MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

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Shreds and Slices

A Combinatorial Proof

In Issue 2, this volume, we issued the following challenge to our audience.

Problem. Find a combinatorial proof of the following identity:

$$(n-r)\binom{n+r-1}{r}\binom{n}{r} = n\binom{n+r-1}{2r}\binom{2r}{r}.$$

Dave Arthur, of Toronto, Ontario, was the first, and so far only person to give a combinatorial proof. His prize is a copy of "Riddles of the Sphinx", by Martin Gardner; we print his solution here. The proposer still seeks a combinatorial proof that proves the identity in *one* step, and so an additional prize will be offered for such a solution.

Solution by Dave Arthur.

Lemma 1. Let us consider the number of ways of choosing two distinct sets, A and B, each with r elements from a set of $n+r-1$ elements.

There are $\binom{n+r-1}{2r}$ ways to choose the elements that will belong to their union, and there are $\binom{2r}{r}$ ways to choose the elements of these that will belong to A. Therefore, the number of ways is

$$\binom{n+r-1}{2r}\binom{2r}{r}.$$

However, we also note there are $\binom{n+r-1}{r}$ ways to choose A, and $\binom{n-1}{r}$ ways to choose B from the remaining elements. It follows that

$$\binom{n+r-1}{2r}\binom{2r}{r} = \binom{n+r-1}{r}\binom{n-1}{r}.$$

Lemma 2. Let us consider the number of ways of choosing two distinct sets, C and D, such that C has 1 element and D has r elements, from a set of n elements.

There are n ways to choose C first, and $\binom{n-1}{r}$ ways to choose D from the remaining elements, so the number of ways is $n\binom{n-1}{r}$. Also, there are $\binom{n}{r}$ ways to choose D first, and $n-r$ ways to choose C from the remaining elements, so it follows that

$$(n-r)\binom{n}{r} = n\binom{n-1}{r}.$$

Therefore, by lemmas 1 and 2,

$$n\binom{n+r-1}{2r}\binom{2r}{r} = n\binom{n+r-1}{r}\binom{n-1}{r} = (n-r)\binom{n}{r}\binom{n+r-1}{r}.$$

Erratum

On page 169, Issue 3 of this Volume, in problem 6 of the Qualifying Round of the 1990 Swedish Mathematical Olympiad, the dimensions of rectangle $ABCD$ are described as 3000 metres by 500 metres. This is a typo: the dimensions should be 300 metres by 500 metres. Thanks to Jim Totten for the correction.

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the last issue be submitted in time for issue 4 of 2000.

High School Solutions

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H228. Verify that the following three inequalities hold for positive reals x, y, and z:

(i) $x(x-y)(x-z)+y(y-x)(y-z)+z(z-x)(z-y) \geq 0$. (This is known as Schur's Inequality.)

(ii) $x^4+y^4+z^4+xyz(x+y+z) \geq 2(x^2y^2+y^2z^2+z^2x^2)$.

(iii) $9xyz+1 \geq 4(xy+yz+zx)$, where $x+y+z=1$.

(Can you derive an ingenious method that allows you to solve the problem without having to prove all three inequalities directly?)

Additional solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

(i) As indicated, this is the special case $n=1$ of Schur's Inequality:

$$x^n(x-y)(x-z)+y^n(y-z)(y-x)+z^n(z-x)(z-y) \;\geq\; 0$$

for n real. For a simple proof, let $x \geq y \geq z$ without loss of generality. Then for $n \geq 0$,

$$x^n(x-y)(x-z) \;\geq\; y^n(y-z)(x-y) \qquad \text{and} \qquad z^n(x-z)(y-z) \;\geq\; 0\,.$$

For $n \leq 0$,

$$z^n(x-z)(y-z) \;\geq\; y^n(y-z)(x-y) \qquad \text{and} \qquad x^n(x-y)(x-z) \;\geq\; 0\,.$$

There is equality if and only if $x=y=z$. It also follows that if n is an even integer, then x, y and z need not be positive.

(ii) Since as known

$$\begin{aligned} &2(y^2z^2+z^2x^2+x^2y^2)-(x^4+y^4+z^4) \\ &\quad = \;(x+y+z)(y+z-x)(z+x-y)(x+y-z) \end{aligned}$$

(related to Heron's formula for the area of a triangle), the inequality reduces to

$$xyz \;\geq\; (y+z-x)(z+x-y)(x+y-z)\,. \tag{1}$$

In terms of the elementary symmetric functions $T_1 = x+y+z$, $T_2 = yz+zx+xy$, and $T_3 = xyz$, (1) becomes $T_1^3+9T_3 \geq 4T_1T_2$, which is the same as (i).

