of (1) we get $df_p(t)/dt \geq 0$ instead of (3), so $(-\infty, 0] \subseteq C$, completing the proof.

Remark. It is easy to see that for $0 < p < 1$ and $1 < p < 2$ the limit $\lim_{t\to\infty} f_p(t)$ exists and equals $2 - 2^p$. Thus by the monotonicity of f_p shown above we have the best possible inequalities

$$\begin{aligned}
2 - 2^p \;<\; \kappa_p(t) - \lambda_p(t) \;\leq\; 0 \qquad &\text{for} \quad 1 < p < 2\,, \\
0 \;\leq\; \kappa_p(t) - \lambda_p(t) \;<\; 2 - 2^p \qquad &\text{for} \quad 0 < p < 1\,.
\end{aligned}$$

For the rest of p it is easy to see that $\lim_{t\to\infty} f_p(t) = \pm\infty$ so no corresponding inequality exists.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong, China; and the proposer. One other reader sent in a counterexample to the incorrect version of this problem published on [1998: 176].

2337. [1998: 177] *Proposed by Iliya Bluskov, Simon Fraser University, Burnaby, BC.*

Let $F(1) = \left\lceil \dfrac{n^2+2n+2}{n^2+n+1} \right\rceil$, and, for each $i > 1$, let

$$F(i) = \left\lceil \frac{n^2+2n+i+1}{n^2+n+i}\, F(i-1) \right\rceil.$$

Find $F(n)$.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

If the given n is $n = 1$, then $F(n) = F(1) = \left\lceil \dfrac{1^2+2+2}{1^2+1+1} \right\rceil = \left\lceil \dfrac{5}{3} \right\rceil = 2$.

Let us then do the more interesting case when the given n is ≥ 2.

We show that for each i with $1 \le i \le n$ we have $F(i) = i + 1$, so that in particular $F(n) = n + 1$. We use induction on i.

For $i = 1$ we have

$$F(i) = F(1) = \left\lceil \frac{n^2 + 2n + 2}{n^2 + n + 1} \right\rceil = \left\lceil 1 + \frac{n+1}{n^2+n+1} \right\rceil .$$

Thus

$$1 < F(1) = \left\lceil 1 + \frac{n+1}{n^2+n+1} \right\rceil \le \left\lceil 1 + \frac{n+1}{n^2+n} \right\rceil = \left\lceil 1 + \frac{1}{n} \right\rceil = 2 ,$$

from which we see that $F(1) = 2$, as required.

If we assume validity for $i = m - 1$ (that is, $F(m-1) = m$, where $2 \le m \le n$) then

$$\begin{aligned} F(m) &= \left\lceil \frac{n^2 + 2n + m + 1}{n^2 + n + m} \cdot m \right\rceil = \left\lceil m + \frac{(n+1)m}{n^2+n+m} \right\rceil \\ &\le \left\lceil m + \frac{(n+1)m}{n^2+n} \right\rceil = \left\lceil m + \frac{m}{n} \right\rceil \le \lceil m + 1 \rceil = m + 1 , \end{aligned}$$

but as $\dfrac{(n+1)m}{n^2+n+m} \ge 0$, we have $m < \left\lceil m + \dfrac{(n+1)m}{n^2+n+m} \right\rceil$.

Thus we have $m < F(m) \le m + 1$, showing that $F(m) = m + 1$, completing the induction step and proving the claim.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

2338. [1998: 234] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose $ABCD$ is a convex cyclic quadrilateral, and P is the intersection of the diagonals AC and BD. Let I_1, I_2, I_3 and I_4 be the incentres of triangles PAB, PBC, PCD and PDA respectively. Suppose that I_1, I_2, I_3 and I_4 are concyclic.

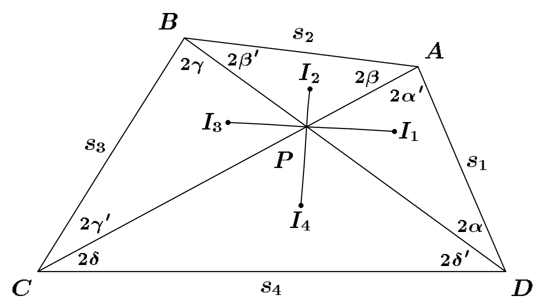
Prove that $ABCD$ has an incircle.

Solution by Peter Y. Woo, Biola University, La Miranda, California, USA.

$ABCD$ does not have to be cyclic. More precisely,

When a convex quadrilateral is subdivided into 4 triangles by its two diagonals, then the incentres of the 4 triangles are concyclic if and only if the quadrilateral has an incircle.

