THE SKOLIAD CORNER

No. 35

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We begin this issue with the problems of the first round of the Old Mutual Mathematical Olympiad 1992. Thanks go to John Grant McLoughlin of the Faculty of Education, Memorial University of Newfoundland, for collecting this contest and forwarding it for use in the Corner.

OLD MUTUAL MATHEMATICAL OLYMPIAD 1992

Time: 1 hour

1. $(0.4)^2 - (0.1)^2$ equals:
(a) 0.04  (b) 1.5  (c) 0.15  (d) 0.6  (e) 0.06

2. The angles of a triangle are in the ratio 2 : 3 : 4. The size of the largest angle in degrees is:
(a) 40°  (b) 80°  (c) 45°  (d) 90°  (e) 72°

3. Reduced to the lowest terms $\frac{a^2 - 2bc}{ab} - \frac{ab - b^2}{ab - a}$ is equal to:
(a) $\frac{a^2 - 2bc}{ab}$  (b) $a - 2b$  (c) $a^2$  (d) $\frac{c}{b}$  (e) none of these

4. If the radius of a circle is increased by 100%, the area is increased by:
(a) 100%  (b) 200%  (c) 300%  (d) 400%  (e) 10000%

5. The area of the largest triangle that can be inscribed in a semicircle of radius $r$ is:
(a) $r^2$  (b) $r^3$  (c) $2r^2$  (d) $2r^3$  (e) $\frac{1}{2}r^2$

6. A manufacturer built a machine which will address 500 envelopes in 8 minutes. He wishes to build another machine so that when both are operating together they will address 500 envelopes in 2 minutes. The equation used to find how many minutes $x$, it would require the second machine to address 500 envelopes alone is:
(a) $8 - x = 2$  (b) $\frac{500}{8} + \frac{500}{x} = 500$  (c) $\frac{1}{8} + \frac{1}{x} = \frac{1}{2}$

(d) $\frac{5}{8} + \frac{x}{8} = 1$  (e) none of these.

7. The expression $\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{4}}$ is equal to:
(a) $\sqrt{\frac{2}{12}}$  (b) $\sqrt{-1}$  (c) $\frac{1}{\sqrt{3}}$  (d) $\sqrt{\frac{1}{7}}$  (e) $-1$
8. Two circles of equal size are contained in a rectangle as shown.

If the radius of each circle is 1 cm, then the area of the shaded portion in cm² is:
(a) π - 4  (b) 4 - 2π  (c) 8 - π  (d) 8 - 2π  (e) 4

9. The yearly changes in the population of a town for four consecutive years are respectively 10% increase, 10% increase, 10% decrease, 10% decrease. The net change over four years to the nearest percent is:
(a) -2  (b) -1  (c) 0  (d) 1  (e) 12

10. If \( \log_{10} 2 = a \) and \( \log_{10} 3 = b \), then \( \log_{10} 12 \) equals:
(a) \( a^2 + b \)  (b) \( 2a + b \)  (c) \( 4b \)  (d) \( 2ab \)  (e) \( a^2b \)

11. Of the following, which is the best approximation to the positive square root of \( \frac{1992}{10000} \):
(a) 0.0045  (b) 0.014  (c) 0.0446  (d) 0.1411  (e) 0.4463

12. A circle and a square have the same perimeter. Then:
(a) their areas are equal  
(b) the area of the circle is \( \frac{\pi}{4} \) times that of the square  
(c) the area of the circle is \( \frac{\pi}{2} \) times that of the square  
(d) the area of the circle is \( \pi \) times the area of the square  
(e) none of these

13. The square of an integer is called a perfect square; for example, 4, 9, 16, 25 are all perfect squares. If \( n \) is a perfect square, then the next perfect square greater than \( n \) is:
(a) \( n + 1 \)  (b) \( n^2 + 1 \)  (c) \( n^2 + 2n + 1 \)  (d) \( n^2 + n \)  (e) \( n + 2\sqrt{n} + 1 \)

14. The last digit of \( 7^{1992} \) is:
(a) 1  (b) 2  (c) 6  (d) 7  (e) 9

15. The figure below shows a triangle \( PQR \), where \( PQ > QR \). The altitude to the base \( PQ \), divides \( PQR \) into two parts \( R_1 \) and \( R_2 \).
16. A cylinder is such that the area of its curved surface is twice its volume. Then its radius is:

(a) \( \frac{1}{2} \) \hspace{1cm} (b) 2 \hspace{1cm} (c) \( \frac{\pi}{2} \) \hspace{1cm} (d) \( \sqrt{\frac{2}{\pi}} \) \hspace{1cm} (e) 1

17. \( (-\frac{1}{8})^{-1/3} \) is equal to:

(a) -2 \hspace{1cm} (b) 2 \hspace{1cm} (c) -\( \frac{1}{2} \) \hspace{1cm} (d) 8 \hspace{1cm} (e) 83

18. A man wishes to travel 1992 kilometres at an average speed of 100 kmh\(^{-1}\). He travels the first half of this distance at 50 kmh\(^{-1}\). How fast must he go over the remaining half?

(a) 150 kmh\(^{-1}\) \hspace{1cm} (b) 200 kmh\(^{-1}\) \hspace{1cm} (c) 400 kmh\(^{-1}\) \hspace{1cm} (d) 496 kmh\(^{-1}\) \hspace{1cm} (e) none of these

19. Six numbers are in arithmetic progression. The sum of the first and last is 5. Then the sum of the third and fourth is:

(a) 5 \hspace{1cm} (b) 6 \hspace{1cm} (c) 7 \hspace{1cm} (d) 12 \hspace{1cm} (e) impossible to determine

20. If \( n \) means \( n^n \), so that \( 3 = 3^3 = 27 \), then \( [2] \) is:

(a) 2 \hspace{1cm} (b) 4 \hspace{1cm} (c) 16 \hspace{1cm} (d) 64 \hspace{1cm} (e) 256

Last issue we gave the problems of the Final Round of the Old Mutual Mathematical Olympiad 1991. My thanks go to John Grant McLoughlin of the Faculty of Education, Memorial University of Newfoundland for forwarding the questions and "official" solutions to the questions of paper 2, and problem 4 of paper 1.

OLD MUTUAL MATHEMATICAL OLYMPIAD 1991
Final Paper 1
Solutions (Time: 2 hours)

1. In the figure shown \( ABC \) and \( AEB \) are semicircles and \( F \) is the mid-point of \( AC \) and \( AF = 1 \) cm. Find the area of the shaded region.
Solution. \( AB \) is the diameter of the semicircle \( AEB \) and the hypotenuse of isosceles right triangle \( AFB \), so \( AB = \sqrt{2} \), and the area of semicircle \( AEB \) is \( \frac{1}{2}\pi (\sqrt{2}/2)^2 = \frac{\pi}{4} \). Now semicircle \( ABC \) has area \( \frac{1}{2}\pi \), while right triangle \( AFB \) has area \( \frac{1}{2} \). Hence the area of the circular sector \( ADB \) and the triangle \( AFB \) is is \( \frac{\pi}{4} \), while \( \frac{1}{2} \). Hence the area of the circular sector \( ADB \) and the triangle \( AFB \) is is \( \frac{\pi}{4}, \frac{1}{2} \), and the shaded area is \( \frac{\pi}{4} - \left( \frac{\pi}{4} - \frac{1}{2} \right) = \frac{1}{2} \).

2. What is the value of \( \sqrt{17 - 12\sqrt{2}} + \sqrt{17 + 12\sqrt{2}} \) in its simplest form?

Solution. Let \( x = \sqrt{17 - 12\sqrt{2}} + \sqrt{17 + 12\sqrt{2}} \). Now

\[
x^2 = 17 - 12\sqrt{2} + 17 + 12\sqrt{2} + 2\sqrt{17 - 12\sqrt{2}} \sqrt{17 + 12\sqrt{2}}
\]
\[
= 34 + 2\sqrt{(17 - 12\sqrt{2})(17 + 12\sqrt{2})}
\]
\[
= 34 + 2\sqrt{(17)^2 - (12)^2(\sqrt{2})^2}
\]
\[
= 34 + 2\sqrt{289 - 288} = 34 + 2 = 36.
\]

So \( x = -6 \), or \( x = 6 \). Since \( x > 0 \), \( x = 6 \).

