THE OLYMPIAD CORNER

No. 198

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this number with the problems of the XXXIX Republic Competition of Mathematics in Macedonia and the problems of the third Macedonian Mathematical Olympiad. My thanks go to Ravi Vakil who collected them for me while he was Canadian Team Leader to the IMO at Mumbai.

XXXIX REPUBLIC COMPETITION OF MATHEMATICS IN MACEDONIA

Class I

1. The sum of three integers a, b and c is 0. Prove that $2a^4+2b^4+2c^4$ is the square of an integer.

2. Prove that if

$$a_0^{a_1} = a_1^{a_2} = \cdots = a_{1995}^{a_{1996}} = a_{1996}^{a_0}, \quad a_1 \in \mathbb{R}^*,$$

then

$$a_0 = a_1 = \cdots = a_{1996}.$$

3. Let h_a, h_b and h_c be the altitudes of the triangle with edges a, b and c, and r be the radius of the inscribed circle in the triangle. Prove that the triangle is equilateral if and only if $h_a+h_b+h_c=9r$.

4. Prove that each square can be cut into n $(n \geq 6)$ squares.

Class II

1. Prove that for positive real numbers a and b

$$2\cdot\sqrt{a}+3\cdot\sqrt[3]{b} \geq 5\cdot\sqrt[5]{ab}.$$

2. The point M is the mid-point on the side $\overline{BC}=a$ of a triangle ABC. Let r_1, r_2, r_3 be the radii of the inscribed circles in the triangles ABC, ABM and ACM respectively. Prove that

$$\frac{1}{r_1}+\frac{1}{r_2} \geq 2\left(\frac{1}{4}+\frac{2}{a}\right).$$

3. Let $A = \{z_1, z_2, \dots, z_{1996}\}$ be a set of complex numbers and for each $i \in \{1, 2, \dots, 1996\}$ suppose $\{z_1 z_1, z_1 z_2, \dots, z_1 z_{1996}\} = A$.

(a) Prove that $|z_i| = 1$ for each i.

(b) Prove that $z \in A$ implies $\overline{z} \in A$.

4. Find the biggest value of the difference $x - y$ if $2(x^2 + y^2) = x + y$.

Class III

1. Solve the equation $x^{1996} - 1996x^{1995} + \cdots + 1 = 0$ (the coefficients in front of $x, \dots, x^{1994}$ are unknown), if it is known that its roots are positive real numbers.

2. Let AH, BK and CL be the altitudes of an arbitrary triangle ABC. Prove that

$$\overline{AK} \cdot \overline{BL} \cdot \overline{CH} = \overline{AL} \cdot \overline{BH} \cdot \overline{CK} = \overline{HK} \cdot \overline{KL} \cdot \overline{LH}.$$

3. An initial triple of numbers $2, \sqrt{2}, \frac{1}{\sqrt{2}}$ is given. A new triple may be obtained from an old one as follows: two numbers a and b of the triple are changed to $\frac{(a+b)}{\sqrt{2}}$ and $\frac{(a-b)}{\sqrt{2}}$ and the third number is unchanged. Is it possible after a finite number of such steps to obtain the triple $1, \sqrt{2}, 1 + \sqrt{2}$?

4. A finite number of points in the plane are given such that not all of them are collinear. A real number is assigned to each point. The sum of the numbers for each line containing at least two of the given points is zero. Prove that all numbers are zeros.

Class IV

1. Let $a_1, a_2, \dots, a_n$ be real numbers which satisfy:

There exists a real number M such that $|a_i| \le M$ for each $i \in \{1, \dots, n\}$.

Prove that $a_1 + 2a_2 + \cdots + na_n \le \dfrac{Mn^2}{4}$.

2. Two circles with radii R and r touch from inside. Find the side of an equilateral triangle having one vertex at the common point of the circles and the other two vertices lying on the two circles.

3. The same problem as problem 3 given for Class III.

4. The same problem as problem 4 given for Class III.

PROBLEMS ON THE THIRD MACEDONIAN MATHEMATICAL OLYMPIAD

1. Let $ABCD$ be a parallelogram which is not a rectangle and E be a point in its plane, such that $AE \perp AB$ and $BC \perp EC$. Prove that $\angle DAE = \angle CEB$. [Ed. We know this is incorrect — can any reader supply the correct version?]

2. Let $\mathcal{P}$ be the set of all polygons in the plane and let $M : \mathcal{P} \to \mathbb{R}$ be a mapping which satisfies:

(i) $M(P) \geq 0$ for each polygon P;

(ii) $M(P) = x^2$ if P is an equilateral triangle of side x;

(iii) If P is a polygon separated into two polygons S and T, then $M(P) = M(S) + M(T)$; and

(iv) If P and T are congruent polygons, then $M(P) = M(T)$.

Find $M(P)$ if P is a rectangle with edges x and y.