Notation. Let P be the point where the diagonals intersect and let the triangles be T_1, T_2, T_3, T_4 (labelled counterclockwise as in the figure), with the respective incentres I_1, I_2, I_3, I_4. Denote the 8 angles formed by the diagonals with the four sides by 2α, $2\alpha'$, 2β, $2\beta'$, 2γ, $2\gamma'$, 2δ, $2\delta'$ (counterclockwise with 2α, $2\alpha'$ in T_1, etc.).



Step 1. In the usual notation (used only here in step 1) for $\triangle ABC$ with sides a, b, c, incentre I, and semiperimeter $s = \frac{a+b+c}{2}$, AI satisfies

$$AI^2 \;=\; bc \tan\frac{B}{2}\tan\frac{C}{2}\,.$$

The proof follows from familiar formulas. In E.W. Hobson's *Treatise on Plane and Advanced Trigonometry*, for example, in section 123 the author shows that

$$AI \;=\; \frac{r}{\sin\frac{A}{2}}\,,\quad \tan\frac{B}{2} \;=\; \frac{r}{s-b}\,,\quad \tan\frac{C}{2} \;=\; \frac{r}{s-c}\,,$$

$$\text{and}\quad \sin^2\frac{A}{2} \;=\; \frac{(s-b)(s-c)}{bc}\,,$$

which, when combined, proves the claim.

Step 2. I_1, I_2, I_3, I_4 are concyclic if and only if

$$\tan\alpha\tan\alpha'\tan\gamma\tan\gamma' \;=\; \tan\beta\tan\beta'\tan\delta\tan\delta'\,.$$

Proof. Since PI_1 and PI_3 bisect vertically opposite angles, as do PI_2 and PI_4, the Intersecting Chords Theorem says that I_1, I_2, I_3, I_4 are concyclic if and only if $PI_1 \cdot PI_3 = PI_2 \cdot PI_4$. The desired equality then follows from step 1.

Step 3. A quadrilateral has an incircle if and only if the sum of one pair of opposite sides equals the sum of the other. This is a standard result of elementary geometry. See, for example, Nathan Altshiller Court's *College Geometry*, p. 135.

Step 4. Let $ABCD$ be any quadrilateral (that is, any four points, no three collinear), *I, I' be incentres of $\triangle ABC$ and $\triangle ADC$, and IN, $I'N'$ be* *perpendiculars to the diagonal AC from I and I'. Then $CN \geq CN'$ if and only if $AD + BC \geq AB + CD$, with equality for both or for neither.*

Proof. $CB - AB = CN - AN$ [because $CN = s - c$ and $AN = s - a$ in the notation of step 1] and $AD - CD = AN' - CN'$. Add these two equalities [noting that $AN = AC - CN$ and $AN' = AC - CN'$].

Step 5. $AD + BC \geq AB + CD$ if and only if

$$\tan\angle BAI \tan\angle DCI' \;\geq\; \tan\angle BCI \tan\angle DAI' ,$$

with equality for both or for neither.

Proof. This follows from step 4 since we have $\tan\angle BAI = \frac{IN}{AC-CN}$, $\tan\angle DCI' = \frac{I'N'}{CN'}$, $\tan\angle BCI = \frac{IN}{CN}$, and $\tan\angle DAI' = \frac{I'N'}{AC-CN'}$. [Note that the incentres here generally do not coincide with those of the main result. The key observation is that IA (for example) bisects the angle between a diagonal and side of the quadrilateral, while the angles α, α', etc. in step 2 each are equal to half the angle between a diagonal and side.]

Proof of the main result. If $ABCD$ has an incircle then $AD + BC = AB + CD$ (step 3), so that $\tan\alpha\tan\gamma = \tan\beta'\tan\delta'$ and $\tan\alpha'\tan\gamma' = \tan\beta\tan\delta$ (step 5). By step 2, I_1, I_2, I_3, I_4 are therefore concyclic. On the other hand, if $ABCD$ does not circumscribe some circle, then let s_i be the side of T_i opposite P (for $i = 1, 2, 3, 4$). Without loss of generality, assume $s_2 + s_4 > s_1 + s_3$. Then by step 5, $\tan\alpha\tan\gamma > \tan\beta'\tan\delta'$ and $\tan\alpha'\tan\gamma' > \tan\beta\tan\delta$, so that by step 2, I_1, I_2, I_3, I_4 are not concyclic.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VJECKOSLAV KOVAČ, student, Univ. Zagreb, Croatia; D.J. SMEENK, Zaltbommel, the Netherlands; JEREMY YOUNG, student, Nottingham, England; and the proposer.

All solvers except Woo and Janous proved the theorem as stated (with $ABCD$ cyclic). Janous remembers having seen the stronger version before, but he did not recall the reference. Can any reader provide a reference?

