MATHMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still mayhem@math.toronto.edu

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Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino  Mayhem High School Problems Editor,
Cyrus Hsia        Mayhem Advanced Problems Editor,
David Savitt      Mayhem Challenge Board Problems Editor.

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. We request that solutions from the previous issue be submitted in time for publication in issue 8 of 1999.

High School Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H229. Here is a simple way to remember how many books there are in the Bible. Remember that there are $x$ books in the Old Testament, where $x$ is a two-digit integer. Then multiply the digits of $x$ to get a new integer $y$, which is the number of books in the New Testament. Adding $x$ and $y$, you end up with 66, the number of books in the Bible. What are $x$ and $y$?
Solution by Katya Permiakova, student, Lisgar Collegiate Institute, Ottawa, Ontario.

Let \( x = 10a + b \), where \( a \) and \( b \) are integers satisfying the conditions \( 1 \leq a \leq 9 \) and \( 0 \leq b \leq 9 \). Then \( y = ab \). Now we are given that \( 10a + b + ab = 66 \). Rearranging terms and solving for \( b \), we get \( b(a + 1) = 66 - 10a \), so \( b = \frac{66 - 10a}{a + 1} = -10 + \frac{76}{a + 1} \). Now in order for \( b \) to be an integer, \( a + 1 \) must divide 76. The only positive divisors of 76 are 1, 2, 4, 19, 38, and 76. Since our choice for \( a \) is limited to the integers between 1 and 9, the only possibilities for \( a \) are 1 and 3 (since that gives us \( a + 1 = 2 \) and \( a + 1 = 4 \), respectively).

If \( a = 1 \), then we have \( b = -10 + \frac{76}{2} = 28 \), but this does not satisfy \( b \leq 9 \). However, if \( a = 3 \), then \( b = -10 + \frac{76}{4} = 9 \), and this is legitimate.

Hence \( x = 39 \) and \( y = 27 \).

Also solved by KEON CHOI, student, A.Y. Jackson Secondary School, North York, Ontario; LINO DEMASI, student, St. Ignatius High School, Thunder Bay, Ontario; and WENDY YU, student, Woburn Collegiate Institute, Scarborough, Ontario.

**H230.** Dick and Cy stand on opposite corners (on the squares) of a \( 4 \times 4 \) chessboard. Dick is telling too many bad jokes, so Cy decides to chase after him. They take turns moving one square at a time, either vertically or horizontally on the board. To catch Dick, Cy must land on the square Dick is on. Prove that:

(i) If Dick moves first, Cy can eventually catch Dick.

(ii) If Cy moves first, Cy can never catch Dick.

(Can you generalize this to a \( 2m \times 2n \) chessboard?)

**Solution.**

(i) Place coordinates on the board so that Cy is standing on \((0,0)\) and Dick is standing on \((3,3)\). We shall show that after a few moves, Cy can catch Dick on a turn. Regardless of what Dick does on his first two moves, Cy can move to \((1,1)\) after two moves. Now it is Dick's turn. At that time, Dick must be on \((1,3), (2,2), (3,1), \) or \((3,3)\). So if on his next move, Dick goes to either \((1,2)\) or \((2,1)\), Cy is standing one square away and so Cy moves into Dick's square on his next move, and catches Dick. So Dick must move to one of \((0,3), (2,3), (3,1), \) or \((3,0)\).

If Dick goes to \((0,3)\) or \((2,3)\), then Cy can go to \((1,2)\), and from here it is easy to see that Dick can last at most two moves before he gets caught (since Cy can trap him into a corner). If Dick goes to \((3,2)\) or \((3,0)\), then Cy can go to \((2,1)\), by the same argument, Cy can catch Dick. Thus no matter what, if Dick moves first, then Cy can eventually catch Dick.
(ii) Colour the $4 \times 4$ board in black and white, as in a regular chessboard (so adjacent squares are of different colours). Thus, if $(0, 0)$ is a black square, then $(3, 3)$ must be a black square as well. So for each move, if a person is on a square of a certain colour, then he will move to a square of the other colour.

Hence, Dick and Cy both start off on a black square. If Cy moves first, then Cy moves onto a white square (while Dick remains on a black square). Then Dick moves to some white square. Now Dick and Cy are both on white squares, and so on Cy's next move he must move onto a black square (while Dick remains on a white square). Thus whenever Dick is on a square of a certain colour, Cy is moving to a square of the other colour. And so, on any given move, Cy can never move to a square that Dick is currently on, and so Cy will not be able to catch Dick.

Also solved by KEON CHOI, student, A.Y. Jackson Secondary School, North York, Ontario; LINO DEMASI, student, St. Ignatius High School, Thunder Bay, Ontario; and WENDY YU, student, Woburn Collegiate Institute, Scarborough, Ontario.

H231. Let $O$ be the centre of the unit square $ABCD$. Pick any point $P$ inside the square other than $O$. The circumcircle of $PAB$ meets the circumcircle of $PCD$ at points $P$ and $Q$. The circumcircle of $PAD$ meets the circumcircle of $PBC$ at points $P$ and $R$. Show that $QR = 2 \cdot OP$.