(iii) In homogeneous form, the inequality is equivalent to

$$9xyz + (x+y+z)^3 \;\geq\; 4(x+y+z)(yz+zx+xy)\,,$$

which is the same as (1).

For a generalization of (1) to

$$\begin{aligned}&(ux+vy+wz)(vx+wy+uz)(wx+uy+vz)\\ &\quad\geq\;\; (y+z-x)(z+x-y)(x+y-z)\,,\end{aligned}$$

where $u+v+w=1$ and $0 \leq u,v,w \leq 1$, see [1].

Reference

1. Klamkin, M.S., *Inequalities for a triangle associated with n given triangles*, Publ. Electrotechn. Fak. Ser. Mat. Fiz. Univ. Beograd, No. 330 (1970), pp. 3–7.

H237. The letters of the word MATHEMATICAL are arranged at random. What is the probability that the resulting arrangement contains no adjacent A's?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

First, the total number of arrangements is

$$\frac{12!}{3!2!2!}\,.$$

Now, to count the number of arrangements with no adjacent A's, we first arrange the other 9 letters, for a total of

$$\frac{9!}{2!2!}$$

ways. For each such arrangement, we choose any 3 of the 10 "spaces" that are between consecutive letters, including those on the ends. This can be done in a total of $\binom{10}{3}$ ways.

Now, we insert the 3 A's into the 3 spaces that we have chosen. This corresponds to a unique arrangement with no adjacent A's. The total number is then

$$\frac{9!}{2!2!}\cdot\binom{10}{3}\,.$$

The required probability is then equal to

$$\frac{\frac{9!}{2!2!}\cdot\binom{10}{3}}{\frac{12!}{3!2!2!}} \;=\; \frac{6}{11}\,.$$

H238. Johnny is dazed and confused. Starting at $A(0,0)$ in the Cartesian grid, he moves 1 unit to the right, then r units up, r^2 units left, r^3 units down, r^4 units right, r^5 units up, and continues the same pattern indefinitely. If r is a positive number less than 1, he will be approaching a point $B(x,y)$. Show that the length of the line segment AB is greater than $\frac{7}{10}$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $P_n = (x_n, y_n)$ denote the point where Johnny is at after n moves, $n = 0, 1, 2, \ldots$. So, $P_0 = A = (0,0)$, $P_1 = (1,0)$, $P_2 = (1,r)$, $P_3 = (1-r^2, r)$, etc. A simple pattern reveals itself, such that for all $m \geq 2$,

$$\begin{aligned} x_{2m-1} = x_{2m} &= 1 - r^2 + r^4 - \cdots + (-1)^{m-1} r^{2m-2}\,, \\ y_{2m} = y_{2m+1} &= r - r^3 + r^5 - \cdots + (-1)^{m-1} r^{2m-1}\,. \end{aligned}$$

Hence,

$$\begin{aligned} x &= \frac{1}{1-(-r^2)} = \frac{1}{1+r^2}\,, \\ y &= \frac{r}{1-(-r^2)} = \frac{r}{1+r^2}\,. \end{aligned}$$

So,

$$AB^2 = x^2 + y^2 = \frac{1}{1+r^2}\,.$$

And since $r < 1$,

$$AB = \frac{1}{\sqrt{1+r^2}} > \frac{1}{\sqrt{2}} = \frac{5\sqrt{2}}{10} > \frac{7}{10}\,.$$

H239. Find all pairs of integers (x,y) which satisfy the equation $y^2(x^2+1) + x^2(y^2+16) = 448$.

Solution. We have

$$\begin{aligned} & y^2(x^2+1) + x^2(y^2+16) &= 448 \\ \Longrightarrow\; & 2x^2y^2 + 16x^2 + y^2 - 448 &= 0 \\ \Longrightarrow\; & 2x^2(y^2+8) + y^2 + 8 &= (2x^2+1)(y^2+8) \\ & &= 456 = 2^3 \times 3 \times 19\,. \end{aligned}$$

Since x and y are integers, both $2x^2+1$ and y^2+8 are positive integers. Since $2x^2+1$ is odd, it must equal one of the odd factors of 456, namely 1, 3, 19, and 57. Checking each of these cases, we find that $x = 0$, ± 1, and ± 3 are the only solutions.

If $x = 0$, then $y^2 + 8 = 456$, for which there is no solution.

If $x = \pm 1$, then $y^2 + 8 = 152$, from which we obtain $y = \pm 12$.

If $x = \pm 3$, then $y^2 + 8 = 24$, from which we obtain $y = \pm 4$.