2340. [1998: 235] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\lambda > 0$ be a real number and a, b, c be the sides of a triangle. Prove that

$$\prod_{\text{cyclic}} \frac{s + \lambda a}{s - a} \geq (2\lambda + 3)^3.$$

[As usual s denotes the semiperimeter.]

I. *Solution by Jeremy Young, student, Nottingham High School, Nottingham, England.*

By the common substitution that a, b, c are the sides of a triangle if and only if $a = y + z$, $b = z + x$, $c = x + y$ for x, y, $z > 0$, the inequality is equivalent to

$$\frac{1}{(2\lambda+3)^3} \prod_{\text{cyclic}} \Big(x + (\lambda+1)y + (\lambda+1)z\Big) \;\geq\; xyz\,. \qquad (1)$$

Now by the weighted AM–GM inequality,

$$\left(\frac{1}{2\lambda+3}\right)x + \left(\frac{\lambda+1}{2\lambda+3}\right)y + \left(\frac{\lambda+1}{2\lambda+3}\right)z \;\geq\; x^{\frac{1}{2\lambda+3}} y^{\frac{\lambda+1}{2\lambda+3}} z^{\frac{\lambda+1}{2\lambda+3}}\,,$$

$$\left(\frac{\lambda+1}{2\lambda+3}\right)x + \left(\frac{1}{2\lambda+3}\right)y + \left(\frac{\lambda+1}{2\lambda+3}\right)z \;\geq\; x^{\frac{\lambda+1}{2\lambda+3}} y^{\frac{1}{2\lambda+3}} z^{\frac{\lambda+1}{2\lambda+3}}\,,$$

and

$$\left(\frac{\lambda+1}{2\lambda+3}\right)x + \left(\frac{\lambda+1}{2\lambda+3}\right)y + \left(\frac{1}{2\lambda+3}\right)z \;\geq\; x^{\frac{\lambda+1}{2\lambda+3}} y^{\frac{\lambda+1}{2\lambda+3}} z^{\frac{1}{2\lambda+3}}\,.$$

Multiplying these three lines together gives (1), the required result.

II. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

Set

$$f(\lambda) \;=\; s(s+\lambda a)(s+\lambda b)(s+\lambda c) - (2\lambda+3)^3\Delta^2$$

where $\lambda \geq 0$ and Δ is the area of the triangle. Then

$$\begin{aligned} f'(\lambda) \;&=\; s \sum_{\text{cyclic}} a(s+\lambda b)(s+\lambda c) - 6(2\lambda+3)^2\Delta^2 \\ &=\; s\left(s^2(a+b+c) + 2\lambda s(ab+bc+ca) + 3\lambda^2 abc\right) - 6(2\lambda+3)^2\Delta^2. \end{aligned}$$

Using $a+b+c = 2s$ and

$$s^2 \geq 3\Delta\sqrt{3}\,, \qquad ab+bc+ca \geq 4\Delta\sqrt{3}\,, \qquad 9abc \geq 8s\Delta\sqrt{3}$$

(see for example Bottema et al, *Geometric Inequalities*, items 4.2, 4.5, 4.13) we have

$$\begin{aligned} f'(\lambda) \;&\geq\; \frac{2}{3}s^2\Delta\sqrt{3}(9+12\lambda+4\lambda^2) - 6(2\lambda+3)^2\Delta^2 \\ &\geq\; 6\Delta^2(2\lambda+3)^2 - 6(2\lambda+3)^2\Delta^2 \;=\; 0\,. \end{aligned}$$

Thus, f is increasing in $[0, \infty)$, and so

$$f(\lambda) \;\geq\; f(0) \;=\; s^4 - 27\Delta^2 \;\geq\; 0\,.$$

Hence by Heron's formula,

$$f(\lambda) \;=\; s(s+\lambda a)(s+\lambda b)(s+\lambda c)-(2\lambda+3)^3 s(s-a)(s-b)(s-c) \;\geq\; 0\,,$$

which reduces to the given inequality.

Also solved by ED BARBEAU, University of Toronto, Toronto, Ontario; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; KEE-WAI LAU, Hong Kong, China; JESSIE LEI, student, Vincent Massey Secondary School, Windsor, Ontario; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Herzig's solution is the same as Young's. Herzig also points out that the inequality is true for all $\lambda \geq -1$, and that equality holds if and only if $\lambda = -1$ or $x = y = z$; that is, $a = b = c$.

2341. [1998: 235] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle. For real $\lambda > 0$, put

$$s(\lambda) := \left|\sum_{\text{cyclic}}\left[\left(\frac{a}{b}\right)^{\lambda}-\left(\frac{b}{a}\right)^{\lambda}\right]\right|$$

and let $\Delta(\lambda)$ be the supremum of $s(\lambda)$ over all triangles.