Construct line $L$ through $O$ parallel to $AD$ and $BC$. All points on this line are the same distance from $A$ as from $B$, and the same distance from $C$ as
from $D$. Thus this line contains the centres of the circumcircles of $PAB$ and $PCD$. Hence, the line $L$ bisects segment $PQ$. So the point $Q$ must be the reflection of $P$ about the line $L$, and it follows that $OP = OQ$. Similarly, if we construct line $M$ through $O$ parallel to $AB$ and $CD$, then $R$ is the reflection of $P$ about the line $M$. Hence, $OP = OR$. Because $PQ$ and $PR$ are perpendicular (since the lines $L$ and $M$ are perpendicular), $PQR$ is a right-angled triangle. Furthermore, $OP = OQ = OR$, which implies that $O$ is the midpoint of the hypotenuse $QR$. Hence, we have $QR = OQ + OR = OP + OP = 2OP$, and so $QR = 2OP$, as desired.

Also solved by LINO DEMASI, student, St. Ignatius High School, Thunder Bay, Ontario.

H232. Lucy and Anna play a game where they try to form a ten-digit number. Lucy begins by writing any digit other than zero in the first place, then Anna selects a different digit and writes it down in the second place, and they take turns, adding one digit at a time to the number. In each turn, the digit selected must be different from all previous digits chosen, and the number formed by the first $n$ digits must be divisible by $n$. For example, 3, 2, 1 can be the first three moves of a game, since 3 is divisible by 1, 32 is divisible by 2 and 321 is divisible by 3. If a player cannot make a legitimate move, she loses. If the game lasts ten moves, a draw is declared.

(i) Show that the game can end up in a draw.

(ii) Show that Lucy has a winning strategy and describe it.

1. Solution by Lino Demasi, student, St. Ignatius High School, Thunder Bay, Ontario.

The number 3,816,547,290 has the property that the number formed by the first $n$ digits is divisible by $n$, for $n = 1, 2, 3, \ldots, 10$. Thus, if the moves are carried out in this order, then the game can end up in a draw.

Here is Lucy's winning strategy. First note that Anna must play an even digit on each of her moves. So Lucy's goal is to play as many even numbers as possible. So Lucy plays a 6 to start. There are three cases to be considered for Anna's second move:

(1) If Anna plays a 4 or a 2, then Lucy plays the other on the third move. Anna must now play an even number because her number now has to be divisible by 4, so if Anna plays an 8, then Lucy plays a 0, and Anna loses. Because on the sixth move, she would have to play an even number and there are none left. If Anna plays a 0, then Lucy plays a 5, and then Anna also loses because she must now play an even number on the sixth move, the only one of which is an 8, but neither 642,058 or 624,058 is divisible by 6.

(2) If Anna plays a 0, then Lucy plays a 9. Then Anna must play a 2 to make the four-digit number divisible by 4. Lucy then plays a 5. Anna must now play an 8 to make her number divisible by 6. Then Lucy can counter with a 3, since 6,092,583 is divisible by 7. The only even number Anna can now play is a 4, but 60,925,834 is not divisible by 8, so she loses.
(3) If Anna plays an 8, then Lucy plays a 4. Anna's only choice now is a 0. Then Lucy plays a 5. Now Anna's only choice is a 2, but 684052 is not divisible by 6, so she loses.

Note that this covers all the cases because Anna must play an even digit on the second move. Thus, Lucy can always force a win.

II. Solution by Wendy Yu, student, Woburn Collegiate Institute, Scarborough, Ontario.

As before, if the game is played in the following order: 3, 8, 1, 6, 5, 4, 7, 2, 9, 0, then the game will end up in a draw.

For Lucy's winning strategy, she can start off with a 4. Then, Anna must counter with an even number. So if she responds with a 2 or an 8, then Lucy's next move is a 0. If Anna's response is a 0 or a 6, then Lucy's next move is a 2. Now, in the case where the number 480 has been written, Anna cannot find a digit to make a four-digit number divisible by 4, so she immediately loses. In the other three cases, there is at least one digit that Anna can pick to remain in the game.

Thus, after four moves, if the game lasts that long, one of the following numbers will be on the board: 4028, 4208, 4620, or 4628. Then Lucy picks a 5, and in each of those four cases, it will be impossible for Anna to then make a move so that the new six-digit number is divisible by 6, since the digits she needs are all taken. Thus, if Lucy follows this strategy, she can always force a win.

Also solved by KEON CHOI, student, A.Y. Jackson Secondary School, North York, Ontario.

Advanced Solutions

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A205. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \leq x$ and $f(x + y) \leq f(x) + f(y)$ for all reals $x$ and $y$.

Solution.

We have

$$f(x) \leq x, \quad (1)$$

$$f(x + y) \leq f(x) + f(y). \quad (2)$$

Let $x = y = 0$ in (2). We have $f(0) \leq 2f(0)$ which implies $0 \leq f(0)$. But for $x = 0$ in (1), we have $f(0) \leq 0$ so $f(0) = 0$.

Now take $y = -x$ in (2) to get $f(0) \leq f(x) + f(-x)$ for all real $x$. In other words, $-f(x) \leq f(-x)$. But $f(-x) \leq -x$ by (1) so $-f(x) \leq -x$. Thus, $f(x) \geq x$ for all real $x$. But then $f(x + y) \geq f(x) + f(y) \geq x + y$ for all real $x$ and $y$.

