PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 December 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

G.P. Henderson, Garden Hill, Campbellcroft, Ontario asks us to point out that problem **2405** is not the problem that he submitted. It is a modification made by the editors. It should be regarded as an unsolved problem proposed by the editors.

2439. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $ABCD$ is a square with side a. Let P and Q be points on sides BC and CD respectively, such that $\angle PAQ = 45^\circ$. Let E and F be the intersections of PQ with AB and AD respectively. Prove that $AE + AF \geq 2\sqrt{2}a$.

2440. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given: triangle ABC with $\angle BAC = 90^\circ$. The incircle of triangle ABC touches BC at D. Let E and F be the feet of the perpendiculars from D to AB and AC respectively. Let H be the foot of the perpendicular from A to BC.

Prove that the area of the rectangle $AEDF$ is equal to $\dfrac{AH^2}{2}$.

2441. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Suppose that D, E, F are the mid-points of the sides BC, CA, AB of $\triangle ABC$. The incircle of $\triangle AEF$ touches EF at X, the incircle of $\triangle BFD$ touches FD at Y, and the incircle of $\triangle CDE$ touches DE at Z.

Show that DX, EY, FZ are collinear. What is the intersection point?

2442. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let $\{a_n\}_1^\infty$, $\{x_n\}_1^\infty$, $\{y_n\}_1^\infty$, $\ldots$, $\{z_n\}_1^\infty$, be a finite number of given sequences of non-negative numbers, where all $a_n > 0$. Suppose that $\sum a_n$ is divergent and all the other infinite series, $\sum x_n$, $\sum y_n$, $\ldots$, $\sum z_n$, are convergent. Let $A_n = \sum_{k=1}^n a_k$.

(a) Show that, for every $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that, simultaneously,

$$0 \le \frac{A_n x_n}{a_n} < \epsilon, \quad 0 \le \frac{A_n y_n}{a_n} < \epsilon, \quad \ldots, \quad 0 \le \frac{A_n z_n}{a_n} < \epsilon.$$

(b) From part (a), it is clear that if $\lim\limits_{n\to\infty} \dfrac{A_n x_n}{a_n}$ exists, it must have value zero. Construct an example of sequences as above such that the stated limit does not exist.

2443. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Without the use of any calculating device, find an explicit example of an integer, M, such that $\sin(M) > \sin(33) (\approx 0.99991)$. (Of course, M and 33 are in radians.)

2444. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Determine $\lim\limits_{n\to\infty} \left(\dfrac{\ln(n!)}{n} - \dfrac{1}{n} \sum\limits_{k=1}^{n} \ln(k) \left(\sum\limits_{j=k}^{n} \dfrac{1}{j} \right) \right)$.

2445. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let A, B be a partition of the set $C = \{q \in \mathbb{Q} : 0 < q < 1\}$ (so that A, B are disjoint sets whose union is C).

Show that there exist sequences $\{a_n\}$, $\{b_n\}$ of elements of A and B respectively such that $(a_n - b_n) \to 0$ as $n \to \infty$.

2446. *Proposed by Catherine Shevlin, Wallsend upon Tyne, England.*

A sequence of integers, $\{a_n\}$ with $a_1 > 0$, is defined by

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } n \equiv 0 \pmod 4, \\ 3a_n + 1 & \text{if } n \equiv 1 \pmod 4, \\ 2a_n - 1 & \text{if } n \equiv 2 \pmod 4, \\ \frac{a_n+1}{4} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Prove that there is an integer m such that $a_m = 1$.

(Compare **OQ**.117 in *OCTOGON*, vol 5, No. 2, p. 108.)

2447. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect at P and Q. A variable line through P meets the circles again at A and B. Find the locus of the orthocentre of triangle ABQ.

2448. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Suppose that S is a circle, centre O, and P is a point outside S. The tangents from P to S meet the circle at A and B. Through any point Q on S, the line perpendicular to PQ intersects OA at T and OB at U. Prove that $OT \times OU = OP^2$.

2449. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Two circles intersect at D and E. They are tangent to the sides AB and AC of $\triangle ABC$ at B and C respectively.. If D is the mid-point of BC, prove that $DA \times DE = DC^2$.

2450. *Proposed by Gerry Leversha, St. Paul's School, London, England.*

Find the exact value of $\dfrac{\displaystyle\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^3}}{\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k!)^2}}$.