1. Show that $\Delta(\lambda)$ is finite if $\lambda \in (0,1]$ and $\Delta(\lambda)$ is infinite for $\lambda > 1$.
2.★ What is the exact value of $\Delta(\lambda)$ for $\lambda \in (0,1)$?

I. *Solution to part* 1 *by Florian Herzig, student, Cambridge, UK.*

Note that

$$s(\lambda) \;=\; \left|\frac{(c^{\lambda}-b^{\lambda})(c^{\lambda}-a^{\lambda})(b^{\lambda}-a^{\lambda})}{(abc)^{\lambda}}\right|\,.$$

First, I show that, for positive reals a and b, the inequality $a^x + b^x \geq (a+b)^x$ holds for all $0 \leq x \leq 1$. Without loss of generality, assume that $a + b = 1$. Then $0 < a, b < 1$ and hence $a^x \geq a$ and $b^x \geq b$. Therefore $a^x + b^x \geq a + b = (a+b)^x$ as claimed.

Suppose $a \leq b \leq c$ without loss of generality. By the above and the triangle inequality, $a^x + b^x \geq (a+b)^x > c^x$. Hence $a^x > c^x - b^x \geq 0$ and $b^x > c^x - a^x \geq 0$. Trivially also $c^x > b^x - a^x \geq 0$. Thus for all $0 < \lambda \leq 1$,

$$s(\lambda) \;\leq\; \frac{a^{\lambda}b^{\lambda}c^{\lambda}}{(abc)^{\lambda}} \;=\; 1\,,$$

and so $\Delta(\lambda) \leq 1$.

If $\lambda > 1$ then consider the triangle with sides $1 + \frac{1}{n}$, $n + \frac{1}{n}$, $n + 1$ for a positive integer n. Then in the expression for $s(\lambda)$ [that is,

$$s(\lambda) \;=\; \left| \left(\frac{n+1}{n^2+1}\right)^{\lambda} + \left(\frac{n^2+1}{n^2+n}\right)^{\lambda} + n^{\lambda} - \left(\frac{n^2+1}{n+1}\right)^{\lambda} - \left(\frac{n^2+n}{n^2+1}\right)^{\lambda} - \left(\frac{1}{n}\right)^{\lambda} \right|$$

— *Ed.*] two terms tend to 0 and two tend to 1 as $n \to \infty$. Consider the remaining two terms:

$$n^{\lambda} - \left(\frac{n^2+1}{n+1}\right)^{\lambda} \;\geq\; n^{\lambda} - \left(n - \frac{1}{2}\right)^{\lambda}. \qquad (1)$$

where the inequality holds for all $n \geq 3$. Since

$$n^{\lambda} \;=\; \left(n - \frac{1}{2} + \frac{1}{2}\right)^{\lambda} \geq \left(n - \frac{1}{2}\right)^{\lambda} + \frac{\lambda}{2}\left(n - \frac{1}{2}\right)^{\lambda-1}$$

[by the Binomial Theorem], the left-hand side of (1) and hence $s(\lambda)$ tends to infinity as $n \to \infty$. It follows that $\Delta(\lambda)$ is infinite.

II. *Partial solution to part 2 by Nikolaos Dergiades, Thessaloniki, Greece (with editorial comments).*

All readers who attempted part 2 agreed that there will be no explicit formula for $\Delta(\lambda)$ for arbitrary $0 < \lambda < 1$. They all gave $\lambda = 1/2$ as a special case (and so did the proposer, in fact), and further agreed that in this case $\Delta(1/2) \approx 0.0740033$.

Dergiades, however, managed to find the exact value of $\Delta(1/2)$. First he puts $a \leq b \leq c$ without loss of generality, in which case (as in Solution I)

$$\begin{aligned} s(\lambda) \;&=\; \frac{(c^{\lambda} - b^{\lambda})(c^{\lambda} - a^{\lambda})(b^{\lambda} - a^{\lambda})}{(abc)^{\lambda}} \\ &=\; \left(\frac{a}{b}\right)^{\lambda} + \left(\frac{b}{c}\right)^{\lambda} + \left(\frac{c}{a}\right)^{\lambda} - \left(\frac{b}{a}\right)^{\lambda} - \left(\frac{c}{b}\right)^{\lambda} - \left(\frac{a}{c}\right)^{\lambda}. \end{aligned}$$