Hence, $f(x) = x$ for all real $x$. This function satisfies the given conditions.
implies that \( f(x) \geq x \) for all real \( x \). Thus combining this with (1) we have \( f(x) = x \) for all real \( x \). Now it can be easily checked that this function satisfies the two conditions (1) and (2) in the problem.

**A206.** Let \( n \) be a power of 2. Prove that from any set of \( 2n - 1 \) positive integers, one can choose a subset of \( n \) integers such that their sum is divisible by \( n \).

*Solution.*

Let \( n = 2^m \), for some integer \( m \). We will prove this result by mathematical induction on \( m \). For \( m = 0 \), \( n = 1 \) and the result clearly holds.

Now assume the result is true for some arbitrary \( k \geq 0 \). In other words, for \( n = 2^k \), any set of \( 2^{k+1} - 1 \) positive integers has a subset of \( n \) integers whose sum is divisible by \( n \).

Now consider any set, \( A \), of \( 2^{k+2} - 1 \) positive integers. Let \( A_1 \) and \( A_2 \) be the two subsets of \( A \) consisting of the first \( 2^{k+1} - 1 \) integers and the last \( 2^{k+1} - 1 \) integers respectively. By the induction hypothesis, each of these sets has \( 2^k \) integers whose sum is divisible by \( 2^k \). Call the subset of \( 2^k \) numbers, from \( A_1 \), \( B_1 \) and the subset of \( 2^k \) numbers from \( A_2 \), \( B_2 \). Now call the set of integers remaining from \( A \) when the \( 2^{k+1} \) integers from \( B_1 \) and \( B_2 \) are removed from \( A \). Now \( A_3 \) also has \( (2^{k+2} - 1) - 2(2^k) = 2^{k+1} - 1 \) elements, so again by the induction hypothesis it has a subset, call it \( B_3 \), of size \( 2^k \) whose sum is divisible by \( 2^k \).

It remains to show that two of the three sets \( B_1 \), \( B_2 \), and \( B_3 \) can be combined to form a set of \( 2^{k+1} \) integers whose sum is divisible by \( 2^{k+1} \). Let the sum of the three sets be \( s_1 \), \( s_2 \), and \( s_3 \) respectively. Each is divisible by \( 2^k \), so let \( t_i = s_i/2^k \), for \( i = 1, 2, \) and 3. Now by the Pigeonhole Principle, at least two of these numbers must have the same parity, even or odd. Without loss of generality, let the two sets be \( A_1 \) and \( A_2 \). The sum of two numbers with the same parity is even, so \( t_1 + t_2 \) is even. Multiplying by \( 2^k \), we have \( s_1 + s_2 \) is divisible by \( 2^{k+1} \). Thus there are \( 2^{k+1} \) integers from the original set \( A \) whose sum is divisible by \( 2^{k+1} \). Our induction on \( m \) is complete.

**A207.** Given triangle \( ABC \), let \( A' \), \( B' \), and \( C' \) be on the sides \( BC \), \( AC \), and \( AB \) respectively such that \( \triangle A'B'C' \sim \triangle ABC \). Find the locus of the orthocentre of all such triangles \( A'B'C' \).

*Solution by Alexandre Trichtchenko, student, Brookfield High School, Ottawa, Ontario.*

Here we give the solution when the triangle \( ABC \) is acute. A similar argument can be given for an obtuse triangle.

Let \( \angle BAC = \alpha \), \( \angle ABC = \beta \), and \( \angle BCA = \gamma \). Let \( A' \), \( B' \), and \( C' \) be the feet of the altitudes from vertices \( A' \), \( B' \), and \( C' \) respectively of triangle \( A'B'C' \). Further, let \( H' \) be the point of the orthocentre of triangle \( A'B'C' \), the point of intersection of its altitudes.
Since triangles $B'H'C''$ and $B'A'B''$ are similar $\angle B'H'C'' = \alpha$. Likewise, $\angle A'H'C'' = \beta$. Thus $\angle B'H'A' = \angle B'H'C'' + \angle A'H'C'' = \alpha + \beta$.

Also, $\angle A'H'B' + \angle B'C'A' = \alpha + \beta + \gamma = 180^\circ$. Hence, the quadrilateral $CB'H'A'$ is cyclic. Since $\angle B'A'H'$ and $\angle B'C'H'$ are inscribed in the circumcircle of $CB'H'A'$ and subtended by the same arc $H'B'$, we have $\angle H'C'B' = \angle H'A'B' = 90^\circ - \beta$. Similarly, $\angle H'AB' = 90^\circ - \beta$. So $\angle H'C'B' = \angle H'AB'$, and so $AH' = CH'$. By similar reasoning, we can show that $AH = BH' = CH'$. Thus $H'$ is the circumcentre of triangle $ABC$ and is independent of the choice of triangle $A'B'C'$. Thus the locus of the orthocentre of all triangles $A'B'C'$ is just the single point $H' = O$, the circumcentre of triangle $ABC$.

Also solved by D.J. SMEENK, Zalkhommel, the Netherlands.