Therefore, there are eight solutions, namely $(\pm 1, \pm 12)$ and $(\pm 3, \pm 4)$.

Also solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

H240. *Proposed by Alexandre Trichtchenko, Brookfield High School, Ottawa, ON.*

A Pythagorean triple (a, b, c) is a triple of integers satisfying the equation $a^2 + b^2 = c^2$. We say that such a triple is *primitive* if $\gcd(a, b, c) = 1$. Let p be an odd integer with exactly n prime divisors. Show that there exist exactly 2^{n-1} primitive Pythagorean triples where p is the first element of the triple. For example, if $p = 15$, then $(15, 8, 17)$ and $(15, 112, 113)$ are the primitive Pythagorean triples with first element 15.

Solution. We may write p uniquely in the form

$$p = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n} ,$$

where $p_1, p_2, \ldots, p_n$ are distinct odd primes, and $e_1, e_2, \ldots, e_n$ are positive integers. We seek all primitive Pythagorean triples (p, b, c). For such a triple, $p^2 + b^2 = c^2$, or $c^2 - b^2 = (c+b)(c-b) = p^2$. Suppose p_i divides both $c+b$ and $c - b$ for some i. Then p_i divides $(c + b) - (c - b) = 2b$, and since p_i is odd, it follows that p_i divides b. Similarly, p_i divides $(c + b) + (c - b) = 2c$, so p_i divides c. Then (p, b, c) fails to be a primitive Pythagorean triple, since p_i divides all three numbers, so for each i, p_i divides at most one of $c + b$ and $c - b$.

This implies that in the factorization $p^2 = (c+b)(c-b)$, for each i, all the factors of p_i must reside in $c + b$ or $c - b$. In other words, for each i, we can make one of two choices of where to place all the factors of p_i, for a total of 2^n factorizations. However, half of them must be discarded, since $c + b$ must be the greater number, and $c - b$ the lesser. (We cannot have $c + b$ and $c - b$ equal, since they have different prime factors.) Each of the other half, however, does lead to a unique solution: If $p^2 = xy$, where $x > y$ and x and y are relatively prime, then $b = (x - y)/2$ and $c = (x + y)/2$. Hence, there are 2^{n-1} such Pythagorean triples.

Also solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Advanced Solutions

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A212. Let A and B be real $n \times n$ matrices such that $A^2 + B^2 = AB$. Prove that if $AB - BA$ is an invertible matrix, then n is divisible by 3.

(International Competition in Mathematics for University Students)

Solution. Let $S = A + \omega B$, where ω is a primitive cube root of unity. Then we have that

$$\begin{aligned} S\overline{S} &= (A+\omega B)(\overline{A+\omega B}) \\ &= (A+\omega B)(A+\overline{\omega}B) \;=\; A^2+\omega BA+\overline{\omega}AB+B^2 \\ &= AB+\omega BA+\overline{\omega}AB \;=\; \omega(BA-AB)\,, \end{aligned}$$

since $\overline{\omega}+1 = -\omega$. Also, $\det(S\overline{S}) = \det S \cdot \det\overline{S}$ is a real number and $\det\omega(BA-AB) = \omega^n\det(BA-AB) \neq 0$, so ω^n must be a real number. Hence, n is divisible by 3.

A213. Show that the number of positive integer solutions to the equation $a+b+c+d = 98$, where $0 < a < b < c < d$, is equal to the number of positive integer solutions to the equation $p+2q+3r+4s = 98$, where $0 < p, q, r, s$.

I. *Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Suppose that a, b, c, and d are positive integers such that $a < b < c < d$ and satisfy $a+b+c+d = 98$. Then letting $p = d-c$, $q = c-b$, $r = b-a$, and $s = a$, we see that p, q, r, and s are positive integers satisfying

$$p+2q+3r+4s = (d-c)+2(c-b)+3(b-a)+4a = a+b+c+d \;=\; 98\,.$$

Conversely, suppose that p, q, r, and s are positive integers satisfying $p+2q+3r+4s = 98$. Then letting $a = s$, $b = r+s$, $c = q+r+s$, and $d = p+q+r+s$, we see that a, b, c and d are positive integers satisfying $a < b < c < d$, and

$$a+b+c+d \;=\; p+2q+3r+4s \;=\; 98\,.$$

This establishes a bijection between the solutions (a,b,c,d) and (p,q,r,s).