Considering a, b and λ as constant and calling this function $F(c)$, he gets

$$F'(c) \;=\; \frac{\lambda\left[c^{2\lambda} - (ab)^{\lambda}\right]\left(b^{\lambda} - a^{\lambda}\right)}{c(abc)^{\lambda}} \;\geq\; 0\,,$$

so F is increasing. Thus to maximize s he puts $c = a + b$, $b = x$ and without loss of generality $a = 1$, which for $\lambda = 1/2$ means that s can be rewritten as the function

$$f(x) \;=\; \frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{1+x}} + \sqrt{1+x} - \sqrt{x} - \frac{\sqrt{1+x}}{\sqrt{x}} - \frac{1}{\sqrt{1+x}}\,,$$

which now must be maximized over $x \geq 1$. Setting the derivative of this to zero results in a sixth degree polynomial equation

$$x^6 - 4x^5 - 32x^4 - 58x^3 - 32x^2 - 4x + 1 \;=\; 0\,,$$

whose root $\rho \approx 8.57318922$ can be substituted into $f(x)$ to obtain the above maximum value of $\Delta(1/2)$. This is where the other solvers stopped. However, Dergiades then transforms the equation for $f(x)$ by putting

$$x \;=\; \tan^2 y \qquad \text{and} \qquad \sin 2y \;=\; z\,; \qquad \text{that is,} \quad z \;=\; \frac{2\sqrt{x}}{1+x}\,,$$

getting the function

$$h(z) \;=\; \frac{\sqrt{1-z}\cdot(2+z-2\sqrt{1+z})}{z}\,.$$

Solving $h'(z) = 0$ gives a **cubic** equation

$$z^3 - 3z^2 + 8z - 4 \;=\; 0$$

which can be solved. He gets the real root

$$r \;=\; 1 - \left(1 + \frac{2}{9}\sqrt{114}\right)^{1/3} + \frac{5}{3}\left(1 + \frac{2}{9}\sqrt{114}\right)^{-1/3}\,,$$

so the exact value of $\Delta(1/2)$ is $h(r)$ (which, by my calculator, is indeed the previously found 0.0740033).

Part 1 also solved by VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. The approximate value of $\Delta(1/2)$ was found by Konečný, Lambrou and the proposer.

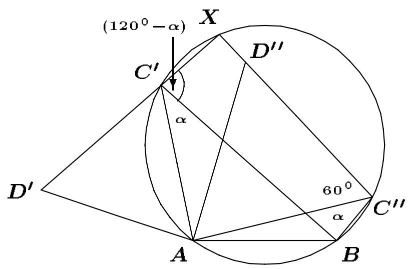
2342. [1998: 235] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given A and B are fixed points of circle Γ. The point C moves on Γ, on one side of AB. D and E are points outside $\triangle ABC$ such that $\triangle ACD$ and $\triangle BCE$ are both equilateral.

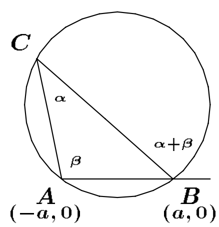
(a) Show that CD and CE each pass through a fixed point of Γ when C moves on Γ.

(b) Determine the locus of the mid-point of DE.

Solution by Jeremy Young, student, Nottingham High School, Nottingham, England.



Let X be the point subtending angles $120°$ and $60°$ with A. By "angles in the same segment", CD always passes through X. Two possible positions of C are shown: C' and C''. Similarly, two corresponding positions of D are shown: D' and D''.



Similarly, define Y relative to B.

Introduce a Cartesian coordinate system with A, B as $(-a, 0)$, $(a, 0)$ respectively. Let C be a point with positive y–coordinate such that $\angle ACB = \alpha$.

By the Sine Rule, we have $AC \;=\; 2a\dfrac{\sin(\alpha+\beta)}{\sin\alpha}$ and $BC \;=\; 2a\dfrac{\sin\beta}{\sin\alpha}$.

Therefore D has position vector

$$\begin{pmatrix} -a \\ 0 \end{pmatrix} + 2a\frac{\sin(\alpha+\beta)}{\sin\alpha}\begin{pmatrix} \cos(\beta+60°) \\ \sin(\beta+60°) \end{pmatrix}.$$

Similarly, E has position vector

$$\begin{pmatrix} a \\ 0 \end{pmatrix} + 2a\frac{\sin\beta}{\sin\alpha}\begin{pmatrix} \cos(\alpha+\beta-60°) \\ \sin(\alpha+\beta-60°) \end{pmatrix}.$$

