PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 December 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2338. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose $ABCD$ is a convex cyclic quadrilateral, and P is the intersection of the diagonals AC and BD. Let I_1, I_2, I_3 and I_4 be the incentres of triangles PAB, PBC, PCD and PDA respectively. Suppose that I_1, I_2, I_3 and I_4 are concyclic.

Prove that $ABCD$ has an incircle.

2339. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A rhombus $ABCD$ has incircle Γ, and Γ touches AB at T. A tangent to Γ meets sides AB, AD at P, S respectively, and the line PS meets BC, CD at Q, R respectively. Prove that

(a) $\dfrac{1}{PQ}+\dfrac{1}{RS}=\dfrac{1}{BT}$,

and

(b) $\dfrac{1}{PS}-\dfrac{1}{QR}=\dfrac{1}{AT}$.

2340. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\lambda > 0$ be a real number and a, b, c be the sides of a triangle. Prove that

$$\prod_{\text{cyclic}} \frac{s+\lambda a}{s-a} \geq (2\lambda+3)^3.$$

[As usual s denotes the semiperimeter.]

2341. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle. For real $\lambda > 0$, put

$$s(\lambda) := \left| \sum_{\text{cyclic}} \left[\left(\frac{a}{b}\right)^{\lambda} - \left(\frac{b}{a}\right)^{\lambda} \right] \right|$$

and let $\triangle(\lambda)$ be the supremum of $s(\lambda)$ over all triangles.

1. Show that $\triangle(\lambda)$ is finite if $\lambda \in (0,1]$ and $\triangle(\lambda)$ is infinite for $\lambda > 1$.
2. ★ What is the exact value of $\triangle(\lambda)$ for $\lambda \in (0,1)$?

2342. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given A and B are fixed points of circle Γ. The point C moves on Γ, on one side of AB. D and E are points outside $\triangle ABC$ such that $\triangle ACD$ and $\triangle BCE$ are both equilateral.

(a) Show that CD and CE each pass through a fixed point of Γ when C moves on Γ.

(b) Determine the locus of the midpoint of DE.

2343. *Proposed by Doru Popescu Anastasiu, Liceul "Radu Greceanu", Slatina, Olt, Romania.*

For positive numbers sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$, $\{z_n\}_{n\geq 1}$ with conditions: for $n \geq 1$, we have

$$(n+1)x_n^2 + (n^2+1)y_n^2 + (n^2+n)z_n^2 = 2\sqrt{n}\left(nx_ny_n + \sqrt{n}x_nz_n + y_nz_n\right),$$

and for $n \geq 2$, we have

$$x_n + \sqrt{n}y_n - nz_n = x_{n-1} + y_{n-1} - \sqrt{n-1}z_{n-1}.$$

Find $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} z_n$.

2344. *Proposed by Murali Vajapeyam, student, Campina Grande, Brazil and Florian Herzig, student, Perchtoldsdorf, Austria.*

Find all positive integers N that are quadratic residues modulo all primes greater than N.

2345. *Proposed by Vedula N. Murty, Visakhapatnam, India.*
Suppose that $x > 1$.

(a) Show that $\ln(x) > \dfrac{3(x^2-1)}{x^2+4x+1}$.

(b) Show that $\dfrac{a-b}{\ln(a)-\ln(b)} < \dfrac{1}{3}\left(2\sqrt{ab}+\dfrac{a+b}{2}\right)$,

where $a > 0$, $b > 0$ and $a \neq b$.

2346. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

The angles of $\triangle ABC$ satisfy $A > B \geq C$. Suppose that H is the foot of the perpendicular from A to BC, that D is the foot of the perpendicular from H to AB, that E is the foot of the perpendicular from H to AC, that P is the foot of the perpendicular form D to BC, and that Q is the foot of the perpendicular from E to AB.

Prove that A is acute, right or obtuse according as $\overline{AH} - \overline{DP} - \overline{EQ}$ is positive, zero or negative.

2347. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove that the equation $x^2 + y^2 = z^{1998}$ has infinitely many solutions in positive integers, x, y and z.

2348. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*
Without the use of trigonometrical formulae, prove that

$$\sin(54^\circ) = \tfrac{1}{2} + \sin(18^\circ)\ .$$

2349. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that $\triangle ABC$ has acute angles such that $A < B < C$. Prove that

$$\sin^2 B \sin\left(\tfrac{A}{2}\right)\sin\left(A+\tfrac{B}{2}\right) > \sin^2 A \sin\left(\tfrac{B}{2}\right)\sin\left(B+\tfrac{A}{2}\right)\ .$$

2350. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that the centroid of $\triangle ABC$ is G, and that M and N are the mid-points of AC and AB respectively. Suppose that circles ANC and AMB meet at (A and) P, and that circle AMN meets AP again at T.

1. Determine $AT : AP$.
2. Prove that $\angle BAG = \angle CAT$.