3. Prove that if α, β and γ are angles of a triangle, then

$$\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} \geq \frac{8}{3 + 2\cos\gamma}.$$

4. A polygon is called "good" if the following conditions are satisfied:

(i) all angles belong to $(0, \pi) \cup (\pi, 2\pi)$;

(ii) two non-neighbouring sides do not have any common point; and

(iii) for any three sides, at least two are parallel and equal.

Find all non-negative integers n such that there exists a "good" polygon with n sides.

5. Find the biggest number n such that there exist n straight lines in space, $\mathbb{R}^3$, which pass through one point and the angle between each two lines is the same. (The angle between two intersecting straight lines is defined to be the smaller one of the two angles between these two lines.)

Next we give the problems of the Ninth Irish Mathematical Olympiad, written Saturday, May 4, 1996. My thanks again go to Ravi Vakil for collecting the problems and sending them to me while he was Canadian Team Leader to the IMO at Mumbai.

NINTH IRISH MATHEMATICAL OLYMPIAD
First Paper — May 4, 1996
Time: 3 hours

1. For each positive integer n, let $f(n)$ denote the greatest common divisor of $n!+1$ and $(n+1)!$ (where ! denotes "factorial"). Find, with proof, a formula for $f(n)$ for each n.

2. For each positive integer n, let $S(n)$ denote the sum of the digits of n (when n is written in base 10). Prove that for every positive integer n,

$$S(2n) \leq 2S(n) \leq 10S(2n).$$

Prove also that there exists a positive integer n with $S(n) = 1996S(3n)$,

3. Let K be the set of all real numbers x with $0 \leq x \leq 1$. Let f be a function from K to the set of all real numbers $\mathbb{R}$ with the following properties:

(i) $f(1) = 1$;

(ii) $f(x) \geq 0$ for all $x \in K$;

(iii) if x, y and $x+y$ are all in K, then $f(x+y) \geq f(x)+f(y)$.

Prove that $f(x) \leq 2x$ for all $x \in K$.

4. Let F be the mid-point of the side BC of the triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on the sides AB and AC with the right angles at D and E, respectively.

Prove that DEF is a right-angled isosceles triangle.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area.

Second Paper — May 4, 1996
Time: 3 hours

6. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined as follows: $F_0 = 0$, $F_1 = 1$ and for all $n \geq 0$, $F_{n+2} = F_n + F_{n+1}$. (So $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8, \ldots$.) Prove that

(i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if $k = 60$ but it is not true for any positive integer $k < 60$.

(ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if $t = 300$ but it is not true for any positive integer $t < 300$.

7. Prove that the inequality $2^{1/2} \cdot 4^{1/4} \cdot 8^{1/8} \cdots (2^n)^{1/2^n} < 4$ holds for all positive integers n.

8. Let p be a prime number and a and n positive integers. Prove that if $2^p + 3^p = a^n$, then $n = 1$.

9. Let ABC be an acute-angled triangle and let D, E, F be the feet of the perpendiculars from A, B, C onto the sides BC, CA, AB, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C onto the lines EF, FD, DE respectively. Prove that the lines AP, BQ, CR (extended) are concurrent.

10. We are given a rectangular chessboard of size 5×9 (so there are five rows of squares, each row containing nine squares). The following game is played: Initially, a number of discs are randomly placed on some of the squares, no square being allowed to contain more than one disc. A complete move consists of moving every disc from the square containing it to another square subject to the following rules:

(i) each disc may be moved one square up or down, or left or right, of the square it occupies, to an adjoining square;

(ii) if a particular disc is moved up or down as part of a complete move, then it must be moved left or right in the next complete move;

(iii) if a particular disc is moved left or right as part of a complete move, then it must be moved up or down in the next complete move;

(iv) at the end of each complete move, no square can contain two or more discs.

The game stops if it becomes impossible to perform a complete move. Prove that if initially 33 discs are placed on the board then the game must eventually stop. Prove also that it is possible to place 32 discs on the board in such a way that the game could go on forever.

Now we turn to solutions by our readers to problems of the 30^{th} Spanish Mathematical Olympiad, Final Round, November 26–27, 1993 [1998: 69–70].

30th SPANISH MATHEMATICAL OLYMPIAD
First Round — November 26–27, 1993

1. Show that, for all $n \in \mathbb{N}$, the fractions

$$\frac{n-1}{n}, \frac{n}{2n+1}, \frac{2n+1}{2n^2+2n},$$

are irreducible.

Solution by Pierre Bornsztein, Courdimanche, France.

Soit $n \in \mathbb{N}^*$.

On a, pour $n \geq 2$, $(n, n-1) = 1$. En plus$(n, 2n+1) = 1$ et donc $\frac{n}{2n+1}$ est irréductible. Enfin $(2n+1, 2n) = 1$ et $(2n+1, n+1) = 1$, car si p divise $2n+1$ et $n+1$ alors p divise $(2n+1) - (n+1) = n$, donc p divise $(n, n+1) = 1$. D'où $(2n+1, 2n(n+1)) = 1$ et donc $\frac{2n+1}{2n^2+2n}$ est irréductible.

