Pythagoras Strikes Again!

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Is it possible that a given triangle is similar to the triangle formed from its medians? In other words, is it possible that the side lengths of a triangle are proportional in some order to its medians' triangle? For a non-trivial example note the triangle with side lengths \((a, b, c) = (23, 7, 17)\) and the medians' lengths \((m_a, m_b, m_c) = \left(7\frac{\sqrt{3}}{2}, 23\frac{\sqrt{3}}{2}, 17\frac{\sqrt{3}}{2}\right)\) in which \(a : m_b = b : m_a = c : m_c\) is possible. You may use the formulas

\[
4m_a^2 = 2b^2 + 2c^2 - a^2
\]

etc. to calculate the lengths of the medians in the above example. The formulas (1) themselves are derived by means of the cosine rule. The type of triangle as described exists and shall be called a self-median triangle [1].

Is it possible, this time, that a given triangle is similar to the triangle formed from its altitudes? That is, the lengths of the sides of a triangle are proportional in some order to the lengths of its altitudes? Again, trivially it's so in an equilateral triangle but non-trivially we have the triangle with the lengths of the sides \((a, b, c) = (6, 9, 4)\) and the lengths of the altitudes \((h_a, h_b, h_c) = \left(\frac{2\Delta}{6}, \frac{2\Delta}{9}, \frac{2\Delta}{4}\right)\), \(\Delta\) being the area of the triangle. Observe that \(a : h_a = b : h_c = c : h_b\) holds. This type of triangle too exists and matching the description of self-median triangles we will call the present type self-altitude triangles [2].

Interestingly, as we shall see, with the exception of the equilateral triangle, both of these triangle types are simply reincarnations of appropriate right triangles. The theorems that follow illustrate this fact.

**Theorems on self-median triangles**

Theorem 1 tells us how an appropriate right triangle yields a self-median triangle.

**THEOREM 1** Let \((a_0, b_0, c_0)\) be a right triangle in which \(a_0^2 + b_0^2 = c_0^2\), \(a_0 > b_0\) and \(c_0 > 2b_0\) hold. Then \((a, b, c) = (a_0 + b_0, a_0 - b_0, c_0)\) is a self-median triangle in which \(a > c > b\) holds.

**Proof:** Let \(m_1, m_2, m_3\) denote the lengths of the medians drawn to the sides \(a_0 + b_0, a_0 - b_0, c_0\) respectively. Then from (1) we can easily deduce that

\[
4m_1^2 = 3(a_0 - b_0)^2, 4m_2^2 = 3(a_0 + b_0)^2, 4m_3^2 = 3c_0^2.
\]
Hence
\[ \frac{a_0 + b_0}{m_2} = \frac{a_0 - b_0}{m_1} = \frac{c_0}{m_3} = \frac{2}{\sqrt{3}}. \]

It should be noted that the lengths \( a_0 + b_0, a_0 - b_0, c_0 \) do not form a triangle for every given right triangle \((a_0, b_0, c_0)\). For example \((a_0, b_0, c_0) = (4, 3, 5)\) yields \((a_0 + b_0, a_0 - b_0, c_0) = (7, 1, 5)\), and there is no triangle with these side lengths. However \((a_0, b_0, c_0) = (12, 5, 13)\) yields the self-median triangle \((a_0 + b_0, a_0 - b_0, c_0) = (17, 7, 13)\). In order to assure the formation of a self-median triangle, the lengths \(a_0 + b_0, a_0 - b_0, c_0\) have to satisfy the triangle inequality \(a_0 - b_0 + c_0 > a_0 + b_0\). This simplifies to \(c_0 > 2b_0\).
In other words the hypotenuse must exceed twice the shorter leg to give us a self-median triangle. It is trivial to check that the other two triangle inequalities are satisfied. Moreover, in the right triangle \((a_0, b_0, c_0)\), we have \(a_0 + b_0 > c_0 > a_0 - b_0\) which is the same as \(a > c > b\).

Theorem 2 tells us how to recover the right triangle from a given self-median triangle.

**THEOREM 2**: Let \((a, b, c)\) be a self-median triangle in which \(a > c > b\) holds. Then \((a_0, b_0, c_0) = (\frac{1}{2}(a + b), \frac{1}{2}(a - b), c)\) is a right triangle in which \(a_0^2 + b_0^2 = c_0^2, a_0 > b_0\) and \(c_0 > 2b_0\) hold.