II. *Solution*. First count the number of ways of distributing 98 identical marbles into 4 distinct boxes, with the first box containing the least number of marbles, the second box containing the second least number, and so on. This value is equal to the number of solutions to the first equation with the appropriate conditions. Another way of counting this is to distribute the same number h to each of the 4 boxes. Then distribute g to each of the second, third and fourth boxes. This ensures that the first box has the least. Distribute f to each of boxes three and four. This ensures that the second box is the second least and so on. The number of ways to do this is equal to the number of solutions to the second equation.

A214. Show that any rational number can be written as the sum of a finite number of distinct unit fractions. A unit fraction is of the form $1/n$, where n is an integer.

Solution.
We solve the problem by step-climbing through several cases.

Case I: Positive rationals in $[0, 1]$.

Let r be a positive rational number in $[0, 1]$. Then we show that r can be written as a sum of distinct unit fractions by providing an algorithm that explicitly finds the unit fractions, namely the greedy algorithm.

Let $r = a/b$, in reduced terms. Let $n = \lceil b/a \rceil$. In other words, n is the integer b/a rounded up, so $b/a \le n < b/a + 1$. Then $1/n$ is the largest unit fraction which is less than or equal to a/b, and when we subtract $1/n$ from a/b, we obtain $r' = \dfrac{a}{b} - \dfrac{1}{n} = \dfrac{an - b}{bn}$.

Since $b/a \le n < b/a + 1$, $b \le an < b + a$, or $0 \le an - b < a$. So, the numerator in r' is less than the numerator in r. If r' is reducible, then we reduce, and the numerator decreases still. We now subtract the largest unit fraction from r', and so on. Since the numerator decreases by at least 1 each step, the algorithm must stop at some point, in fact after at most $a - 1$ steps. Then r is the sum of the unit fractions produced by the algorithm.

For example, for $r = 6/7$, we have that

$$\frac{6}{7} - \frac{1}{2} = \frac{5}{14}, \qquad \frac{5}{14} - \frac{1}{3} = \frac{1}{42},$$

so that

$$\frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}.$$

Case II: Positive rationals greater than 1.
Let r be a positive rational greater than 1. Since the harmonic series $\sum_{i=1}^{\infty} 1/i$ diverges, there exists a positive integer k such that

$$\sum_{i=1}^{k} \frac{1}{i} \le r < \sum_{i=1}^{k+1} \frac{1}{i}.$$

If $r = \sum_{i=1}^{k} 1/i$, then we have an expression of r as a sum of distinct unit fractions, so assume that $r > \sum_{i=1}^{k} 1/i$. Consider the rational

$$s = r - \sum_{i=1}^{k} \frac{1}{i}, \qquad \text{so that} \qquad 0 < s < \frac{1}{k+1}.$$

Then s is a positive rational in $[0, 1]$, so by Case I, s is the sum of distinct unit fractions

$$s = \sum_{j=1}^{m} \frac{1}{n_j}.$$

Since $s < 1/(k+1)$, each n_j is at least $k+1$ (otherwise $n_j \leq k \Longrightarrow 1/n_j \geq 1/k \Longrightarrow s \geq 1/k$, contradiction), so that

$$r = \sum_{i=1}^{k} \frac{1}{i} + \sum_{j=1}^{m} \frac{1}{n_j}$$

is an expression of r as a sum of distinct unit fractions.

Case III: Negative rationals and 0.

If $r < 0$, then $-r$, by Cases I and II, can be written as the sum of distinct unit fractions. Negate each term to get such a sum for r. Finally, 0 can be written as $0 = \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{6}$.

A215. For a fixed integer $n \geq 2$, determine the maximum value of $k_1 + \cdots + k_n$, where $k_1, \ldots, k_n$ are positive integers with $k_1^3 + \cdots + k_n^3 \leq 7n$.

(Polish Mathematical Olympiad)

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $T = \sum_{i=1}^{n} k_i^3$ and $S = \sum_{i=1}^{n} k_i$. We claim that

$$S \leq n + \left\lfloor \frac{6n}{7} \right\rfloor ,$$

where $\lfloor x \rfloor$ denotes, as usual, the greatest integer less than or equal to x. This maximum is attained if and only if $\lfloor 6n/7 \rfloor$ of the k_i equal 2 and the rest equal 1.

First note that if $k_i \geq 2$ for all i, then $T \geq 8n$, a contradiction. Hence $k_i = 1$ for at least one i, say $k_1 = 1$. If $k_j \geq 3$ for some $j > 1$, then we consider T' obtained from T as follows: Replace k_1 and k_j by $k_1' = 2$ and $k_j' = k_j - 1$ respectively, and leave all the other k_i unchanged. Then clearly the value of S is unchanged. On the other hand, $T' \leq T$, which means that $(k_j - 1)^3 + 8 \leq k_j^3 + 1$. Rewriting this, we obtain $(k_j + 1)(k_j - 2) \geq 0$. This is true as long as $k_j \geq 3$. Hence $T' \leq 7n$, and to obtain the maximum value of S, we may assume, without loss of generality, that $k_i = 1$ or 2 for all i.