3. Solve the following system of equations:

$$x \cdot |x| + y \cdot |y| = 1, \qquad \lfloor x \rfloor + \lfloor y \rfloor = 1\,,$$

in which $|t|$ and $\lfloor t \rfloor$ represent the absolute value and the integer part of the real number t.

Solutions by Pierre Bornsztein, Courdimanche, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

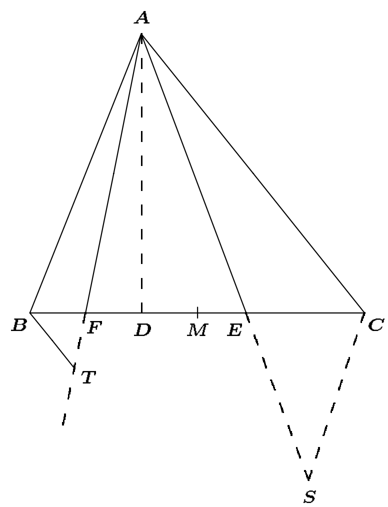
We show that there are only two solutions: $(x, y) = (1, 0)$ and $(0, 1)$. Clearly, x and y cannot both be negative. Hence by symmetry, there are two cases to be considered:

(i) If $x \geq 0$ and $y \geq 0$, then from $x^2 + y^2 = 1$ we get $0 \leq x$, $y \leq 1$. If $x < 1$ and $y < 1$, then $\lfloor x \rfloor + \lfloor y \rfloor = 0$, a contradiction. Hence $x = 1$ or $y = 1$. Then from $x^2 + y^2 = 1$ we obtain the two solutions $(1, 0)$ and $(0, 1)$.

(ii) If $x \geq 0$ and $y < 0$ then $x^2 - y^2 = 1$, and from $\lfloor x \rfloor \leq x$, $\lfloor y \rfloor \leq y$ we get $x + y \geq 1$. Since $x - y > x + y \geq 1$ we have $x^2 - y^2 = (x-y)(x+y) > 1$, a contradiction. Therefore, there are no solutions in this case.

4. Let AD be the internal bisector of the triangle ABC ($D \in BC$), E the point symmetric to D with respect to the mid-point of BC, and F the point of BC such that $\angle BAF = \angle EAC$. Show that $\frac{BF}{FC} = \frac{c^3}{b^3}$.

Solutions by Pierre Bornsztein, Courdimanche, France; by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya's solution.



Let T be the point on AF such that $BT \parallel AC$ and let S be the point on AE such that $CS \parallel AB$. Then $\angle ABT + \angle BAC = 180°$ and $\angle ACS + \angle BAC = 180°$.

Thus we have $\angle ABT = \angle ACS$. Since $\angle BAT = \angle BAF = \angle CAE = \angle CAS$ we have $\triangle ABT \sim \triangle ACS$ so that

$$\frac{BT}{CS} = \frac{AB}{AC} = \frac{c}{b}. \tag{1}$$

As $BT \parallel AC$ we get

$$\frac{BF}{FC} = \frac{BT}{AC} = \frac{BT}{b}. \tag{2}$$

As $AB \parallel CS$ we have

$$\frac{BE}{EC} = \frac{AB}{CS} = \frac{c}{CS}. \tag{3}$$

Let M be the mid-point of BC.

Since E is the point symmetric to D with respect to M we have

$$EC = BD \quad \text{and} \quad BE = DC,$$

so that $\dfrac{BE}{EC} = \dfrac{DC}{BD}$.

Since AD is the bisector of $\angle BAC$, we get $\dfrac{DC}{BD} = \dfrac{AC}{AB} = \dfrac{b}{c}$. Thus we have

$$\frac{BE}{EC} = \frac{b}{c}. \tag{4}$$

From (3) and (4) we have

$$\frac{c}{CS} = \frac{b}{c}, \quad \text{so that} \quad CS = \frac{c^2}{b}.$$

Hence, from (1), we get $BT = \dfrac{c^3}{b^2}$; whence from (2), $\dfrac{BF}{FC} = \dfrac{c^3}{b^3}$.

5. Find all the natural numbers n such that the number

$$n(n+1)(n+2)(n+3)$$

has exactly three prime divisors.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

The only such n's are $n = 2$, 3 and 6.

