THE ACADEMY CORNER

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THE BERNOULLI TRIALS 1998

Hints and Answers

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The questions were printed earlier this year [1998: 257].

1. False. The diameter of the larger circle is 50.

Let $CD = x$.
Then $AC = 9 + x$
and $CE = 4 + x$.
Consider similar triangles in $AED$.

2. True. Let $X$ be the number of rounds until his first error, $Y$ the number of subsequent rounds until his second error. Then

$$E(X + Y) = E(X) + E(Y) = 2E(X)$$

But $E(X) = 2$, which can be found by summing the series.

3. True. Use 1, 2, 4, 8, ..., $2^{10}$ and a binary expansion of any number from 8 to 1998.

4. True. Write the points in coordinates and use the Pythagorean formula. Everything will cancel!
5. True. The volume of the tetrahedron is $\frac{\sqrt{2}}{12}$. To prove this, imbed the tetrahedron into a cube so that the vertices of the tetrahedron are four of the eight vertices of the cube. Show that the tetrahedron has a volume which is one third the volume of the cube. On the other hand, the volume of the sphere is $4\pi r^3/3$ where $r = 1/\pi$.

6. False. Actually $1503^2 = 2250009$ is the smallest.

7. True. Draw a graph of the function in the unit square.

8. True. The choice $k = 9$ works: $\frac{9n(n+1)}{2} + 1 = \frac{(3n+1)(3n+2)}{2}$.

9. True. The maximum area is achieved with a cyclic quadrilateral. The area of such a cyclic quadrilateral can be determined by Brahmagupta's formula

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{4!} = 2\sqrt{6}.$$ 

10. False. Write $\alpha(x) = x/(1 + x)$. The equation can be written as

$$f[\alpha(x)] = \alpha[f(x)].$$

This is clearly satisfied by $f(x) = \alpha^n(x)$. In particular, the choice of $f(x) = x/(1 + x)$ works.

11. True. $3^3 = 3^5 > 2 \times 2^2 = 2 \times 2^{3^2}$.

So $2^{3^2} > 2^{2 \times 2^2} = 4^{3^2} > 3^{3^2}$.

Finally, we have $2^{3^{2^{3^2}}} > 2^{2 \times 2^{3^2}} = 4^{2^{3^2}} > 3^{2^{3^2}}$.

Obviously, this can be continued.

12. True. Write

$$\int_0^\infty \frac{x}{e^x - 1} dx = \int_0^\infty \frac{xe^{-x}}{1-e^{-x}} dx$$

$$= \int_0^\infty x(e^{-x} + e^{-2x} + e^{-3x} + \cdots) dx$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$
13. False.

The answer is one square metre exactly.
To prove this consider the figure to the right:

The thin triangle has the required dimensions and its area is

\[ \frac{1}{2} \times 4 \times 5 - \left( 2 \times 2 + \frac{1}{2} \times 2 \times 3 + \frac{1}{2} \times 2 \times 2 \right) = 1. \]

14. False. Suppose there were such a function.

The function \( e^x - e^{-x} \) can be seen to be a one-to-one decreasing function. So \( f \) must be one-to-one:

\[ f(x) = f(y) \implies f[f(x)] = f[f(y)] \implies x = y. \]

As \( f \) is continuous and one-to-one, it must be strictly increasing or strictly decreasing.

But either way, \( f[f(x)] \) must then be a strictly increasing function. Contradiction.

15. False.

Consider a third random point \( X \) in the circle.

The region \( R_1 \) corresponds to the points for \( X \) where the angle at \( X \) is obtuse. The region \( R_2 \) corresponds to the points for \( X \) where the angle at \( B \) is obtuse.

As the probabilities are the same, the average areas of \( R_1 \) and \( R_2 \) must be the same.

16. False. The second player wins by forcing bilateral symmetry on remaining petals. For example, if the first player starts by taking petal 1, the second player takes petals 7 and 8 together. If the first player chooses 1 and 2 the second player takes petal 8, etc. This ensures that the first player can never take the last petal.