Suppose among all the k_i, that there are m 2's and $n - m$ 1's. Then $T = 8m + (n - m) \leq 7n$, which implies that $m \leq \lfloor 6n/7 \rfloor$. Clearly, the maximum value of S is attained when $m = \lfloor 6n/7 \rfloor$, in which case we obtain $S = 2\lfloor 6n/7 \rfloor + n - \lfloor 6n/7 \rfloor = n + \lfloor 6n/7 \rfloor$.

A216. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions:

$$f(1000) = 999 ,$$

$$f(x) \cdot f(f(x)) = 1 \text{ for all } x \in \mathbb{R} .$$

Determine $f(500)$.

(Polish Mathematical Olympiad)

Solution. By the second condition, $f(1000)f(f(1000)) = 1$, so we have that $999f(999) = 1$ or $f(999) = 1/999$.

Since f is a continuous function, by the Intermediate Value Theorem, there exists an $a \in [999, 1000]$ such that $f(a) = 500$.

Then $f(a)f(f(a)) = 1$, giving $500f(500) = 1$, so $f(500) = 1/500$.

In fact, $f(x) = 1/x$ for all $x \in [1/999, 999]$. To complete the function, for any x outside this range, set $f(x)$ to any value, within the interval $[1/999, 999]$. Then for any x, $1/999 \leq f(x) \leq 999$, and so $f(f(x)) = 1/f(x)$.

Note: Because $f(1000) = 999 \neq 1/1000$, $f(x)$ can never equal 1000.

Challenge Board Solutions

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C77. Let F_i denote the i^{th} Fibonacci number, with $F_0 = 1$ and $F_1 = 1$. (Then $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, etc.)

(a) Prove that each positive integer is uniquely expressible in the form $F_{a_1} + \cdots + F_{a_k}$, where the subscripts form a strictly increasing sequence of positive integers no pair of which are consecutive.

(b) Let $\tau = \frac{1}{2}(1 + \sqrt{5})$, and for any positive integer n, let $f(n)$ equal the integer nearest to $n\tau$. Prove that if $n = F_{a_1} + \cdots + F_{a_k}$ is the expression for n from part (a) and if $a_2 \neq 3$, then $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$.

(c) Keeping the notation from part (b), if $a_2 = 3$ (so that $a_1 = 1$), it is not always true that the formula $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$ holds. For example, if $n = 4 = F_3 + F_1 = 1 + 3$, then the closest integer to $n\tau = 6.47\ldots$ is 6, not $F_2 + F_4 = 2 + 5 = 7$. Fortunately, in the cases where the formula fails, we can correct the problem by setting $a_1 = 0$ instead of $a_1 = 1$: For example, $4 = F_0 + F_3 = 1 + 3$ as well, and indeed $6 = F_1 + F_4 = 1 + 5$. Determine for which sequences of a_i this correction is necessary.

Solution.

(a) Suppose we know that every positive integer less than F_n is uniquely expressible in the desired form. (The base case $n = 1$ is vacuous.) If N satisfies $F_n \leq N < F_{n+1}$, then by our inductive assumption, $N - F_n$ is expressible in the desired form, say $N - F_n = F_{a_1} + \cdots + F_{a_k}$. Since

$N - F_n < F_{n+1} - F_n = F_{n-1}$, we must have that $a_k \leq n - 2$, and so the expression

$$N = F_{a_1} + \cdots + F_{a_k} + F_n$$

is of the desired form. It remains to show that this is the *unique* expression for N as such a sum; observe, by induction, that it suffices to show that any such sum for N must include F_n.

Given an expression

$$N = F_{a_1} + \ldots + F_{a_k}$$

of the necessary form, if we assume that $a_k < n$, then we know that a_k is at most $n - 1$, so a_{k-1} is at most $n - 3$, and so on. Thus, N is at most

$$F_1 + F_3 + F_5 + \cdots + F_{n-1}$$

if n is even and

$$F_2 + F_4 + \cdots + F_{n-1}$$

if n is odd. However, it is a straightforward induction argument to prove that in fact for any positive integer m,

$$F_1 + F_3 + F_5 + \cdots + F_{2m-1} = F_{2m} - 1$$

and

$$F_2 + F_4 + \cdots + F_{2m} = F_{2m+1} - 1\,,$$

so we obtain a contradiction.