Let $P(n) = n(n+1)(n+2)(n+3)$. Then $P(1) = 2^3 \times 3$ and thus $n = 1$ is not a solution. Hence we assume $n \geq 2$. Note first that for all $k \in \mathbb{N}$, $(k, k+1) = (2k-1, 2k+1) = 1$ and hence if n is odd, then each of n, $n+1$ and $n+2$ must be a distinct prime power. We are led to two cases:

(a) If n is odd, then since $n+1$ is even we must have $n = p^a$, $n+1 = 2^b$ and $n+2 = q^c$ where a, b, c, p, $q \in \mathbb{N}$ with p and q being distinct odd primes. Note that $n+3 = 2^b+2 = 2(2^{b-1}+1)$ where $b \geq 2$. Since the only possible prime divisors of $n+3$ are 2, p or q, we have either

$$\text{(i) } 2^{b-1}+1 = p^\alpha \quad \text{for some} \quad \alpha \in \mathbb{N}$$

or

$$\text{(ii) } 2^{b-1}+1 = q^\beta \quad \text{for some} \quad \beta \in \mathbb{N}.$$

In case (i) we have $2p^\alpha = n+3 = p^a+3$. Clearly $\alpha \leq a$ and thus $p^\alpha \mid p^a$. Hence $p^\alpha \mid 3$ which implies $p = 3$, $\alpha = 1$. Thus $b = 2$ and $n = 3$. Indeed, $n = 3$ is a solution since $P(3) = 2^3 \times 3^2 \times 5$.

In case (ii) we have $2q^\beta = n+3 = q^c+1$, which is clearly impossible since $q \nmid 1$.

(b) If n is even, then by the same argument, we have $n+1 = p^a$, $n+2 = 2^b$ and $n+3 = q^c$ where a, b, c, p, $q \in \mathbb{N}$ with p and q being distinct odd primes. Note that $n = 2^b-2 = 2(2^{b-1}-1)$ where $b \geq 2$. If $b = 2$, then $n = 2$, which is indeed a solution since $P(2) = 2^3 \times 3 \times 5$. If $b > 2$, then we must have either

$$\text{(iii) } 2^{b-1}-1 = p^\alpha \quad \text{for some} \quad \alpha \in \mathbb{N}$$

or

$$\text{(iv) } 2^{b-1}-1 = q^\beta \quad \text{for some} \quad \beta \in \mathbb{N}.$$

In case (iii) we have $2p^\alpha = n = p^a - 1$ which is clearly impossible since $p \not| 1$.

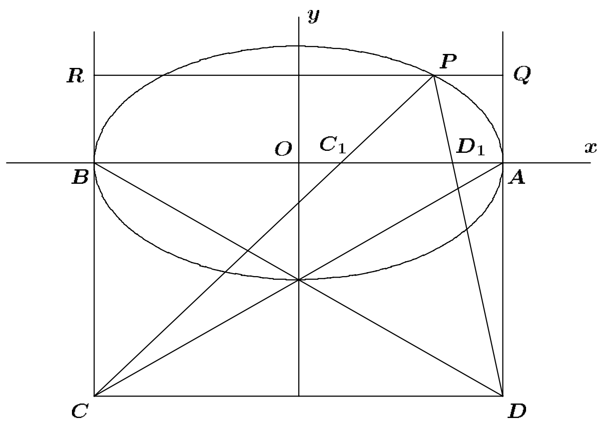
In case (iv) we have $2q^\beta = n = q^c - 3$. Clearly $\beta \le c$ and thus $q^\beta \mid 3$ which implies $q = 3$, $\beta = 1$. Thus $b = 3$ and $n = 6$, which is indeed a solution since $P(6) = 2^4 \times 3^3 \times 7$.

To summarize, $n(n+1)(n+2)(n+3)$ has exactly three prime divisors if and only if $n = 2$, 3 or 6.

6. An ellipse is drawn taking as major axis the biggest of the sides of a given rectangle, such that the ellipse passes through the intersection point of the diagonals of the rectangle.

Show that, if a point of the ellipse, external to the rectangle, is joined to the extreme points of the opposite side, then three segments in geometric progression are determined on the major axis.

Solutions by Pierre Bornsztein, Courdimanche, France; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.



An equation of the ellipse E is $b^2x^2 + a^2y^2 - a^2b^2 = 0$. The rectangle $ABCD$ has sides $AB = CD = 2a$, $AD = BC = 2b$. A point P of E is $P(a\cos\varphi, b\sin\varphi)$, with $\sin\varphi > 0$. Let the line through P parallel to AB meet BC and AD in R and Q, respectively. Let PD and PC meet the major axis at D_1, C_1 respectively.

From $\triangle DPQ$ and $\triangle DD_1A$, we have

$$D_1A : PQ \;=\; 2b : (2b + b\sin\varphi)\,, \qquad D_1A : a(1-\cos\varphi) \;=\; 2 : (2+\sin\varphi)\,;$$

$$\Longrightarrow \quad D_1A \;=\; \frac{2a(1-\cos\varphi)}{2+\sin\varphi}\,. \tag{1}$$

In the same way, from $\triangle PCD$ and $\triangle PC_1D_1$, we have

$$C_1D_1 \;=\; \frac{2a\sin\varphi}{2+\sin\varphi}\,, \tag{2}$$

and from $\triangle CPR$ and $\triangle CC_1B$, we have

$$BC_1 = \frac{2a(1+\cos\varphi)}{2+\sin\varphi}\,. \tag{3}$$

From (1), (2) and (3) $D_1A : C_1D_1 : BC_1 = (1-\cos\varphi) : \sin\varphi : (1+\cos\varphi)$. And as $\sin^2\varphi = 1-\cos^2\varphi$ the three segments are in geometric progression indeed.

