THE SKOLIAD CORNER

No. 38

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In this issue we give an example of a rather different contest, the Newfoundland and Labrador Teachers' Association Senior Mathematics League, for 1998–1999. The League started in 1987 as a contest in the St. John's area and grew to a province-wide event, with schools competing in local leagues at several sites. The same contest occurs simultaneously at the sites, and the top schools in each district coming together for the provincial finals. The emphasis is on cooperative problem solving. A school team consists of four students who work together on problems. Individual work is rewarded, but to foster collaboration there are bonus marks for correct work as a team. A typical competition consists of ten questions and a relay. Unlike most contests, the problems are presented separately, answers collected, and solutions discussed before going on to the next problem. Individual student solutions may earn 1 mark each, while a correct team solution gains 5 marks. Incorrect answers score 0. The relay has four points, with the answer to each part being an input to the next. One point is awarded if only part 1 is correct, two for parts 1 and 2, three for parts 1, 2, and 3 and five marks for all four parts of the relay. Bonus points are awarded for each minute remaining in the first ten minutes allotted for the relay. (If an incorrect answer is submitted early, the team is told the answer is wrong but not why, and they may go back to rework it.) A tie-breaker may be required. This is based on speed, but to deter silly guesses, an incorrect answer means the team cannot answer for one minute. My thanks go to John Grant McLoughlin, Memorial University of Newfoundland, for forwarding me the contest and background information.

NLTA SENIOR MATH LEAGUE
GAME 1 — 1998–99

1. Find a two-digit number that equals twice the product of its digits.

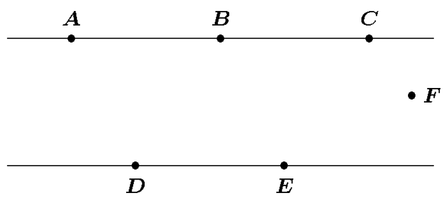
2. The degree measures of the interior angles of a triangle are A, B, C where $A \leq B \leq C$. If A, B, and C are multiples of 15, how many possible values of (A, B, C) exist?

3. Place an operation $(+, -, \times, \div)$ in each square so that the expression using 1, 2, 3, …, 9 equals 100.

$$1 \square 2 \square 3 \square 4 \square 5 \square 6 \square 7 \square 8 \square 9 = 100$$

You may also freely place brackets before/after any digits in the expression. Note that the squares must be filled in with operational symbols only.

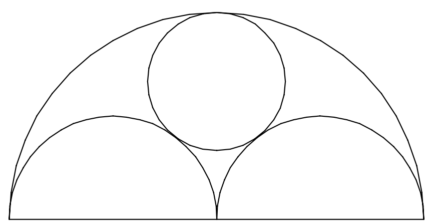
4. A, B and C are points on a line that is parallel to another line containing points D and E, as shown. Point F does not lie on either of these lines.



How many distinct triangles can be formed such that all three of their vertices are chosen from A, B, C, D, E, and F?

5. Michael, Jane and Bert enjoyed a picnic lunch. The three of them were to contribute an equal amount of money toward the cost of the food. Michael spent twice as much money as Jane did buying food for lunch. Bert did not spend any money on food. Instead, Bert brought \$6 which exactly covered his share. How much (in dollars) of Bert's contribution should be given to Michael?

6. Two semicircles of radius 3 are inscribed in a semicircle of radius 6. A circle of radius R is tangent to all three semicircles, as shown. Find R.



7. If $5^A = 3$ and $9^B = 125$, find the value of AB.

8. The legs of a right-angled triangle are 10 and 24 cm respectively.

Let A = the length (cm) of the hypotenuse,
B = the perimeter (cm) of the triangle,
C = the area (cm^2) of the triangle.

Determine the lowest common multiple of A, B, and C.

9. A lattice point is a point (x, y) such that both x and y are integers. For example, $(2, -1)$ is a lattice point, whereas, $(3, \frac{1}{2})$ and $(-\frac{1}{3}, \frac{2}{3})$ are not.

How many lattice points lie inside the circle defined by $x^2 + y^2 = 20$? (Do NOT count lattice points that lie on the circumference of the circle.)

10. The quadratic equation $x^2 + bx + c = 0$ has roots, r_1 and r_2, that have a sum which equals 3 times their product. Suppose that $(r_1 + 5)$ and $(r_2 + 5)$ are the roots of another quadratic equation $x^2 + ex + f = 0$. Given that the ratio of $e : f = 1 : 23$, determine the values of b and c in the original quadratic equation.

RELAY

R1. Operations $*$ and $\diamond$ are defined as follows:

$$A * B \;=\; \frac{A^B + B^A}{A + B} \quad \text{and} \quad A \diamond B \;=\; \frac{A^B - B^A}{A - B}\,.$$

Simplify $N = (3 * 2) * (3 \diamond 2)$. Write the value of N in Box #1 of the relay answer sheet.

R2. A square has a perimeter of P cm and an area of Q cm^2? Given that $3NP = 2Q$, determine the value of P. Write the value of P in Box #2 of the relay answer sheet.