(b) Since $f(n)$ is the unique integer N such that $|N - n\tau| < 1/2$, we need only show that

$$|F_{a_1+1} + \cdots + F_{a_k+1} - \tau(F_{a_1} + \cdots + F_{a_k})| < \frac{1}{2}\,.$$

Recall that $F_i = \frac{\tau^{i+1} - \sigma^{i+1}}{\sqrt{5}}$, where $\sigma = \frac{1}{2}(1 - \sqrt{5})$. Hence, we want to prove that

$$\left|\sum_{i=1}^{k} \left((\tau^{a_i+2} - \sigma^{a_i+2}) - \tau(\tau^{a_i+1} - \sigma^{a_i+1})\right)\right| < \frac{\sqrt{5}}{2}\,.$$

Using the fact that $\sigma\tau = -1$, this becomes the same as showing that

$$\left|\sum_{i=0}^{k} (\sigma^{a_i+2} + \sigma^{a_i})\right| < \frac{\sqrt{5}}{2}\,,$$

and since $\sigma^2 + 1 = -\sigma\sqrt{5}$, we are further reduced to showing that

$$\left|\sum_{i=0}^{k} \sigma^{a_i+1}\right| < \frac{1}{2}\,.$$

Let $\mathcal{P}$ be the sum of the positive terms in the sum $\sum \sigma^{a_i+1}$, and let $\mathcal{N}$ be the sum of the negative terms.

Then $\mathcal{P}$ is at most $\displaystyle\sum_{i=1}^{\infty} \sigma^{2i} \;=\; \frac{\sigma^2}{1-\sigma^2} \;<\; 0.62$,

and $\mathcal{N}$ is at least $\displaystyle\sum_{i=1}^{\infty} \sigma^{2i+1} \;=\; \frac{\sigma^3}{1-\sigma^2} \;>\; -0.39$. In particular, the only way that $|\mathcal{P}+\mathcal{N}| \geq 1/2$ is if $\mathcal{P} \;\geq\; \dfrac{1}{2} + |\mathcal{N}|$.

If $a_2 \neq 3$, then either σ^2 or σ^4 is not in the sum for $\mathcal{P}$. Since $\sigma^2 > 0.38$ and $\sigma^4 > 0.14$, it follows that $\mathcal{P} < 1/2$, so $\mathcal{P} < 1/2 + |\mathcal{N}|$ and the desired formula holds.

(c) Keeping our notation from (b), if $\mathcal{P}+\mathcal{N} > 1/2$, then $a_1 = 1$ and $a_2 = 3$. Setting $a_1' = 0$ and $a_i' = a_i$ for $i > 1$, we find that

$$\left|\sum_{i=0}^{k} \sigma^{a_i'+1}\right| \;=\; \left|(\mathcal{P}-\sigma^2)+(\mathcal{N}+\sigma)\right| \;=\; |\mathcal{P}+\mathcal{N}-1| \;<\; \frac{1}{2},$$

so the suggested correction does indeed work when necessary.

We would like to decide precisely when this correction is necessary; that is, when

$$\mathcal{S} \;=\; \sum_{i=0}^{k} \sigma^{a_i'+1} \;>\; -\frac{1}{2}.$$

Let us retain the assumption that $a_1 = 1$ (so that the definition of the a_i' makes sense) but drop the assumption that $a_2 = 3$. This will turn out to be more convenient, in the end.

Observing, by direct calculation, that $\displaystyle\sum_{i=0}^{\infty} \sigma^{3i+1} \;=\; -\frac{1}{2}$, let j be the smallest (non-negative) integer i such that $a_{i+2}' \neq 3(i+1)$.

(If $a_{i+2}' = 3(i+1)$ for all $i \leq k$, then put $j = k-1$.) Then

$$\mathcal{S}+\frac{1}{2} \;=\; \frac{1}{2}+\sigma+\cdots+\sigma^{3j+1}+\sum_{i=j+2}^{k} \sigma^{a_i'+1} \;=\; \frac{1}{2}\,\sigma^{3j+3}+\sum_{i=j+2}^{k} \sigma^{a_i'+1}.$$

We will attempt to show (in what will amount to a clumsy verification) that $\mathcal{S}+1/2$ has the same sign as $\frac{1}{2}\sigma^{3j+3}$, and so the correction is necessary precisely when j is odd.

Remembering the definition of j and the fact that the a_i are non-consecutive, using $a_{j+1}' = 3j$, we break into three cases: $a_{j+2}' = 3j+2$, $a_{j+2}' = 3j+4$, or $a_{j+2}' \geq 3j+5$.