7. Let $a \in \mathbb{R}$ be given. Find the real numbers $x_1, \ldots, x_n$ which satisfy the system of equations

$$\begin{aligned}
x_1^2 + ax_1 + \left(\tfrac{a-1}{2}\right)^2 &= x_2\,,\\
x_2^2 + ax_2 + \left(\tfrac{a-1}{2}\right)^2 &= x_3\,,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\\
x_{n-1}^2 + ax_{n-1} + \left(\tfrac{a-1}{2}\right)^2 &= x_n\,,\\
x_n^2 + ax_n + \left(\tfrac{a-1}{2}\right)^2 &= x_1\,.
\end{aligned}$$

Solutions by Pierre Bornsztein, Courdimanche, France; by Pavlos Maragoudakis, Pireas, Greece; and by Panos E. Tsaoussoglou, Athens, Greece. We give the write-up of Maragoudakis, although the methods were the same.

The first equation becomes

$$x_1^2 + (a-1)x_1 + \left(\frac{a-1}{2}\right)^2 = x_2 - x_1 \quad \text{or} \quad \left(x_1 + \frac{a-1}{2}\right)^2 = x_2 - x_1\,.$$

Similarly

$$\begin{aligned}
\left(x_1 + \tfrac{a-1}{2}\right)^2 &= x_2 - x_1\,,\\
\left(x_2 + \tfrac{a-1}{2}\right)^2 &= x_3 - x_2\,,\\
\cdots\cdots\cdots\cdots &\\
\left(x_{n-1} + \tfrac{a-1}{2}\right)^2 &= x_n - x_{n-1}\,,\\
\left(x_n + \tfrac{a-1}{2}\right)^2 &= x_1 - x_n\,.
\end{aligned}$$

Adding, we get:

$$\left(x_1 + \tfrac{a-1}{2}\right)^2 + \left(x_2 + \tfrac{a-1}{2}\right)^2 + \cdots + \left(x_{n-1} + \tfrac{a-1}{2}\right)^2 + \left(x_n + \tfrac{a-1}{2}\right)^2 = 0\,.$$

So $x_1 = x_2 = \cdots = x_{n-1} = x_n = \frac{1-a}{2}$.

We next give solutions to two of the problems from Peru's Selection Test for the XII IberoAmerican Olympiad [1998: 131].

1. Given an integer $a_0 > 2$, the sequence $a_0, a_1, a_2, \ldots$ is defined as follows:

$$\begin{array}{ll} a_{k+1} = a_k(1+a_k), & \text{if } a_k \text{ is an odd number} \\ a_{k+1} = \frac{a_k}{2}, & \text{if } a_k \text{ is an even number.} \end{array}$$

Prove that there is a non-negative integer p such that $a_p > a_{p+1} > a_{p+2}$.

Solution by Pierre Bornsztein, Courdimanche, France.

On montre facilement par récurrence sur k que pour tout $k \in \mathbb{N}$, $a_k \in \mathbb{N}$.

Alors pour tout $k \geq 0$, $a_{k+1} \neq a_k$ si a_k est impair alors $a_{k+1} = a_k + a_k^2 > a_k$. Cependant si a_k est pair alors $a_{k+1} < a_k$. Donc $a_p > a_{p+1} > a_{p+2}$ si et seulement si $a_p \equiv 0 \pmod 4$.

Lemme : Soit $n \in \mathbb{N}$, $n \geq 2$. S'il existe $k \geq 0$ tel que $a_k = 2 + 2^n q$ où q impair, $q \geq 1$, alors il existe $p \geq k$ tel que $a_p > a_{p+1} > a_{p+2}$.

Preuve du Lemme : Par récurrence sur n

Pour $n = 2$, si $a_k = 2 + 4q$ où q impair, $q \geq 1$.

Alors

$$\begin{array}{rcl} a_{k+1} & = & 1 + 2q \quad \text{impair, d'où} \\ a_{k+2} & = & (1+2q)(2+2q) \;=\; 2(1+2q)(1+q) \end{array}$$

avec $1 + q$ pair, donc $a_{k+2} \equiv 0 \pmod 4$, et il suffit de choisir $p = k + 2$.

