THE ACADEMY CORNER
No. 22
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APICS Mathematics Competition 1998
held at Saint Mary's University, Halifax, Nova Scotia,

Rules:

- Teams of two are to work in cooperation and to submit one set of answers each.
- No notes, calculators, or other such aids are permitted.
- You may not communicate with noncontestants (except invigilators) or other teams.
- There are nine questions.

1. Fred and Cathy play the following game. They are given the polynomial \( f(x) = ax^3 + bx^2 + cx + d \). They take turns, Cathy first, in replacing \( a \), then \( b \), then \( c \) and finally \( d \) with positive integers. Fred wins if the resulting polynomial has at least two distinct roots. Who should win and what is the winning strategy?

2. Define the integer sequence \( \{T_n\} \) by \( T_0 = 0 \), \( T_1 = 1 \), \( T_2 = 2 \) and \( T_{n+1} = T_n + T_{n-1} + T_{n-2} \) \((n \geq 2)\). Compute

\[
S := \sum_{n=0}^{\infty} \frac{T_n}{2^n}.
\]

3. Let \( X_1, X_2, \ldots, X_n \) be independent, integer-valued random variables with \( p = \text{probability}\{X_k \text{ is even}\} \). Form the sum \( S_n \) of the random variables. Show that the probability that the sum is even is

\[
\left[1 + (2p - 1)^n\right]/2.
\]
4. Show that there do not exist four points in the Euclidean plane such that the pairwise distances between them are all odd integers.

5. If \( \{a_n\} \) is a sequence of positive integers such that
\[
\lim_{n \to \infty} \frac{a_n}{a_1 + a_2 + \cdots + a_n} = 0,
\]
show that there is a sequence \( \{b_n\} \) of positive integers such that for every positive integer \( n \geq 2 \)
\[
\frac{b_n}{b_1 + b_2 + \cdots + b_n} \leq \frac{1}{3},
\]
and for some positive integer \( N \) we have \( a_n = b_n \) for all \( n \geq N \).

6. For \( a > 1 \) evaluate
\[
\int_0^a x a^{(-|\log_a x|)} \, dx,
\]
where \([t]\) denotes the greatest integer less than or equal to \( t \).

7. Let \( ABCD \) be a cyclic quadrilateral, inscribed in a circle \( \omega \). Let \( A', B', C', D' \) be the points where the tangents at \( A \) and \( B \), at \( B \) and \( C \), at \( C \) and \( D \) and at \( D \) and \( A \), respectively, intersect. Prove that the lines \( AC, BD, A'C' \) and \( B'D' \) are concurrent; that is, they intersect at one point.

8. The expression
\[
\frac{\left(\ldots\left((x - 2)^2 - 2\right)^2 - 2\right)^2 - \ldots - 2\right)^2}{n - \text{times}}
\]
is multiplied out and coefficients of equal powers are collected. Find the coefficient of \( x^2 \).

9. Let \( f(n) = 2n^2 + 14n + 25 \). We see that \( f(0) = 25 = 5^2 \). Find two positive integers \( n \) such that \( f(n) \) is a perfect square.

The winning teams were:

1. Ian Caines and Alex Fraser — Dalhousie University;
2. Dave Morgan and Shannon Sullivan — Memorial University;
3. Tara Small and Kit Yan Wong — University of New Brunswick.

All six students received a free subscription to \textit{CRUX with MAYHEM}, in addition to some other prizes.

We will publish solutions later this year. Will your name be attached to a solution? Send them to me as soon as you can!

Thanks to Karl Dilcher, Dalhousie University, Halifax, Nova Scotia, for sending me the \LaTeX file.