To begin with, in the first case,

$$\left|\sum_{i=j+3}^{k} \sigma^{a'_i+1}\right| \;<\; \left|\sigma^{3j+5}\right|\sum_{i=0}^{\infty}\sigma^{2i} \;=\; \left|\sigma^{3j+4}\right| \;<\; \left|\frac{3}{2}\,\sigma^{3j+3}\right| \;=\; \left|\frac{1}{2}\,\sigma^{3j+3} + \sigma^{a'_{j+2}+1}\right|\,,$$

as desired. Similarly, in the second case,

$$\left|\sum_{i=j+3}^{k} \sigma^{a'_i+1}\right| \;<\; \left|\sigma^{3j+7}\right|\sum_{i=0}^{\infty}\sigma^{2i} \;=\; \left|\sigma^{3j+6}\right| \;<\; \left|\frac{1}{2}\,\sigma^{3j+3} + \sigma^{3j+5}\right| \;=\; \left|\frac{1}{2}\,\sigma^{3j+3} + \sigma^{a'_{j+2}+1}\right|\,.$$

Finally, in the third case,

$$\left|\sum_{i=j+2}^{k} \sigma^{a'_i+1}\right| \;<\; \left|\sigma^{3j+6}\right|\sum_{i=0}^{\infty}\sigma^{2i} \;=\; \left|\sigma^{3j+5}\right| \;<\; \left|\frac{1}{2}\sigma^{3j+3}\right|\,,$$

and we are done.

Thus, we have shown: *If $a_1 = 1$, then if j is the smallest* non-negative *integer i such that $a_{i+2} \neq 3(i+1)$, then the correction is necessary if and only if j is odd.*

Remark: In the solution of part (b), we never fully used the fact that we were using the representation from part (a). In particular, the proof of (b) actually showed that if $n = F_{a_1} + \cdots + F_{a_k}$ and the a_i are distinct positive integers, then as long as 1 and 3 are not both among the a_i, we can conclude that $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$. Unfortunately, as should not be a surprise, not every integer can be expressed in this way, $n = 4$ being the smallest example.

Problem of the Month

Jimmy Chui, student, Earl Haig S.S.

Problem. A rectangular wine rack, $PQRS$, holds five rows of identical bottles (Figure 1). The bottom row contains enough room for three bottles (A, B, and C) but not enough room for a fourth bottle. The second row, consisting of just two bottles (D and E), holds B in place somewhere between A and C, and pushes A and C to the sides of the rack. The third row, consisting of three bottles (F, G, and H), lies on top of those two bottles, and F and H rest against the sides of the rack. The fourth layer holds two bottles (I and J) and the fifth layer contains three bottles (K, L and M). Prove that the fifth row is perfectly horizontal regardless of how A, B and C are positioned.

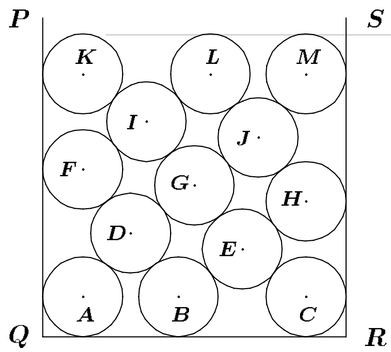

Figure 1.

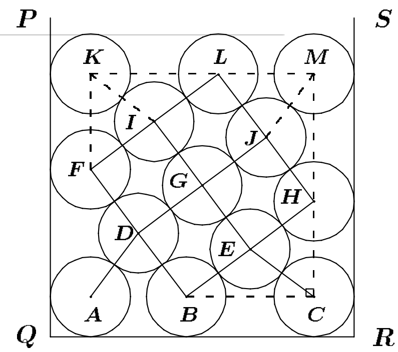

Figure 2.

Solution. All the bottles are identical, and so F and K are the same distance away from the wall (Figure 2). Since FK is vertical, we only need to show that $\angle FKL = 90°$.

The distance between the centres of touching bottles is constant; it is the diameter of a bottle. Therefore, IF, IK, and IL are all equal, and hence I is the circumcentre of triangle FKL. In order for $\angle FKL$ to be a right angle, then from properties of right angle triangles, the circumcentre I is also the mid-point of FL. Hence, we will show that I is the mid-point of FL.