Soit $n \geq 2$ fixé. Supposons la propriété vraie pour ce n. Soit a_k tel que $a_k = 2 + 2^{n+1} q$, où q impair, $q \geq 1$. Alors

$$\begin{array}{rcl} a_{k+1} & = & 1 + 2^n q, \quad \text{impair et} \\ a_{k+2} & = & (1+2^n q)(2+2^n q) \;=\; 2 + 2^n(q + 2q + 2^n q^2)\,, \end{array}$$

et comme $q+2q+2^n q^2$ est impair, d'après l'hypothèse de récurrence il existe $p \geq k + 2 \geq k$ tel que $a_p \equiv 0 \pmod 4$, d'où le résultat pour $n + 1$.

Ce qui achève la recurrence et prouve le Lemme.

On distingue quatre cas.

1. $a_0 \equiv 0 \pmod 4$: il suffit de choisir $p = 0$.

2. $a_0 \equiv 3 \pmod 4$: alors a_0 est impair, d'où $a_1 = a_0(1 + a_0)$ et $1 + a_0 \equiv 0 \pmod 4$. Donc $a_1 \equiv 0 \pmod 4$, et il suffit de choisir $p = 1$.

3. $a_0 \equiv 2 \pmod 4$: alors il existe $n \geq 2$, il existe q, impair, $q \geq 1$ (car $a_0 > 2$) tel que $a_0 = 2 + 2^n q$. Le Lemme permet de conclure.

4. $a_0 \equiv 1 \pmod 4$: alors $a_0 = 1 + 4k$, $k \geq 1$, car $a_0 > 2$. D'où, a_0 impair et

$$\begin{array}{rcl} a_1 & = & (1+4k)(2+4k) \\ & = & 2 + 4(3k + 4k^2) \end{array}$$

donc $a_1 \equiv 2 \pmod 4$ et on est ramené au $3^{\text{ème}}$ cas.

Donc, dans tous les cas, il existe $p \geq 0$ tel que $a_p \equiv 0 \pmod 4$, c.à.d. tel que $a_p > a_{p+1} > a_{p+2}$.

2. A positive integer is called "almost-triangular" if the number is itself triangular or is the sum of different triangular numbers. How many almost-triangular numbers are there in the set $\{1, 2, 3, \ldots, 1997\}$?

Note: The triangular numbers are a_1, a_2, a_3, $\ldots$, a_k, $\ldots$, where $a_1 = 1$, and $a_k = k + a_{k-1}$, for all $k \geq 2$.

Solution by Pierre Bornsztein, Courdimanche, France.

On pose $T_i = i^{\text{ème}}$ nombre triangulaire $= \frac{i(i+1)}{2}$, $i \geq 1$. On a $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, $T_5 = 15$, $T_6 = 21$, $T_7 = 28$, $T_8 = 36$, $T_9 = 45$, $T_{10} = 55$, $T_{11} = 66$.

On vérifie facilement que si $n \leq 33$ alors n est presque triangulaire sauf pour $n \in I = \{2, 5, 8, 12, 23, 33\}$.

Supposons $n \geq 34$.

On vérifie que : si $34 \leq n \leq 66$ alors n convient et que $n = \sum T_i$ avec $T_i < 66$.

On en déduit que si $n = 66 + a = T_{11} + a$, où $a \in \{1, \ldots, 66\}$ et $a \notin I$, alors n convient (on est sûr que l'on peut décomposer a sans utiliser T_{11}). On vérifie que si $n = 66 + a$ avec $a \in I$, alors n convient puisque :

$$\begin{aligned} 68 &= 55 + 10 + 3 & \qquad 78 &= 36 + 28 + 10 + 3 + 1 \\ 71 &= 55 + 10 + 6 & 89 &= 55 + 28 + 6 \\ 74 &= 55 + 10 + 6 + 3 & 99 &= 55 + 28 + 10\,. \end{aligned}$$

Par conséquent si $n \leq 132$ alors n convient et $n = \sum T_i$ avec $T_i \leq 66$. Or $T_{13} = 91$ et donc tout $n = 91 + a$ convient pour $a \in \{34, \ldots, 132\}$. Donc si $n \leq 223$, alors n convient et $n = \sum T_i$ avec $T_i \leq 91$.

Or $T_{19} = 190$. On en déduit que tout $n = 190 + a$, où $a \in \{34, \ldots, 223\}$ convient. Donc si $n \leq 413$ alors n convient et $n = \sum T_i$ avec $T_i \leq 190$.

Or $T_{25} = 325$. Donc tout $n = 325 + a$ où $a \in \{34, \ldots, 413\}$ convient. Donc si $n \leq 738$ alors n convient et $n = \sum T_i$ avec $T_i \leq 325$.

Or $T_{37} = 703$. Donc tout $n = 703 + a$ où $a \in \{34, \ldots, 738\}$ convient, donc si $n \leq 1441$ alors n convient et $n = \sum T_i$ avec $T_i \leq 703$. Or $T_{52} = 1378$.

Donc tout $n = 1378 + a$ où $a \in \{34, \ldots, 1441\}$ convient, donc si $n \leq 2819$ alors n convient.