A208. Let $p$ be an odd prime, and let $S_k$ be the sum of the products of the elements $\{1, 2, \ldots, p-1\}$ taken $k$ at a time. For example, if $p = 5$, then $S_3 = 1 \times 2 \times 3 + 1 \times 2 \times 4 + 1 \times 3 \times 4 + 2 \times 3 \times 4 = 50$. Show that $p|S_k$ for all $2 \leq k \leq p-2$.

Solution.

Consider the monic polynomial of degree $p-1$, $x^{p-1} - 1 \equiv 0 \pmod{p}$. There are precisely $p-1$ incongruent solutions modulo $p$ of this polynomial equation, namely, $x = 1, 2, \ldots, p-1$. Each of these follows from Fermat's Little Theorem, which states that $a^{p-1} \equiv 1 \pmod{p}$, where $p$ is a prime and $a$ and $p$ are relatively prime. Thus, modulo $p$, we have

$$x^{p-1} - 1 \equiv (x-1)(x-2) \cdots [x-(p-1)] \equiv x^{p-1} - S_1x^{p-2} + S_2x^{p-3} - \cdots + (-1)^{p-2}S_{p-2} + (-1)^{p-1}S_{p-1} \pmod{p}.$$  

Equating coefficients, we have $p|S_k$ for all $2 \leq k \leq p-2$.

Note: We also get Wilson's Theorem for free. (Why?)

Also solved by ALEXANDRE TRICHTCHENKO, student, Brookfield High School, Ottawa, Ontario.
Challenge Board Solutions

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C77. — Corrected problem (not a solution!)

Let $F_i$ denote the $i^{th}$ Fibonacci number, with $F_0 = 1$ and $F_1 = 1$. (Then $F_2 = 2, F_3 = 3, F_4 = 5$, etc.)

(a) Prove that each positive integer is uniquely expressible in the form $F_{a_1} + \cdots + F_{a_k}$, where the subscripts form a strictly increasing sequence of positive integers no pair of which are consecutive.

(b) Let $r = \frac{1}{2}(1 + \sqrt{5})$, and for any positive integer $n$, let $f(n)$ equal the integer nearest to $nr$. If $n = F_{a_1} + \cdots + F_{a_k}$ is the expression for $n$ from part (a) and if $a_2 \neq 3$, prove that $f(n) = F_{a_1 + 1} + \cdots + F_{a_k + 1}$.

(c) Keeping the notation from part (b), if $a_2 = 3$ (so that $a_1 = 1$), it is not always true that the formula $f(n) = F_{a_1 + 1} + \cdots + F_{a_k + 1}$ holds. For example, if $n = 4 = F_3 + F_1 = 3 + 1$, then the closest integer to $nr = 6.47\ldots$ is 6, not $F_4 + F_2 = 5 + 2 = 7$. Fortunately, in the cases where the formula fails, we can correct the problem by setting $a_1 = 0$ instead of $a_1 = 1$: for example, $4 = F_3 + F_0 = 3 + 1$ as well, and indeed $6 = F_4 + F_1 = 5 + 1$. Determine for which sequences of $a_i$ this correction is necessary.

C78. Let $n$ be a positive integer. An $n \times n$ matrix $A$ is a magic matrix of order $m$ if each entry is a non-negative integer and each row and column sum is $m$. (That is, for all $i$ and $j$, $\sum_k A_{ik} = \sum_k A_{kj} = m$.) Let $A$ be a magic matrix of order $m$. Show that $A$ can be expressed as the sum of $m$ magic matrices of order 1.

1. Solution by Christopher Long, graduate student, Rutgers University.

Consider the magic matrix $A$ as the adjacency matrix of a weighted bipartite graph $G$ between two sets ("left" and "right") of $n$ vertices: if the $(i, j)^{th}$ entry of $A$ is greater than 0, place an edge in $G$ between the $i^{th}$ vertex on the left and the $j^{th}$ vertex on the right and give the edge a weight equal to the $(i, j)^{th}$ entry of $A$. If the $(i, j)^{th}$ entry of the $A$ is 0, do not place an edge at all. The condition that the matrix $A$ is a magic matrix implies that total weight of all the edges emanating from any single vertex of $G$, left or right, is equal to $m$.

Given a subset $S$ of the left-hand vertices of $G$, let us compute the size of its neighbourhood (the collection of all vertices on the right which are joined by an edge to a vertex in $S$). Remove from $G$ all of the edges whose left-hand vertices are not in $S$. Then the total weight of the remaining edges is exactly $m|S|$, and the neighbourhood of $S$ is exactly the set of right-hand vertices whose weight is still non-zero. (By the weight of a vertex, we mean the weight of all the edges touching that vertex.) But each right-hand vertex has
weight at most \(m\), so by the Pigeonhole Principle the number of right-hand vertices with non-zero weight must be at least \(m |S| / m = |S|\). That is, every subset on the left has a neighbourhood on the right which is at least as big. Thus, the conditions of the following famous theorem (phrased traditionally, and thus thoroughly objectionable) are satisfied, with the left-hand vertices as boys, the right-hand vertices as girls, and edges as acceptable marriages:

**Theorem (Hall's Marriage Theorem).** Suppose there are \(n\) boys and \(n\) girls, and that each boy knows precisely which (possibly more than one) of the girls he is willing to marry. Suppose further that given any set \(S\) of boys, the total number of different girls that boys in \(S\) are willing to marry is at least \(S\). Then there exists a way of pairing all the boys with the girls in such a way that each boy is willing to marry the girl to whom he is paired.