3. In a certain mathematics examination, the average grade of the students passing was \( x \)%, while the average of those failing was \( y \)% \( y \). The average of all students taking the examination was \( z \)% \( z \). Find, in terms of \( x \), \( y \) and \( z \), the percentage who fail.

Solution. The total of the scores for those passing the exam is \( \frac{x}{100} \cdot p \), where \( p \) is the number of students who pass. Similarly, with \( f \) the number who fail \( \frac{y}{100} \cdot f \) is the total of the failing scores. The total of all students scores is \( \frac{z(p+f)}{100} = \frac{xp}{100} + \frac{yf}{100} \) so \( zp + zf = xp + yf \),

\[
(z - x)p = (y - z)f \text{ and } p = \frac{y - x}{z - x}f.
\]

Now

\[
p + f = \frac{y - x}{z - x}f + f = \left( \frac{y - z}{z - x} + 1 \right) f
\]
\[
= \frac{y - x}{z - x}f.
\]

The percentage who fail is

\[
\frac{f}{p + f} \times 100 = \frac{f}{p} \times \frac{100}{\frac{y - z}{z - x}f} = \frac{z - x}{y - x} \times 100.
\]
4. In the figure shown $\overline{AB} = \overline{AD} = \sqrt{130}$ cm and $\overline{BEDC}$ is a square.

Also the area of $\triangle AEB = \text{area of square } \overline{BEDC}$.

Find the area of $\overline{BEDC}$.

**Solution.** Let $\overline{BOD}$ be a diagonal of the square, with $O$ its centre. Now triangles $\triangle BEA$ and $\triangle BCE$ have the same altitude, from $B$, namely $\overline{BO}$. Since the area of square $\overline{BEDC}$ equals the area of triangle $\triangle AEB$, and the area of the square is twice the area of $\triangle BCA$, we have $\overline{AE} = 2\overline{EC}$. Clearly $\overline{BO} = \overline{EO} = \frac{1}{2}\overline{EC}$, so $\overline{AE} = 4\overline{EO}$.

By Pythagoras, $\overline{AO}^2 + \overline{BO}^2 = \overline{AB}^2 = 130$. Thus,

$$(\overline{AE} + \overline{EO})^2 + \overline{EO}^2 = 130$$

or

$$(4\overline{EO} + \overline{EO})^2 + \overline{EO}^2 = 130$$

$$25\overline{EO}^2 = 130$$

$$\overline{EO}^2 = \frac{130}{25} = 5.$$ 

But the area of the square is just twice the area of the square with side length $\overline{EO}$, so the area is $2 \times 5 = 10$.

**Final Paper 2**

**Solutions (Time: 2 hours)**

1. If the pattern below of dot-figures is continued, how many dots will there be in the $100^{th}$ figure?

```
   O
  O O
 O O O
```

**Solution.** By inspection the number of dots in the $n^{th}$ figure is $2n + 1$. So the $100^{th}$ has 201 dots.
2. It is required to place a small circle in the space left by a large circle as shown. If the radius of the large one is $a$ and that of the small one is $b$, find the ratio $a/b$.

![Diagram](image)

**Solution.** Let $O$ be the centre of the larger circle and let $O_1$ be the centre of the small circle. The distance of $O$ from $P$, the point of intersection of the two tangents shown, is $\sqrt{2}a$, by Pythagoras, and similarly the distance from $O_1$ to $P$ is $\sqrt{2}b$. But $OO_1 = a + b$. So $\sqrt{2}b + a + b = \sqrt{2}a$, giving $\frac{a}{b} = \frac{\sqrt{2} + 1}{\sqrt{2} - 1} = 3 + 2\sqrt{2}$.

3. Find all solutions to the simultaneous equations

\[
\begin{align*}
x + y &= 2, \\
xy - z^2 &= 1,
\end{align*}
\]

and prove that there are no other solutions.

**Solution.** We find all real solutions. The first observation is that

\[
x y \geq 1
\]  

in order that $xy - z^2 = 1$. Thus $x$ and $y$ are both positive or both negative. But for $x + y = 2$, $x$ and $y$ must both be positive. Now for any $x, y > 0$, $\sqrt{xy} \leq \frac{x+y}{2}$ (this is just the Arithmetic-Geometric Mean Inequality). Thus $\sqrt{xy} \leq \frac{3}{2} = 1$, and $xy \leq 1$. From (1) we have $xy = 1$ and $z = 0$. But the AM/GM inequality is strict unless $x = y$ so $x = y = 1$ and $z = 0$ is the only solution.