Finalement : les seuls $n \in \{1, \ldots, 1997\}$ qui ne sont pas presque triangulaires sont 2, 5, 8, 12, 23, 33. Il y a donc 1991 nombres presque triangulaires dans $\{1, \ldots, 1997\}$.

Remarque : "Tout nombre $n \geq 34$ est presque triangulaire". Ce résultat serait dû à R. Graham et P. Erdős, "L'enseignement mathématiques" 1980.

We now turn to solutions from the readers to problems for the Third Grade of the 38$^{\text{th}}$ Mathematics Competition of the Republic of Slovenia [1998: 132].

1. Let n be a natural number. Prove: if $2n+1$ and $3n+1$ are perfect squares, then n is divisible by 40.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; by Pavlos Maragoudakis, Pireas, Greece; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Amengual Covas.

Since $40 = 2^3 \cdot 5$, it is sufficient to prove that n is divisible by 8 and 5. Set

$$2n+1 \;=\; x^2 \tag{1}$$

and

$$3n+1 \;=\; y^2 \tag{2}$$

where x, y are natural numbers.

We note that the number x^2 is odd, and thus also the number x is odd; consequently $x = 2a+1$, where a is a natural number.

Equation (1) implies the equality $2n+1 = (2a+1)^2$, whence $n = 2a^2 + 2a$.

The number n, as the sum of two even numbers, is even. It follows from equation (2) that the number y^2 is odd, and thus also the number y is odd; consequently $y = 2b+1$, where b is a natural number.

1° We subtract (1) from (2) and find that

$$n \;=\; y^2 - x^2 \;=\; (2b+1)^2 - (2a+1)^2 \;=\; 4(b+a+1)(b-a)\,.$$

Since both of $b+a$, $b-a$ are either even or odd, one of the numbers $b+a+1$, $b-a$ is even, whence the number n is divisible by 8.

2° We can eliminate n between (1) and (2) to get

$$3x^2 - 2y^2 \;=\; 1\,.$$

Since the square of an odd number ends in 1, 5, or 9, each of the numbers x^2 and y^2 ends in 1, 5, or 9. Therefore the number $3x^2$ ends in 3, 5, or 7 and $2y^2$ ends in 2, 0 or 8.

Since $3x^2 - 2y^2 = 1$, $3x^2$ must have ended in 3 and $2y^2$ must have ended in 2, whence both of the numbers x^2 and y^2 end in 1.

Hence $n = y^2 - x^2$ ends in 0, and consequently n is divisible by 5.

Comment: A related problem appears in Arthur Engel's *Problem-Solving Strategies*, Springer-Verlag 1998, page 131. If $2n + 1$ and $3n + 1$ are squares, then $5n + 3$ is not a prime.

2. Show that $\cos(\sin x) > \sin(\cos x)$ holds for every real number x.

Solutions by Mohammed Aassila, Univérsité Louis Pasteur, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the two solutions of Maragoudakis.

First Solution. $-\frac{\pi}{2} < -1 \leq \sin x \leq 1 < \frac{\pi}{2} \Longrightarrow \cos(\sin x) > 0$ for $x \in \mathbb{R}$.

If $\cos x \leq 0$, then $\cos x \in (-\frac{\pi}{2}, 0]$, so $\sin(\cos x) \leq 0 < \cos(\sin x)$.

If $\cos x > 0$, then $\cos x \in (0, \frac{\pi}{2})]$. It is known that $\sin y < y$ for $y \in (0, \frac{\pi}{2})$. So

$$\sin(\cos x) \; < \; \cos x \, . \tag{1}$$

Also $\cos y \geq 1 - \frac{y^2}{2}$ for $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Since $\sin x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have

$$\cos(\sin x) \; \geq \; 1 - \frac{\sin^2 x}{2} \; = \; \frac{1 + \cos^2 x}{2} \, . \tag{2}$$

But, using (1) and (2), $\dfrac{1 + \cos^2 x}{2} \geq \cos x \Longrightarrow \cos(\sin x) > \sin(\cos x)$.

Second Solution.

$$\begin{aligned}
&\cos(\sin x) - \sin(\cos x) \\
&\quad = \cos(\sin x) - \cos\left(\frac{\pi}{2} - \cos x\right) \\
&\quad = 2\sin\left(\frac{\sin x - \cos x + \frac{\pi}{2}}{2}\right)\sin\left(\frac{\frac{\pi}{2} - \sin x - \cos x}{2}\right) \\
&\quad = 2\sin\left(\frac{\sqrt{2}}{2}\sin\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}\right)\left(\sin\left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}\sin\left(x + \frac{\pi}{4}\right)\right)\right) \, .
\end{aligned}$$