The proof of the Marriage Theorem is an excellent exercise, and can also be found in almost any graph theory book, so we omit it here. In our case, if the pairing obtained from the Marriage Theorem pairs the vertex \(i\) on the left with the vertex \(\sigma_i\) on the right, then we know that the \((i, \sigma_i)\)th entry of \(A\) is positive. Let \(A'\) be the matrix whose \((i, \sigma_i)\)th entry is 1 for all \(i\) and whose other entries are all 0. Then \(A'\) is a magic matrix of order 1 and \(A - A'\) is a magic matrix of order \(m - 1\), and the result follows by induction.

**II. Solution.**

As in the previous solution, we prove the result by induction by showing that there exists a permutation \(\sigma\) of \(\{1, 2, \ldots, n\}\) such that the \((i, \sigma_i)\)-entry of \(A\) is positive—that is, by constructing a “magic submatrix” of order 1 in \(A\). We do this for all magic matrices \(A\) by another induction, this time on \(NZ(A)\), the number of non-zero entries of \(A\). If the order of the magic matrix is \(m > 0\), then there is a non-zero entry in every row, so the total number of non-zero entries is at least \(n\). However, if the number of non-zero entries is exactly \(n\), then certainly \(A\) is \(m\) times a magic matrix of order 1, and this completes our base case.

Now assume the result holds for \(NZ(A) < k\), where \(k > n\). Consider a magic matrix \(A\) of order \(m\) with exactly \(k\) non-zero entries. As \(k > n\), by the Pigeonhole Principle, there is a row with at least two non-zero entries, and each is less than \(m\). Let \((i_1, j_1)\) be the position of one of them, and let \((i_2, j_2)\) be the position of the other. As the \((i_1, j_2)\)th entry of \(A\) is less than \(m\), there is a non-zero entry at some position \((i_2, j_2)\) in the same column (and it is also less than \(m\)). By the same argument, there is a non-zero entry in the same row \((i_2, j_3)\) as \((i_2, j_2)\). We continue this process to get sequences \((i_k), (j_k)\), such that \(i_k \neq i_{k+1}, j_k \neq j_{k+1}\), and such that the \((i_k, j_k)\)th and \((i_k, j_{k+1})\)th entries of \(A\) are all non-zero.

Our goal is to find a loop of an even number of distinct non-zero entries in the matrix, connected by alternating horizontal and vertical moves. Once we have such a loop, find the point \((i, j)\) in the loop whose entry is minimal. Suppose the \((i, j)\)th entry is \(q\). Decrease the \((i, j)\)th entry by \(q\) to 0, increase
the next entry in the loop by \( q \), decrease the next by \( q \), and continue travelling once around the entire loop, alternately adding and subtracting \( q \) in this fashion. This will yield a magic matrix \( B \) with fewer non-zero entries than \( A \), so by the induction hypothesis \( B \) contains a magic submatrix \( B' \) of order 1. However, by the construction of \( B \), the non-zero entries of \( B \) all correspond to non-zero entries of \( A \), so \( B' \) is also a magic submatrix of \( A \). We will thus be done by induction.

Let us proceed with finding this loop. As there are only a finite number of entries of \( A \), at some point in the sequence \((i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \ldots \) there will be a repeated term. If the first repeated term is of the form \((i_l, j_l)\), and the first appearance of this term is \((i_k, j_k)\) (with \( k < l \)), then the loop \((i_k, j_k), (i_k, j_{k+1}), \ldots, (i_{l-1}, j_l), (i_l, j_l) = (i_k, j_k)\) is exactly the kind of loop we are looking for, and we are done. We are similarly finished if the first repeated term is of the form \((i_l, j_{l+1})\) and the first occurrence of that term is of the form \((i_k, j_{k+1})\).

Suppose instead that the first repeated term is of the form \((i_l, j_{l+1})\) and that the first appearance of the term is \((i_k, j_k)\). Then, replacing \( j_k \) by \( j_l \), we obtain the loop \((i_l, j_l) = (i_k, j_l), (i_k, j_{k+1}), \ldots, (i_{l-1}, j_l), (i_l, j_l)\), and we are done. In the case when the first repeated term is of the form \((i_l, j_l)\) and the first appearance of the term is \((i_k, j_{k+1})\), a similar trick works. Thus, we have exhausted all cases, and the proof is complete.

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**Shreds and Slices**

**Erratum**

There is a mistake in the 1987 Swedish Mathematical Olympiad, as printed in *CRUX and MAYHEM* [1998:298]. The expression \("-a + 2b - 3c"\) in problem 2 of the Qualifying Round should read \("-a + 2b + 3c"\). Thanks to Solomon Golomb for pointing this out.

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**Goodbye, Richard!**

We regret to inform our readership that Richard Hoshino, long time Mayhem staff member and High School Editor, will be leaving us after this issue. We all thank Richard for his strong dedication and numerous contributions to Mayhem over the years, and we hope that he has gained as much in the experience as we have. We wish Richard the best of luck in his new responsibilities, and we will name his successor in the next issue.
1998–1999 Olympiad Correspondence
Problems

Please mail your solutions to Professor E.J. Barbeau, Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3.