4. If $a, b, c$ and $d$ are numbers such that

\[
\begin{align*}
a + b &< c + d, \\
b + c &< d + e, \\
c + d &< e + a, \\
\text{and } d + c &< a + b.
\end{align*}
\]

Prove that the largest number is $a$ and the smallest is $b$. 


Solution.

From \( d + e < a + b \) and \( a + b < c + d \) we see \( d + e < c + d \) so \( e < c \). \hspace{1cm} (1)

From \( a + b < c + d \) and \( c + d < e + a \) we get \( b < e \). \hspace{1cm} (2)

From \( b + c < d + e \) and \( d + e < a + b \) we get \( c < a \). \hspace{1cm} (3)

Now from \( d + e < a + b \) we read \( d - a < b - e < 0 \) by (2), so \( d < a \).

From (1), (2) and (3) \( b < e < c < a \), so with \( d < a \), \( a \) is the largest number.

Similarly from \( a + b < c + d \) we see \( 0 < a - c < d - b \) from (3) and \( b < d \). This makes \( b \) the smallest number.

5. The diagram below \( [\text{rotated through 90°}] \) shows a container whose lower part is a hemisphere and whose upper part is a cylinder.

![Diagram](image)

The cylindrical part has internal diameter of 20 cm and is 220 cm long. Water is poured into it and rises to a height of 20 cm in the cylindrical part. The top is then sealed with a flat cover and the container is turned upside down. The water is now 200 cm high in the cylindrical part.

(i) Calculate the volume of the hemisphere in terms of \( \pi \).

(ii) Find the total height of the container.

[Note: The volume of a sphere of radius \( R \) is \( \frac{4}{3} \pi R^3 \).]

Solution. (Diagrams are not to scale)

![Diagram](image)

The volume \( V \) of water is \( V = \pi \cdot 10^2 \cdot 200 \), from the information about the upside down figure.

But \( V = \frac{1}{2} \cdot \frac{4}{3} \pi r^3 + \pi \cdot 10^2 \cdot 20 \) from the initial configuration.
So the volume of the hemisphere \( V_H \) is
\[
V_H = \frac{1}{2} \frac{4}{3} \pi r^3 = \pi \cdot 10^2 \cdot 200 - \pi \cdot 20^2 \cdot 20 = \pi \times 18000.
\]

Now
\[
\frac{2}{3} \pi r^3 = \pi \times 18000 \\
r^3 = 27000 \\
r = 30.
\]

The total height is \( 220 + 30 = 250 \) cm

That completes the *Skoliad Corner* for this issue. Please send me contest materials and suggestions for other features of the Corner.

**Challenge**

What is the 10\(^{th}\) term in the following sequence, and why?

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{\pi} (\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2} - 2 (\sqrt{5} - 1) \sqrt{5} + \sqrt{3}) )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{\pi} (\sqrt{30} - 6\sqrt{5} - \sqrt{5} - 1) )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{\pi} (\sqrt{10} + \sqrt{2} - 2\sqrt{5} - \sqrt{3}) )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{\pi} (\sqrt{10} + 2\sqrt{5} - \sqrt{15} + \sqrt{3}) )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{4} (\sqrt{6} - \sqrt{2}) )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1}{4} (\sqrt{5} - 1) )</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{1}{\pi} (2 (\sqrt{5} - 1) \sqrt{5} - \sqrt{5} - \sqrt{30} + \sqrt{10} - \sqrt{6} + \sqrt{3}) )</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{1}{\pi} (\sqrt{15} + \sqrt{5} - \sqrt{10} - 2\sqrt{5}) )</td>
</tr>
<tr>
<td>9</td>
<td>( \frac{1}{\pi} (2\sqrt{5} + \sqrt{5} - \sqrt{10} + \sqrt{2}) )</td>
</tr>
<tr>
<td>10</td>
<td>?</td>
</tr>
</tbody>
</table>