**Proof**: Since \(a > c\), we find that \(r(m_a^2 - m_c^2) = 3(c^2 - a^2) < 0\), or, \(m_a < m_c\). Thus \(a > c > b \iff m_b > m_c > m_a\). From this and the definition of self-median triangle we have \(\frac{a}{m_a} = \frac{b}{m_b} = \frac{c}{m_c}\). We square two of these equations, use the formulas (1) and simplify. This yields \((a^2 - b^2)(a^2 + b^2 - 2c^2) = 0\). The hypothesis \(a > c > b\) shows that this is equivalent to \(a^2 + b^2 = 2c^2\). We consistently arrive at this same equation (no matter which two equations are squared) which is the same as \([\frac{1}{2}(a + b)]^2 + [\frac{1}{2}(a - b)]^2 = c^2\) or \(a_0^2 + b_0^2 = c_0^2\), with \(a_0 > b_0\). Now in triangle \((a, b, c)\), we have \(c > a - b\) so that \(c_0 > 2b_0\) holds and the proof is complete.

**Remark 1**: The proof of Theorem 2 says much more than its statement. Since the argument is reversible it has given two characterizations of self-median triangles: Suppose the side lengths of triangle \((a, b, c)\) satisfy \(a > c > b\). Then the triangle is self-median (i) if and only if \(a : m_b = b : m_a = c : m_c\) or (ii) if and only if \(a^2 + b^2 = 2c^2\). Also, it is easy to see that if triangle \((a, b, c)\) has \(a : m_a = b : m_b = c : m_c\) then it must be equilateral. This useful observation enables us to give an elegant proof of Theorem 3.

Theorems 1 and 2 considered the determination of self-median triangles in each of which the sides have distinct length measures. Theorem 3 shows that if an isosceles triangle is self-median then it must be equilateral.

**THEOREM 3**: The equilateral triangle is the only self-median triangle not covered by the Theorems 1 and 2.

**Proof**: Any self-median triangle \((a, b, c)\) not covered by the Theorems 1 and 2 must have at least two sides equal, say \(a = b\). Then \(m_a = m_b\). Let
\[ m_1, m_2, m_3 \] denote the medians \( m_a, m_b, m_c \) in some order so that the definition of a self-median triangle enables us to write
\[
\frac{a}{m_1} = \frac{b}{m_2} = \frac{c}{m_3}.
\]

If \( m_1 = m_c \) then \( m_2 = m_3 = m_a = m_b \). But then \( a = b \) forces \( m_a = m_b = m_c \) and hence the triangle is equilateral. If \( m_1 = m_2 = m_3 = m_a = m_b \) then \( m_a = m_c \). This again forces the triangle to be equilateral as mentioned in Remark 1.

**Exercise 1:** Find the right triangle that generates the self-median triangle \((23, 7, 17)\).

**Exercise 2:** Suppose \((a, b, c)\) is a right triangle with \(a^2 + b^2 = c^2\). Show that the triangle \((a\sqrt{2}, b\sqrt{2}, c)\) is self-median.

**Exercise 3:** Show that there are exactly two self-median triangles in each of which (i) all the side lengths are integers and (ii) two side lengths are 7 and 17.

### Theorems on self-altitude triangles

Theorem 4 tells us how an appropriate right triangle yields a self-altitude triangle.

**Theorem 4:** Let \((a_0, b_0, c_0)\) be a right triangle in which \(a_0^2 + b_0^2 = c_0^2\) and \(a_0 > 2b_0\) hold. Then \((a, b, c) = (a_0, c_0 + b_0, c_0 - b_0)\) is a self-altitude triangle in which \(b > a > c\) holds.

**Proof:** Let \(h_a, h_b, h_c\) denote the altitudes to the sides \(a, b, c\) and \(\triangle\) denote the area of this triangle. Then \(ah_a = bh_b = ch_c = 2\triangle\) or
\[ a_0h_a = (c_0 + b_0)h_b = (c_0 - b_0)h_c = 2\triangle. \]

Now
\[ \frac{a_0}{h_a} = \frac{a_0^2}{2\triangle} = \frac{a_0^2 - b_0^2}{(c_0 - b_0)h_c} = \frac{c_0^2 - b_0^2}{(c_0 + b_0)h_b}, \]
or
\[ \frac{a_0}{h_a} = \frac{c_0 + b_0}{h_c} = \frac{c_0 - b_0}{h_b} \]
implicating that the triangle \((a, b, c)\) is self-altitude. Furthermore, in triangle \((a_0, b_0, c_0)\), we have that \(c_0 + b_0 > a_0 > c_0 - b_0\) holds. This in triangle \((a, b, c)\) is equivalent to \(b > a > c\).