Set 1

1. $ABC$ is an isosceles triangle with $\angle A = 100^\circ$ and $AB = AC$. The bisector of angle $B$ meets $AC$ in $D$. Show that $BD + AD = BC$.

2. Let $I$ be the incentre of triangle $ABC$. Let the lines $AI$, $BI$, and $CI$ produced intersect the circumcircle of triangle $ABC$ at $D$, $E$, and $F$ respectively. Prove that $EF$ is perpendicular to $AD$.

3. Let $PQR$ be an arbitrary triangle. Points $A$, $B$, and $C$ external to the triangle are determined for which

$$\angle AQR = \angle ARQ = 15^\circ, \quad \angle QPC = \angle RPB = 30^\circ, \quad \angle PQC = \angle PRB = 45^\circ.$$ 

Prove that: (a) $AC = AB$; (b) $\angle BAC = 90^\circ$.

4. Let $a$ and $b$ be two positive real numbers. Suppose that $ABC$ is a triangle and $D$ a point on side $AC$ for which $\angle BCA = 90^\circ$, $|AD| = a$, and $|DC| = b$. Let $|BC| = x$ and $\angle ABD = \theta$. Determine the values of $x$ and $\theta$ for the configuration in which $\theta$ assumes its maximum value.

5. Let $C$ be a circle with centre $O$ and radius $k$. For each point $P \neq O$, we define a mapping $P \rightarrow P'$ where $P'$ is that point on $OP$ produced for which $|OP| \cdot |OP'| = k^2$.

In particular, each point on $C$ remains fixed, and the mapping at other points has period 2. This mapping is called inversion in the circle $C$ with centre $O$, and takes the union of the sets of circles and lines in the plane to itself. (You might want to see why this is so. Analytic geometry is one route.)

(a) Suppose that $A$ and $B$ are two points in the plane for which $|AB| = d$, $|OA| = r$, and $|OB| = s$, and let their respective images under the inversion be $A'$ and $B'$. Prove that $|A'B'| = \frac{k^2d}{rs}$.

(b) Using (a), or otherwise, show that there exists a sequence $\{X_n\}$ of distinct points in the plane with no three collinear for which all distances between pairs of them are rational.
6. Solve each of the following two systems of equations:

(a) \( x + xy + y = 2 + 3\sqrt{2}, \)  
\[ x^2 + y^2 = 6. \]

(b) \( x^2 + y^2 + \frac{2xy}{x + y} = 1, \)  
\[ \sqrt{x + y} = x^2 - y. \]

Set 2

7. For a positive integer \( n \), let \( r(n) \) denote the sum of the remainders when \( n \) is divided by 1, 2, \ldots, \( n \) respectively.

(a) Prove that \( r(n) = r(n - 1) \) for infinitely many positive integers \( n \).

(b) Prove that \( \frac{n^2}{10} < r(n) < \frac{n^2}{4} \) for each integer \( n \geq 7 \).

8. Counterfeit coins weigh \( a \) and genuine coins weigh \( b \) (\( a \neq b \)). You are given two samples of three coins each and you know that each sample has exactly one counterfeit coin. What is the minimum number of weighings required to be certain of isolating the two counterfeit coins by means of an accurate scale (not a balance)?

(a) Solve the problem assuming \( a \) and \( b \) are known.

(b) Solve the problem assuming \( a \) and \( b \) are not known.

9. Similar isosceles triangles \( EBA, FCB, GDC \), and \( HAD \) are erected externally on the four sides of the planar quadrilateral \( ABCD \) with the sides of the quadrilateral as their bases. Let \( M, N, P, \) and \( Q \) be the respective midpoints of the segments \( EG, HF, AC \), and \( BD \). What is the shape of \( PQMN \)?

10. Given two points \( A \) and \( B \) in the Euclidean plane, let \( C \) be free to move on a circle with \( A \) as centre. Find the locus of \( P \), the point of intersection of \( BC \) with the internal bisector of angle \( A \) of triangle \( ABC \).

11. Let \( ABC \) be a triangle; let \( D \) be a point on \( AB \) and \( E \) a point on \( AC \) such that \( DE \) and \( BC \) are parallel and \( DE \) is a tangent to the incircle of the triangle \( ABC \). Prove that \( 8DE \leq AB + BC + CA \).

12. Suppose that \( n \) is a positive integer and that \( x + y = 1 \). Prove that

\[ x^{n+1} \sum_{k=0}^{n} \binom{n+k}{k} y^k + y^{n+1} \sum_{k=0}^{n} \binom{n+k}{k} x^k = 1. \]
J.I.R. McKnight Problems Contest 1985

1. If \( S_n = 3 + 8x + 15x^2 + 24x^3 + \cdots + (n^2 + 4n + 3)x^{n-1} \), determine \( S_n \) by first evaluating \((1-x)S_n\). Hence find the limit of \( S_n \) as \( n \) approaches infinity, given \( x = \frac{1}{3} \).

2. (a) \( P \) and \( Q \) are points \((ap^2, 2ap)\) and \((aq^2, 2aq)\) on the parabola \( y^2 = 4ax \). Show that the equation of the chord \( PQ \) is \( 2x - (p + q)y + 2aq = 0 \).