Hence, the mid-point has position vector

$$\begin{aligned} &\frac{a}{\sin\alpha}\begin{pmatrix} \sin(\alpha+\beta)\cos(\beta+60°)+\sin\beta\cos(\alpha+\beta-60°) \\ \sin(\alpha+\beta)\sin(\beta+60°)+\sin\beta\sin(\alpha+\beta-60°) \end{pmatrix} \\ =\;& \frac{a}{2\sin\alpha}\begin{pmatrix} \sin(\alpha+2\beta) \\ 2\cos(\alpha-60°)-\cos(\alpha+2\beta) \end{pmatrix} \\ =\;& \frac{a}{2\sin\alpha}\begin{pmatrix} \sin(\alpha+2\beta) \\ -\cos(a+2\beta) \end{pmatrix} + \frac{a}{\sin\alpha}\begin{pmatrix} 0 \\ \cos(\alpha-60°) \end{pmatrix}, \end{aligned}$$

where $0 \le \beta \le 18° - \alpha$. Thus the required locus is an arc of a circle, radius $R/2$ (where $R = a/2\sin\alpha$ is the circumradius of $\triangle ABC$) and centre $(0, 2R\cos(\alpha-60°))$, which is exterior to the equilateral triangle with AB as base.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, *UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.*

Most solvers used pure geometric methods. Bellot Rosado and López Chamorro solved the problem entirely with the use of complex numbers.

2343. [1998: 235] *Proposed by Doru Popescu Anastasiu, Liceul "Radu Greceanu", Slatina, Olt, Romania.*

For positive numbers sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$, $\{z_n\}_{n\geq 1}$ with conditions: for $n \geq 1$, we have

$$\begin{aligned}(n+1)x_n^2 + (n^2+1)y_n^2 + (n^2+n)z_n^2 \\ = \; 2\sqrt{n}\left(nx_ny_n + \sqrt{n}x_nz_n + y_nz_n\right) \,, \end{aligned} \qquad (1)$$

and for $n \geq 2$, we have

$$x_n + \sqrt{n}y_n - nz_n \;=\; x_{n-1} + y_{n-1} - \sqrt{n-1}z_{n-1} \,. \qquad (2)$$

Find $\lim\limits_{n\to\infty} x_n$, $\lim\limits_{n\to\infty} y_n$ and $\lim\limits_{n\to\infty} z_n$.

Solution by Christo Saragiotis, Thessaloniki, Greece.

Equation (1) can be re-written as

$$\left(\sqrt{n}x_n - ny_n\right)^2 + \left(x_n - nz_n\right)^2 + \left(y_n - \sqrt{n}z_n\right)^2 \;=\; 0 \,.$$

Thus, $y_n = \dfrac{x_n}{\sqrt{n}}$, $z_n = \dfrac{x_n}{n}$ and $y_n = \sqrt{n}z_n$ for all $n \geq 1$. Substituting these into equation (2) yields $x_n = x_{n-1}$ for all $n \geq 2$.

Therefore, $\lim\limits_{n\to\infty} x_n = x_1$ and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = 0$.

Also solved by THEODORE CHRONIS, Athens, Greece; OSCAR CIAURRI, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VJEKOSLAV KOVAČ, student, University of Zagreb, Croatia; MICHAEL LAMBROU, University of Crete, Crete, Greece; LAURENT LESSARD, student, Le Collège français, Toronto, Ontario; GERRY LEVERSHA, St. Paul's School, London, England; and the proposer. There were two partially incorrect solutions.

The solutions of almost all the solvers were virtually identical to the one given above.

2344. [1998: 235] *Proposed by Murali Vajapeyam, student, Campina Grande, Brazil and Florian Herzig, student, Perchtoldsdorf, Austria.*

Find all positive integers N that are quadratic residues modulo all primes greater than N.

Solution by Mansur Boase, student, Cambridge, England.

Clearly any square is a quadratic residue modulo any prime. Suppose N is not a square. Then we can write N as $m^2 p_1 p_2 \cdots p_r$ where $p_i \neq p_j$ for any pair (i,j) with $i \neq j$, and where r is some positive integer. Without loss of generality let us assume $p_1 < p_2 < \cdots < p_r$. We shall show that N cannot be a perfect square modulo all primes congruent to $1 \pmod 4$. Let q be a prime congruent to $1 \pmod 4$. Then, introducing Legendre symbols, we have

$$\begin{aligned} \left(\frac{N}{q}\right) = \left(\frac{m^2 p_1 p_2 \cdots p_r}{q}\right) &= \left(\frac{m^2}{q}\right)\left(\frac{p_1}{q}\right)\left(\frac{p_2}{q}\right)\cdots\left(\frac{p_r}{q}\right) \\ &= \left(\frac{p_1}{q}\right)\left(\frac{p_2}{q}\right)\cdots\left(\frac{p_r}{q}\right) \\ &= \left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right)\cdots\left(\frac{q}{p_r}\right). \end{aligned}$$

The latter equality follows from the Law of Quadratic Reciprocity since $q \equiv 1 \pmod 4$.