Again, not any right triangle \((a_0, b_0, c_0)\) yields a self-altitude triangle. For example, \((a_0, b_0, c_0) = (3, 4, 5)\) yields \((a, b, c) = (3, 9, 1)\) and there is no triangle with these side lengths. However, \((a_0, b_0, c_0) = (12, 5, 13)\) yields \((a, b, c) = (12, 18, 8)\). We may divide these side lengths by their gcd 2 and
take the primitive self-altitude triangle \((6, 9, 4)\). Here the triangle inequality to be satisfied is \(a_0 + (c_0 - b_0) > c_0 + b_0\) which simplifies to \(a_0 > 2b_0\). It is trivial to check that the other two triangle inequalities are satisfied.

Theorem 5 tells us how to recover the right triangle from a given self-altitude triangle.

**Theorem 5:** Let \((a, b, c)\) be a self-altitude triangle in which \(b > a > c\) holds. Then

\[
(a_0, b_0, c_0) = \left( a, \frac{1}{2}(b - c), \frac{1}{2}(b + c) \right)
\]

is a right triangle in which \(a_0^2 + b_0^2 = c_0^2\) and \(a_0 > 2b_0\) hold.

**Proof:** From the hypothesis \(b > a > c\) we have \(h_b < h_a < h_c\). From the definition of self-altitude triangle we have

\[
\frac{a}{h_a} = \frac{b}{h_b} = \frac{c}{h_c} \iff \frac{a^2}{2\Delta} = \frac{bc}{2\Delta} = \frac{bc}{2\Delta},
\]

where \(\Delta\) is the area of triangle \((a, b, c)\). Thus

\[
a^2 = bc = \left[ \frac{1}{2}(b + c) \right]^2 - \left[ \frac{1}{2}(b - c) \right]^2
\]

which is \(a_0^2 + b_0^2 = c_0^2\). From \(a > b - c\) follows \(a_0 > 2b_0\) as required.

**Remark 2:** The proof of Theorem 5 gives two characterizations of self-altitude triangles: Suppose the side lengths of triangle \((a, b, c)\) satisfy \(b > a > c\). Then the triangle is self-altitude

(i) if and only if \(a : h_a = b : h_c = c : h_b\), or

(ii) if and only \(a^2 = bc\).

It is easy to see that if in triangle \((a, b, c)\) we have \(a : h_a = b : h_b = c : h_c\) then it must be equilateral. This observation enables us to give an elegant proof of Theorem 6.

Theorems 4 and 5 considered the determination of self-altitude triangles in each of which the sides have distinct length measures. Theorem 6 shows that if an isosceles triangle is self-altitude then it must be equilateral.

**Theorem 6:** The equilateral triangle is the only self-altitude triangle not covered by the Theorems 4 and 5.

**Proof:** We omit. It is similar to that of Theorem 3.

**Exercise 4:** Find the right triangle that generates the self-altitude triangle \((35, 49, 25)\).

**Exercise 5:** Let \(ABC\) be a right triangle with right angle at \(C\). \(CD\) is drawn perpendicular to \(AB\) with the point \(D\) on \(AB\). Prove that the triangle whose
side lengths are $AD$, $DB$, $CD$ is self-altitude. In terms of $a$, $b$, $c$ give an answer to the question: when do the lengths $AD$, $DB$, $CD$ form a triangle?

Remarks 1 and 2 suggest the following.

**OPEN PROBLEM:** Suppose $AD$, $BE$, $CF$ are three concurrent cevians of triangle $ABC$. Assume that $\frac{BC}{AD} = \frac{CA}{BE} = \frac{AB}{CF}$ holds. Prove or disprove that the triangle $ABC$ is equilateral.

**Acknowledgement:** I thank the referee for his painstaking efforts to improve the presentation. He also suggested Theorems 3 and 6.

**References**


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**Advance Notice**

At the summer 1999 meeting of the Canadian Mathematical Society, to be held in St. John's, Newfoundland, there will be a Mathematics Education Session on the topic “What Mathematics Competitions Do for Mathematics”.

Invited speakers include Edward Barbeau, Toronto; Tony Gardner, Birmingham, England; Ron Dunkley, Waterloo; and Rita Janes, St. John’s. Anyone interested in giving a paper at this session should contact one of the organizers, Bruce Shawyer or Ed Williams, at the Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, Newfoundland, Canada.

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