(b) If \( O \) is the origin and the chords \( OP \) and \( OQ \) are perpendicular, prove that the chord \( PQ \) cuts the \( x \)-axis in the same point for all possible positions of \( P \) and \( Q \).

3. In the figure, angle \( A \) has a measure of \( 60^\circ \). At a distance of 10 cm from the vertex, a perpendicular is erected and a square is constructed on it with side \( s_1 \). In toward the vertex of the angle a second square of side \( s_2 \) is formed. Then similarly a square of side \( s_3 \), and so on ad infinitum. Find the sum of the areas of these squares in simplest radical form and then give an approximation to the nearest hundredth of a square centimetre.

4. A wire of length \( L \) is to be cut into two pieces, one of which is bent to form a circle and the other to form a square. How should the wire be cut if the sum of the areas enclosed by the two pieces is to be a maximum?

5. (a) Sketch the hyperbola represented by the equation

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a > b,
\]

and draw its asymptotes.

(b) Draw a tangent to the hyperbola at any point on the hyperbola, and prove that the portion of the tangent between the points where it meets the asymptotes is bisected by the point of contact of the tangent.
(c) Prove that the segment of the tangent in (b) forms with the asymptotes a triangle of constant area.

6. Prove that if \( \cos x + \cos y = a \) and \( \sin x - \sin y = b \), then
\[
\cos(x - y) = \frac{a^2 - b^2}{a^2 + b^2}.
\]

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**Swedish Mathematics Olympiad 1989**

1989 Qualifying Round

1. Find the integer \( t \) and the hundreds digit \( a \) such that
\[
(3(320 + t))^2 = 492a04.
\]

2. Form all possible six digit numbers, each using the digits 1, 2, 3, 4, 5, 6 exactly once. What is the sum of all these numbers?

3. Let \( ABC \) be an acute-angled triangle and let \( P \) be a point on the side \( BC \). Let \( P' \) be the reflection of \( P \) in the side \( AB \), and let \( P'' \) be the reflection of \( P \) in the side \( CA \). Show that the distance \( PP'' \) is least when \( P \) is the foot of the perpendicular from \( A \) to \( BC \).

4. Show that if \( x, y, \) and \( z \) are positive real numbers and \( x^y = y^z = z^x \), then \( x = y = z \).

5. The equations \( x^2 + px + q = 0 \) and \( qx^2 + mx + 1 = 0 \), where \( m, p, \) and \( q \) are real, and \( q > 0 \), have roots \( x_1, x_2, \) and \( x_1, 1/x_2 \) respectively. Show that \( mp \geq 4 \).

6. Assume that \( a_1 < a_2 < \cdots < a_{995} \) are 995 real numbers. Form all sums \( a_i + a_j \), for \( 1 \leq i \leq j \leq 995 \). Show that at least 1989 different numbers are obtained. Show also that exactly 1989 different numbers are obtained if and only if \( a_1, a_2, \ldots, a_{995} \) is an arithmetic progression.

The 1989 Final Round has already appeared in a previous issue of *CRUX*, in Olympiad Corner #125, 1991.
Dividing Points Equally

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We start off with a simple problem and follow it through with some related questions that can be reduced to this problem.

Problem 1. Given $2n$ points in the plane, is it possible to draw a straight line so that there are an equal number of points on either side of the line?

Solution. The answer is yes. Since there are a finite number of points, namely $2n$ of them, there are a finite number of lines that pass through a pair of points, at most $\binom{2n}{2}$. Each line passing through a pair of points has a certain direction. So, the $2n$ points have at most $\binom{2n}{2}$ distinct directions. It is not too hard then to pick a direction, call it $\lambda$, different from all of these.

Intuitively, it is possible to take a line with direction $\lambda$ and slide it along (preserving its direction of course) until half the points are on either side. This is possible since the line can pass through only one point at a time. If it passed through more than one point, then the direction between the two points must be the direction $\lambda$ of this line. This contradicts the choice of $\lambda$ to be different from any of those pairs from the $2n$ points. Hence, by sliding the line with direction $\lambda$ along, we pass each point one at a time until we have passed exactly half, and then we are done.

We follow this up with some generalizations and related problems for the reader to solve.

Problem 1A. Prove that given $mn$ points in the plane, we can find $m - 1$ parallel lines that divide the plane into $m$ regions with $n$ points in each region.

Problem 1B. Show that it is possible to divide $2n$ points in the plane by two intersecting lines so that for each line, half the points lie on either side of it. Show that it is possible to divide them by $m$ concurrent lines so that for each line, half the points lie on either side of it.

Problem 1C. Take $4n$ points in the plane and take any three distinct lines, each pair of which divides the set of points into equal quarters. Show that these three lines cannot be concurrent. (Hint: Show that the three lines must form a triangle containing $n$ of the points.)

Problem 1D. Is it possible to divide $2n$ points in the plane by a circle so that half of the points are inside and half are outside? (Note: This does not follow immediately from Problem 1 by inversion.)
Problem 1E. Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.

Can every set of $4n$ points in the plane, no three of which are collinear, be evenly quartered by two mutually perpendicular lines?


In the next two problems, we consider three dimensions.

**Problem 2.** There are $2n$ chocolate chips in a roll of frozen cookie dough. Show that it is possible to divide them into two sets of $n$ chips by a plane cut.