Suppose $p_1 = 2$. Then if $q \equiv 1 + 4p_2p_3 \cdots p_r \pmod{8p_2p_3 \cdots p_r}$, we have $q \equiv 1 \pmod{p_i}$ for $2 \leq i \leq r$, whence

$$\left(\frac{q}{p_i}\right) = \left(\frac{1}{p_i}\right) = 1 \qquad \text{for } 2 \leq i \leq r.$$

Also, since all the p_2, p_3, ..., p_r are odd, $q \equiv 5 \pmod 8$, and therefore, $\left(\frac{2}{q}\right) = (-1)^{(q^2-1)/8} = -1$. Thus $\left(\frac{N}{q}\right) = -1$, which would be a contradiction. It thus remains to show that a prime greater than N exists satisfying $q \equiv 1 + 4p_2p_3 \cdots p_r \pmod{8p_2p_3 \cdots p_r}$. But by Dirichlet's Theorem there are infinitely many such primes. Thus we cannot have $p_1 = 2$, and all the primes are odd.

There are $(p_i - 1)/2$ quadratic residues modulo the prime p_i, $(p_i \geq 3)$ and this is a positive integer. Suppose n is a quadratic non-residue modulo p_1. Then by the Chinese Remainder Theorem, there exists a solution modulo $4p_1p_2 \cdots p_r$ to the set of congruences:

$$x \equiv 1 \pmod 4, \quad x \equiv n \pmod{p_1}, \quad x \equiv 1 \pmod{p_i} \quad \text{for } 2 \leq i \leq r.$$

as $(2, p_i) = 1$ for all i, $1 \leq i \leq r$, and $(p_i, p_j) = 1$ for all $i \neq j$. Suppose the prime q satisfies $q \equiv x \pmod{4p_1p_2 \cdots p_r}$ and hence q satisfies the set of congruences above. Then

$$\begin{aligned} \left(\frac{N}{q}\right) &= \left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right)\cdots\left(\frac{q}{p_r}\right) \\ &= \left(\frac{n}{p_1}\right)\left(\frac{1}{p_2}\right)\cdots\left(\frac{1}{p_r}\right) \\ &= -1, \end{aligned}$$

which would be a contradiction. Thus we must prove that there can be no primes greater than N congruent to $x \pmod{4p_1p_2\cdots p_r}$, as all such numbers satisfy the requirement of being congruent to $1 \pmod 4$. But by Dirichlet's Theorem, as $(x, 4p_1p_2\cdots p_r) = 1$ there must be infinitely many primes in this arithmetic progression. [$n \neq 0$, so the set of congruences above shows that x is relatively prime to each of 4, p_1, p_2, $\ldots$, p_r, and hence also to the product.]

Thus N can satisfy the conditions of the problem only if N is a perfect square.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; and the proposer. There was one incorrect solution submitted.

Lau comments that Theorem 3 of Chapter 5 of Ireland and Rosen's "A Classical Introduction to Modern Number Theory" (Springer-Verlag, 1982) proves that if N is a non-square integer, then there are infinitely many primes p for which N is a quadratic nonresidue. Combining this with the observation that the squares are quadratic residues modulo all primes greater than them solves the problem.

2345. [1998: 236] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Suppose that $x > 1$.

(a) Show that $\ln(x) > \dfrac{3(x^2-1)}{x^2+4x+1}$.

(b) Show that $\dfrac{a-b}{\ln(a)-\ln(b)} < \dfrac{1}{3}\left(2\sqrt{ab} + \dfrac{a+b}{2}\right)$,

where $a > 0$, $b > 0$ and $a \neq b$.

Solution by Kee-Wai Lau, Hong Kong, China.

(a) For $x \geq 1$, let $f(x) = \ln(x) - \dfrac{3(x^2-1)}{x^2+4x+1}$. Then straightforward computations show that $f'(x) = \dfrac{(x-1)^4}{x\left(x^2+4x+1\right)^2}$, and so $f'(x) > 0$ for all $x > 1$. Since $f(1) = 0$, we have $f(x) > 0$ for $x > 1$, and the inequality follows.

(b) Clearly, we can assume that $a > b$. Then the desired inequality follows readily by substituting $x = \sqrt{a/b}$ into the inequality in (a).

Also solved by THEODORE CHRONIS, Athens, Greece; OSCAR CIAURRI; Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD B. EDEN, student, Ateneo de Manila University, Quezon City, Philippines; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; LAURENT LESSARD, student, Le Collège Français, Toronto, Ontario; GERRY LEVERSHA, St. Paul's School, London, England; PHILIP McCARTNEY, Northern Kentucky University, Highland Heights, Kentucky, USA; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta;

PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, England; and the proposer. There was one incorrect solution.

Solutions virtually identical to the one given above were submitted by about half of the solvers.