It is easy to prove that

$$0 \; < \; \frac{\pi}{4} - \frac{\sqrt{2}}{2} \; \leq \; \frac{\sqrt{2}}{2}\sin\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4} \, ,$$

$$\frac{\pi}{4} - \frac{\sqrt{2}}{2}\sin\left(x + \frac{\pi}{4}\right) \; \leq \; \frac{\pi}{4} + \frac{\sqrt{2}}{2} < \frac{\pi}{2} \, ,$$

so that

$$\sin\left(\frac{\sqrt{2}}{2}\sin\left(x-\frac{\pi}{4}\right)+\frac{\pi}{4}\right), \qquad \sin\left(\frac{\pi}{4}-\frac{\sqrt{2}}{2}\sin\left(x+\frac{\pi}{4}\right)\right) \;>\; 0\,.$$

3. The polynomial $p(x) = x^3 + ax^2 + bx + c$ has only real roots. Show that the polynomial $q(x) = x^3 - bx^2 + acx - c^2$ has at least one non-negative root.

Solutions by Pierre Bornsztein, Courdimanche, France; by Pavlos Maragoudakis, Pireas, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

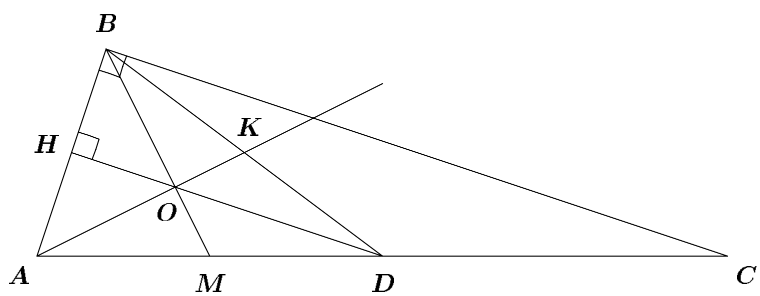
Les hypothèses sont inutiles car :

$$\lim_{x\to+\infty} q(x) \;=\; +\infty \quad \text{et} \quad q(0) \;=\; -c^2 \le 0\,.$$

Donc, d'après le théorème des valeurs intermediaires (q est continue sur $\mathbb{R}^+$), il existe $\alpha \in \mathbb{R}^+$ tel que $q(\alpha) = 0$.

4. Let the point D on the hypotenuse AC of the right triangle ABC be such that $|AB| = |CD|$. Prove that the bisector of the angle $\angle A$, the median through B and the altitude through D of the triangle ABD have a common point.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Courdimanche, France; by Masoud Kamgarpour, Carson Graham Secondary School, North Vancouver, BC; and by Pavlos Maragoudakis, Pireas, Greece. We give the solution of Kamgarpour.



Let M be the foot of the median from B, K the point of intersection of the angle bisector with BD and H the foot of the altitude from D in $\triangle ABD$.

We use Ceva's Theorem to prove that AM, DH and AK have a common point.

$$\frac{MA}{MD} \;=\; 1 \qquad \text{(because } BM \text{ is a median)}$$

$$\frac{KD}{KB} \;=\; \frac{AD}{AB} \qquad \text{(Bisector Property)}$$

$$\triangle AHD \sim \triangle ABC \quad \Longrightarrow \quad \frac{HB}{HA} = \frac{DC}{DA} = \frac{AB}{AD}.$$

Thus

$$\frac{MA}{MD} \cdot \frac{KD}{KB} \cdot \frac{HB}{HA} = \frac{AD}{AB} \cdot \frac{AB}{AD} = 1.$$

That completes the *Corner* this issue. Send me your Olympiad contests, your nice solutions, and generalizations.

Challenge Answer

In the February 1999 issue [1999: 32], we issued the challenge:

What is the 10^{th} term in the following sequence, and why?

n	x_n
0	0
1	$\frac{1}{16}\left(\sqrt{30}+\sqrt{10}-\sqrt{6}-\sqrt{2}-2\left(\sqrt{3}-1\right)\sqrt{5+\sqrt{5}}\right)$
2	$\frac{1}{8}\left(\sqrt{30-6\sqrt{5}}-\sqrt{5}-1\right)$
3	$\frac{1}{8}\left(\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}}\right)$
4	$\frac{1}{8}\left(\sqrt{10+2\sqrt{5}}-\sqrt{15}+\sqrt{3}\right)$
5	$\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$
6	$\frac{1}{4}\left(\sqrt{5}-1\right)$
7	$\frac{1}{16}\left(2\left(\sqrt{3}+1\right)\sqrt{5-\sqrt{5}}-\sqrt{30}+\sqrt{10}-\sqrt{6}+\sqrt{2}\right)$
8	$\frac{1}{8}\left(\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}}\right)$
9	$\frac{1}{8}\left(2\sqrt{5+\sqrt{5}}-\sqrt{10}+\sqrt{2}\right)$
10	?

The answer, sent in by Luyun Zhong-Qiao, Columbia International College, Hamilton, Ontario, is $\frac{1}{2}$. He notes that $T_n = \sin(3n^\circ)$.
[Ed.: note there was a typo in T_7.]