R3. List all two-digit numbers that have digits whose product is P. Call the sum of these two-digit numbers S. Write the value of S in Box #3 of the relay answer sheet.

R4. How many integers between 6 and 24 share no common factors with S that are greater than 1? Write the number in Box #4 of the relay answer sheet.

TIE-BREAKER

Find the maximum value of

$$f(x) \;=\; 14 - \sqrt{x^2 - 6x + 25}\,.$$

Last number we gave the problems of Part I of the Alberta High School Mathematics Competition written in November 1998. Next we give the official solutions. My thanks go to the organizing committee, chaired by Ted Lewis, University of Alberta, Edmonton, for supplying the contest problems and the official solutions.

THE ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION
Part I — November 1998

1. A restaurant usually sells its bottles of wine for 100% more than it pays for them. Recently it managed to buy some bottles of its most popular wine for half of what it usually pays for them, but still charged its customers what it would normally charge. For these bottles of wine, the selling price was what percent more than the purchase price?

Solution. The answer is: (c) 300. Let the old buying price be $2x$. Then the new buying price is x and the selling price is $4x$. The latter exceeds the former by $3x$.

2. How many integer solutions n are there to the inequality $34n \geq n^2 + 289$?

Solution. The answer is: (b) 1. The only solution is $n = 17$ since the inequality can be rewritten as $0 \geq (n - 17)^2$.

3. A university evaluates five magazines. Last year, the rankings were MacLuck with a rating of 150, followed by MacLock with 120, MacLick with 100, MacLeck with 90, and MacLack with 80. This year, the ratings of these magazines are down 50%, 40%, 20%, 10% and 5% respectively. How does the ranking change for MacLeck?

Solution. The answer is: (a) up 3 places. The new ratings are 75 for MacLuck, 72 for MacLock, 80 for MacLick, 81 for MacLeck and 76 for MacLack. *This problem illustrates that quality and ranking are two different things*.

4. Parallel lines are drawn on a rectangular piece of paper. The paper is then cut along each of the lines, forming n identical rectangular strips. If the strips have the same length to width ratio as the original, what is this ratio?

Solution. The answer is: (a) $\sqrt{n} : 1$. Suppose the length to width ratio for the original rectangle is $a : b$ with $a > b$. Then the ratio for the strips is $b : \frac{a}{n} = a : b$. Hence $\frac{a}{b} = \sqrt{n}$.

5. "The smallest integer which is at least $a\%$ of 20 is 10." For how many integers a is this statement true?

Solution. The answer is: (e) 5. The largest possible value of a is 50 and the smallest is 46.

6. Let $S = 1 + 2 + 3 + \cdots + 10^n$. How many factors of 2 appear in the prime factorization of S?

Solution. The answer is: (c) $n - 1$. We have $S = \frac{1}{2}10^n(10^n + 1) = 2^{n-1}5^n(10^n + 1)$.

7. When $1 + x + x^2 + x^3 + x^4 + x^5$ is factored as far as possible into polynomials with integral coefficients, what is the number of such factors, not counting trivial factors consisting of the constant polynomial 1?

Solution. The answer is: (c) 3. We have $(1+x+x^2)+(x^3+x^4+x^5) = (1 + x + x^2)(1 + x^3) = (1 + x + x^2)(1 + x)(1 - x + x^2)$. The two quadratic factors are irreducible over polynomials with integral coefficients since the only possible linear factors are $1 + x$ and $1 - x$, but neither divides them.

8. In triangle ABC, $AB = AC$. The perpendicular bisector of AB passes through the midpoint of BC. If the length of AC is $10\sqrt{2}$ cm, what is the area of ABC in cm^2 ?

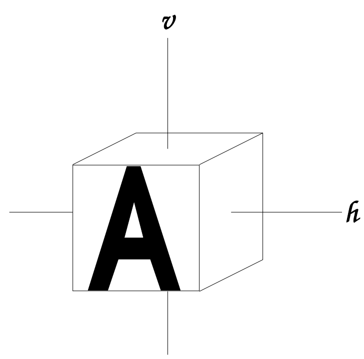
Solution. The answer is: (e) none of these. The line joining the midpoints of AB and BC is parallel to AC. Hence $\angle CAB = 90°$ and the area of triangle ABC is $\frac{1}{2}AB \cdot AC = 100$.

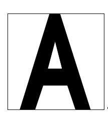
9. If $f(x) = x^x$, what is $f(f(x))$ equal to?

Solution. The answer is: **(d)** $x^{(x^{(x+1)})}$. We have

$$f(f(x)) = (x^x)^{(x^x)} = x^{x x^x}.$$

10. A certain TV station has a logo which is a rotating cube in which one face has an A on it and the other five faces are blank. Originally the A-face is at the front of the cube as shown on the right. Then you perform the following sequence of three moves over and over: rotate the cube $90°$ around the vertical axis v, so that the front face moves to the left; then rotate the cube $90°$ around a horizontal axis h, so that the new front face moves down; then rotate the cube $90°$ around the vertical axis again, so that the new front face moves to the left. Suppose you perform this sequence of three moves a total of 1998 times. What will the front face look like when you have finished?