Herzig remarked that the right hand side of the inequality in (a) *is a Padé approximation to* $\ln x$ *at* $x = 1$. *Janous commented that this inequality is "quite sharp" in positive neighbourhoods of* $x = 1$. *Lambrou remarked that part* (b) *is a repetition of problem* **2206** [1997: 46]. *He and Konečný both pointed out that the inequality in* (a) *is reversed for* $0 < x < 1$. (Ed: *This is obvious from the solution given above.*) *Leversha considered some generalizations of* (b) *by setting* $x = \left(\frac{a}{b}\right)^{1/n}$ *in the inequality of* (a)*; for example, putting* $n = 4$ *would give*

$$\frac{a-b}{\ln a - \ln b} \;<\; \frac{1}{12}\left(\sqrt{a}+\sqrt{b}+4\sqrt[4]{ab}\right)\left(\sqrt{a}+\sqrt{b}\right),$$

which is sharper than the inequality in (b). *Although a few solvers commented that the inequality in* (b) *is known as* Pólya's Inequality, *Sieffert (who was the proposer of problem* **2206** *mentioned above) stated that it should really be called the* Pólya–Szegö Inequality, *since it first appeared in a joint paper by them in 1951.*

2347. [1998: 236] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove that the equation $x^2 + y^2 = z^{1998}$ has infinitely many solutions in positive integers, x, y and z.

Solution by Richard I. Hess, Rancho Palos Verdes, California, USA. Let k be any positive integer. Multiply both sides of the equality $3^2 + 4^2 = 5^2$ by $5^{1996}k^{1998}$. The result is

$$(3 \cdot 5^{998}k^{999})^2 + (4 \cdot 5^{998}k^{999})^2 \;=\; (5k)^{1998}.$$

Hence $(3 \cdot 5^{998}k^{999}, 4 \cdot 5^{998}k^{999}, 5k)$ is an infinite family of solutions to the given equation.

Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; NIELS BEJLEGAARD, Stavanger, Norway; M. BENITO and E. FERNÁNDEZ, Logroño, Spain (two solutions); MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, Connecticut, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA (two solutions); MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (three solutions); BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; CHRISTOS SARAGIOTIS, Thessaloniki, Greece; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; STAN WAGON, Macalester College, St. Paul, Minnesota, USA (two solutions); JEREMY YOUNG, student, Nottingham High School, Nottingham, England; and the proposer.

2349. [1998: 236] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that $\triangle ABC$ has acute angles such that $A < B < C$. Prove that

$$\sin^2 B \sin\left(\frac{A}{2}\right) \sin\left(A + \frac{B}{2}\right) \;>\; \sin^2 A \sin\left(\frac{B}{2}\right) \sin\left(B + \frac{A}{2}\right) .$$

Solution by Florian Herzig, student, Cambridge, UK.

Let AD, BE be the angle bisectors in the triangle. By the angle bisector property, $EC = \frac{ab}{a+c}$. Hence in triangle BEC,

$$\frac{\sin\frac{B}{2}}{\sin(A+\frac{B}{2})} \;=\; \frac{EC}{BC} \;=\; \frac{b}{a+c} .$$

A similar result holds in triangle ADC. Since also $\sin B : \sin A = b : a$, the given inequality is equivalent to $\frac{b^2}{a^2} > \frac{b+c}{a} \cdot \frac{b}{a+c}$, or $b(a+c) > a(b+c)$; that is, $b > a$, and this is, of course, true.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; JEREMY YOUNG, student, Nottingham High School, Nottingham, England; and the proposer.

Most of the solutions were along the same lines as the featured one, which was chosen for its brevity.

2350. [1998: 236] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that the centroid of $\triangle ABC$ is G, and that M and N are the mid-points of AC and AB respectively. Suppose that circles ANC and AMB meet at (A and) P, and that circle AMN meets AP again at T.

1. Determine $AT : AP$.

2. Prove that $\angle BAG = \angle CAT$.

Identical independent solutions by Florian Herzig, student, Cambridge, UK and by Vjekoslav Kovač, student, U. of Zagreb, Croatia.

Invert in a circle with centre A, and denote the image of a point X by X'. Then C' is the mid-point of AM' since $AC : AM = AM' : AC'$. Similarly B' is the mid-point of AN'. P' is the intersection of lines $M'B'$ and $N'C'$, so that it is the centroid of $\triangle AM'N'$. T' is then the point of intersection of line AP' and line $M'N'$ (that is, the mid-point of segment $M'N'$). By the property of medians,

$$AT : AP \;=\; AP' : AT' \;=\; 2:3\,.$$

Note that $\triangle ABC \sim \triangle ANM \sim \triangle AM'N'$ and hence

$$\angle CAT \;=\; \angle C'AT' \;=\; \angle M'AP' \;=\; \angle BAG$$

as we wanted to show.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.