Note that the four quadrilaterals $GDFI$, $GILJ$, $GJHE$, $GEBD$ are all rhombi (they have side length of a bottle diameter). So $\overrightarrow{FI} = \overrightarrow{BE}$ and $\overrightarrow{IL} = \overrightarrow{EH}$. Furthermore, since EB, EC, and EH are all equal, E is the circumcentre of triangle BCH. However, we know that BCH is a right triangle. Hence, E is the mid-point of BH. Thus, I is the mid-point of FL, and it follows that $\angle FKL = 90°$.

Similarly, $\angle HML$ is a right angle. Therefore, the top row is perfectly horizontal, QED.

Four Ways to Count

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Problem. Evaluate

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n},$$

where n is a positive integer.

Solution 1. Let the given sum be equal to S. Now,

$$\begin{aligned} S &= 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} \\ &= 0\binom{n}{n} + 1\binom{n}{n-1} + 2\binom{n}{n-2} + 3\binom{n}{n-3} + \cdots + n\binom{n}{0} \\ &= n\binom{n}{0} + (n-1)\binom{n}{1} + (n-2)\binom{n}{2} + (n-3)\binom{n}{3} + \cdots + 0\binom{n}{n}. \end{aligned}$$

Adding the first and third equations and dividing by 2, we obtain

$$\begin{aligned} S &= \frac{n}{2}\left(\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}\right) \\ &= \frac{n}{2} \cdot 2^n \\ &= n2^{n-1}. \end{aligned}$$

Solution 2. We claim that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

We will prove this by mathematical induction. The base case $n = 1$ is trivial. Now we assume, for some $n = k$, that

$$\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \cdots + k\binom{k}{k} = k2^{k-1}.$$

Then,

$$\begin{aligned}
&\binom{k+1}{1} + 2\binom{k+1}{2} + 3\binom{k+1}{3} + \cdots + k\binom{k+1}{k} + (k+1)\binom{k+1}{k+1} \\
&\quad = 1\left(\binom{k}{0} + \binom{k}{1}\right) + 2\left(\binom{k}{1} + \binom{k}{2}\right) + 3\left(\binom{k}{2} + \binom{k}{3}\right) \\
&\qquad + \cdots + k\left(\binom{k}{k-1} + \binom{k}{k}\right) + (k+1)\binom{k}{k} \\
&\quad = 1\binom{k}{0} + 3\binom{k}{1} + 5\binom{k}{2} + 7\binom{k}{3} + \cdots + (2k+1)\binom{k}{k} \\
&\quad = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \binom{k}{3} + \cdots + \binom{k}{k} \\
&\qquad + 2\left(0\binom{k}{0} + 1\binom{k}{1} + 2\binom{k}{2} + 3\binom{k}{3} + \cdots + k\binom{k}{k}\right) \\
&\quad = 2^k + 2(k2^{k-1}) \\
&\quad = (k+1)2^k\,.
\end{aligned}$$

Hence, the claim is true for $n = k + 1$, and by the principle of mathematical induction, for all positive integers n.

Solution 3. Let

$$f(x) \;=\; \sum_{i=0}^{n} \binom{n}{i} x^i \;=\; 1 + \sum_{i=1}^{n} \binom{n}{i} x^i\,.$$

Then

$$f'(x) \;=\; \sum_{i=1}^{n} i\binom{n}{i} x^{i-1}\,,$$

so that

$$f'(1) \;=\; \sum_{i=1}^{n} i\binom{n}{i}\,.$$

But

$$f(x) \;=\; (1+x)^n$$

by the Binomial Theorem. Then

$$f'(x) \;=\; n(1+x)^{n-1}\,,$$

so that

$$f'(1) \;=\; n2^{n-1}\,.$$

Hence,

$$\sum_{i=1}^{n} i\binom{n}{i} \;=\; n2^{n-1}\,.$$

Solution 4. Consider a set of n people. We wish to count the number of different teams that can be formed, with the condition that there is one and only one leader.

One way is to find the total number of team members first, and then select a leader from those chosen. Suppose you choose i members. (Note that $1 \leq i \leq n$.) There are $\binom{n}{i}$ such subsets. In each of these subsets, there are i possible leaders. Hence, the total number of teams that can be formed with i members is $i\binom{n}{i}$. Therefore, the total number of teams with any number of members is merely the sum

$$\sum_{i=1}^{n} i\binom{n}{i} .$$

Another way is to choose the leader first, and the rest of the members afterwards. The leader can be chosen in n ways. The members can be chosen out of the other $n-1$ people in any way, and there are 2^{n-1} ways of doing so. Hence the total number of teams is $n2^{n-1}$.

Thus,

$$\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1} .$$

Comment. This question appears in the strangest of places, and it is pleasant to see four very different, yet equally elegant, solutions.