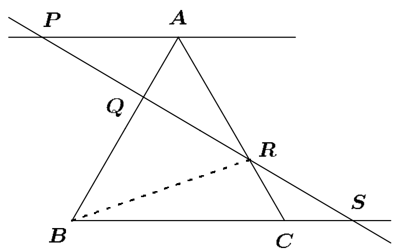
Solution. The answer is: **(a)** 

. When the sequence is performed, the A-face first goes to the left, pointing up, then stays at the left but pointing to the front, and finally goes to the back, pointing to the left. When the sequence is performed again, the A-face first goes to the right, pointing to the back, then stays at right, pointing up, and finally returns to the front, pointing up.

11. How many triples (x, y, z) of real numbers satisfy the simultaneous equations $x + y = 2$ and $xy - z^2 = 1$?

Solution. The answer is: **(a)** 1. We have $1 + z^2 = x(2 - x)$ which is equivalent to $(x - 1)^2 + z^2 = 0$. Hence $x = y = 1$ and $z = 0$.

12. In the diagram, ABC is an equilateral triangle of side length 3 and PA is parallel to BS. If $PQ = QR = RS$, what is the length of BR?



Solution. The answer is: **(d)** $\sqrt{7}$. Triangles PRA and SRC are similar. Since $PR = 2RS$ and $AC = 3$, we have $CR = 1$. Let the foot of perpendicular from R to BC be T. Since $\angle ACB = 60°$, we have $RT = \frac{1}{2}\sqrt{3}$ and $CT = \frac{1}{2}$, so that $BT = \frac{5}{2}$. By Pythagoras' Theorem, $BR^2 = RT^2 + BT^2$.

13. Let a, b, c and d be the roots of $x^4 - 8x^3 - 21x^2 + 148x - 160 = 0$. What is the value of $\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}$?

Solution. The answer is: **(b)** $-\frac{1}{20}$. We have $a + b + c + d = 8$ while $abcd = -160$. The desired expression is equal to $\frac{a+b+c+d}{abcd}$.

14. Wei writes down, in order of size, all positive integers b with the property that b and 2^b end in the same digit when they are written in base 10. What is the 1998^{th} number in Wei's list?

Solution. The answer is: **(b)** 19976. Since 2^b is even, so is b. When b is even, 2^b ends alternately in 4 and 6. For $2 \leq b \leq 20$, the only matches are $b = 14$ and $b = 16$. Since everything repeats in a cycle of 20, Wei's list is 14, 16, 34, 36, ... The $(2n-1)$-st number is $10(2n-1) + 4$ and the $2n$-th number is $10(2n-1) + 6$.

15. Suppose $x = 3^{1998}$. How many integers are between $\sqrt{x^2 + 2x + 4}$ and $\sqrt{4x^2 + 2x + 1}$?

Solution. The answer is: **(b)** $3^{1998} - 1$. Note that

$$x + 1 \; < \; \sqrt{x^2 + 2x + 4} \; < \; x + 2\,,$$

while

$$2x \; < \; \sqrt{4x^2 + 2x + 1} \; < \; 2x + 1\,.$$

Hence the number of integers between the two radicals is $x - 1$.

16. The lengths of all three sides of a right triangle are positive integers. The area of the triangle is 120. What is the length of the hypotenuse?

Solution. The answer is: **(c)** 26. Suppose that the three side lengths are $x \leq y \leq z$. Then $x^2 + y^2 = z^2$ since we have a right triangle. Hence $\frac{1}{2}xy = 120$ and $xy = 2^4 3 \cdot 5$.

Of the numbers 3^2+80^2, 5^2+48^2, 15^2+16^2, 6^2+40^2, 10^2+24^2 and 12^2+20^2, we only have an integral value for z when $x=10$ and $y=24$, namely, $z^2=2^2(5^2+12^2)=(2\cdot 13)^2$.

In the February 1999 number of the *Corner* we gave the solutions to the problems of the Old Mutual Mathematical Olympiad 1991, Final Paper 1. One of our readers proposes an alternate "variable free" solution to one of them.

2. [1998: 477; 1999: 28] What is the value of

$$\sqrt{17-12\sqrt{2}}+\sqrt{17+12\sqrt{2}}$$

in its simplest form?

Alternate solution by Luyun Zhong Qiho, Mathematics Teacher, Hamilton, Ontario.

$$\begin{aligned}
&\sqrt{17-12\sqrt{2}} \;+\; \sqrt{17+12\sqrt{2}}\\
&= \sqrt{9-2(3)\left(2\sqrt{2}\right)+8} \;+\; \sqrt{9+2(3)\left(2\sqrt{2}\right)+8}\\
&= \sqrt{\left(3-2\sqrt{2}\right)^2} \;+\; \sqrt{\left(3+2\sqrt{2}\right)^2}\\
&= \left(3-2\sqrt{2}\right)+\left(3+2\sqrt{2}\right) \qquad \text{since } 3>2\sqrt{2}\\
&= 3+3\\
&= 6\,.
\end{aligned}$$

That completes the *Skoliad Corner* for this month. Send me suitable contest materials, novel solutions, and suggestions for future directions.