**Solution.** The $2n$ chocolate chips can be thought of as points in space. We want to roll the frozen cookie dough into a position where if the chocolate chips fell straight down, they would not hit each other. Essentially, we want to project the $2n$ points onto a plane so that each point is mapped to a distinct position. Then we are back to Problem 1 of dividing the points in the plane by a line. (Why?) This problem then becomes one of showing that there is such a projection.

In 3 dimensions, we need something that is similar to “slope”. This is where the concept of a *direction vector* comes into play. The direction vector between two points from $A = (a_1, a_2, a_3)$ to $B = (b_1, b_2, b_3)$ is given by $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$ (or any multiple of this). To reduce two points having multiple direction vectors, we usually consider the *unit direction vector* which is found simply by taking the given direction vector and dividing each coordinate by the vector’s norm or length, namely $\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$. (The reader may wish to verify that for any line, any two points on the line will give the same direction vector up to a ± sign.)

Since there are a finite number of points, there are a finite number of unit direction vectors. We can then pick a unit direction vector different from all of them; call it $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. By projecting each point in this direction onto a plane perpendicular to this vector, each point projects to a different point in the plane.

Thus, take the dough so that the direction vector $\lambda$ points vertically down on the kitchen table. Take a knife and cut along the line that would divide the points in half in the plane.

**Problem 3.** Suppose the Earth has a population of 6 billion people in the near future. Is it possible to draw an imaginary equator around the world so that each hemisphere contains an equal number of humans?

Here is a lemma that we will need.

**Lemma 3A.** $2n$ points and a special point labelled $S$ are given in the plane. Any line passing through $S$ passes through at most one of the $2n$ points. It is then possible to divide the plane by a straight line that passes
through $S$ so that $n$ points are on either side of the line.

**Solution.** Consider the lines passing through $S$ and one other point. There are $2n$ of them. Pick any other line through $S$ which is not one of these. Say there are $k$ points on one side and $2n - k$ points on the other. If $k = n$ then we are finished. Otherwise, rotate this line counter-clockwise about $S$. This line will go through all of the points one at a time as it rotates. Let $f(\theta)$ be the difference between the number of points on one side labelled $A$ and the number of points on the other side labelled $B$ as shown when the angle is $\theta$ from the original line. Points on the line are to be ignored.

Thus, at the beginning, $f(0^\circ) = (2n - k) - k = 2(n - k)$, and when it has completed a rotation of $180^\circ$, we have $f(180^\circ) = k - (2n - k) = -2(n - k)$. Make sure that sides $A$ and $B$ stay in the same orientation.

Thus these values have different signs and if this were a problem of a continuous function, then we could easily claim by the Intermediate Value Theorem that there is a value $0^\circ < \theta < 180^\circ$ such that $f(\theta) = 0$. Nonetheless, we can still conclude this, since we know that when the value of $f$ changes, it changes by a value of 1.

To see this, note that each time the line passes through a point, the value on one side, say $A$ decreases by 1 and the number of points on side $B$ stays the same. The value of $f$ then decreases by one. Once the line immediately passes by this point, the number of points on side $B$ increases by 1 and the number of points on side $A$ remains the same. Again, the value of $f$ decreases by 1. This will happen for each point that the line crosses.
Note that when the line passes completely over a point the value of $f$ changes by a value of 2. Thus the parity of $f$ changes when the line goes through a point from the state when there are no points on the line and the parity stays the same when the line passes by a point.

Since $2(n - k)$ and its negative are both integers, at some step in the value of $f$, 0 is reached. Now we must check that this value is not obtained when the line crosses through one of the $2n$ points. Since the values of 0 and $2(n - k)$ have the same parity, when the value of 0 is achieved, it is achieved in the state where the line does not pass through any of the $2n$ points. Hence, for this line, there is an angle $\theta$ counter-clockwise away from the original line where the points are equally divided.

**Solution to Problem 3.**

We must assume, of course, that the people are points occupying a distinct and fixed location on a spherical Earth. Now for each pair of people draw a great circle passing through them. That is, a circle given by the intersection of the sphere with a plane passing through the centre of the sphere; this would be our definition of an equator. There are a finite number of such circles as there are a finite number of pairs. Thus we may always choose another point, call it $N$, not on any of the great circles. The point diametrically opposite $N$, call it $S$, must not be on any of the great circles either. (Why?) If a point was on a great circle then the point diametrically opposite it would be also, by definition.

Now place the spherical world on a plane with the point $S$ tangent to it. From point $N$ project each point onto the plane by drawing a line from $N$ through each point to intersect the plane – this projection is called the stereographic map. Now consider lines through $S$ in the plane. Any such line can have at most one other point. If a line contains $S$ and two other points then that would mean that $S$ was on the great circle through the preimage of these points.

By Lemma 3A there is a line through $S$ which cuts the plane into two parts containing an equal number of points. This line projected back onto the sphere is a great circle dividing the world into two equal populations.

**Exercises**

1. Give solutions to the problems listed above.

2. Is it possible to divide any $4n$ points in the plane by two intersecting lines so that each of the four sectors contains $n$ points?

3. Is it possible to divide the Earth into 4 quarters with an equal number of people in each? (Assume a population of 6 billion.)