THE SKOLIAD CORNER

No. 30

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In the last number, we gave the problems of Part I of the Alberta High School Mathematics Competition. Here are the "official" solutions.

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION Part I — November, 1996

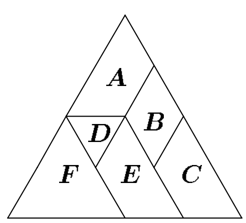
1. An eight-inch pizza is cut into three equal slices. A ten-inch pizza is cut into four equal slices. A twelve-inch pizza is cut into six equal slices. A fourteen inch pizza is cut into eight equal slices. From which pizza would you take a slice if you want as much pizza as possible?

Solution. **(b)** The area of a circle is proportional to the square of its radius. It follows that we are comparing $16/3$, $25/4$, $36/6$ and $49/8$.

2. One store sold red plums at four for a dollar and yellow plums at three for a dollar. A second store sold red plums at four for a dollar and yellow plums at six for a dollar. You bought m red plums and n yellow plums from each store, spending a total of ten dollars. How many plums in all did you buy?

Solution. **(d)** The total expenditure is $10 = \frac{m}{4} + \frac{n}{3} + \frac{m}{4} + \frac{n}{6} = \frac{(m+n)}{2}$.

3.



Six identical cardboard pieces are piled on top of one another, and the result is shown in the diagram.

The third piece to be placed is:

Solution. **(b)** Clearly, F, E, C, B and A are in that order from top to bottom. If D is pointing up, it is under A. If it is pointing down, it is under B.

4. A store offered triple the GST in savings. A sales clerk calculated the selling price by first reducing the original price by 21% and then adding the 7% GST based on the reduced price. A customer protested, saying that the store should first add the 7% GST and then reduce that total by 21%. They agreed on a compromise: the clerk just reduced the original price by the 14% difference. How do the three ways compare with one another from the customer's point of view?

Solution. **(e)** Since 1.07 times 0.79 is 0.8103, both the customer's way and the clerk's way yield a discount approximately 19%.

5. If m and n are integers such that $2m - n = 3$, then what will $m - 2n$ equal?

Solution. **(c)** We have $m - 2n = m - 2n + 2m - n - 3 = 3(m - n - 1)$.

6. If x is $x\%$ of y, and y is $y\%$ of z, where x, y and z are positive real numbers, what is z?

Solution. **(a)** Since y is $y\%$ of z, $z = 100$. The situation is possible if and only if $y = 100$ also.

7. About how many lines can one rotate a regular hexagon through some angle x, $0° < x < 360°$, so that the hexagon again occupies its original position?

Solution. **(e)** The axes of rotational symmetry are the three lines joining opposite vertices, the three lines joining the midpoints of opposite sides, and the line through the centre and perpendicular to the hexagon.

8. AB is a diameter of a circle of radius 1 unit. CD is a chord perpendicular to AB that cuts AB at E. If the arc CAD is $2/3$ of the circumference of the circle, what is the length of the segment AE?

Solution. **(b)** By symmetry, ACD is an equilateral triangle. Hence its centroid is the centre O of the circle. Since $AO = 1$, $AE = AO + OE = 3/2$.

9. One of Kerry and Kelly lies on Mondays, Tuesdays and Wednesdays, and tells the truth on the other days of the week. The other lies on Thursdays, Fridays and Saturdays, and tells the truth on the other days of the week. At noon, the two had the following conversation:

Kerry : I lie on Saturdays.

Kelly : I will lie tomorrow.

Kerry : I lie on Sundays.

On which day of the week did this conversation take place?

Solution. **(b)** Kerry is clearly lying, and is the one who tells the truth on Saturday. Hence the conversation takes place Monday, Tuesday or Wednesday, and Kelly's statement is true only on the last day.

10. How many integer pairs (m, n) satisfy the equation:
$m(m + 1) = 2n$?

Solution. **(c)** Either both m and $m + 1$ are powers of 2, or both are negatives of powers of 2. The two solutions are $(m, n) = (1, 1)$ and $(-2, 1)$.

11. Of the following triangles given by the lengths of their sides, which one has the greatest area?

Solution. **(b)** Note that $5^2 + 12^2 = 13^2$. With two sides of length 5 and 12, the area is maximum when there is a right angle between them.

12. If $x < y$ and $x < 0$, which of the following numbers is never greater than any of the others?

Solution. **(d)** If $y > 0$ or $y = 0$, then $x - y = x - |y|$ is the minimum. If $y < 0$, then $x + y = -|x + y| = x - |y|$ is the minimum.

13. An x by y flag, with $x < y$, consists of two perpendicular white stripes of equal width and four congruent blue rectangles at the corners. If the total area of the blue rectangles is half that of the flag, what is the length of the shorter side of each blue rectangle?

Solution. **(a)** Let z be the length of the shorter side of a blue rectangle. Then the longer side has length $z + \frac{(y-x)}{2}$. From $8z(z + (y - x)) = xy$, we have $z = \frac{1}{4}(x - y + \{x^2 + y^2\}^{1/2})$ or $z = \frac{1}{4}(x - y - \{x^2 + y^2\}^{1/2})$. Clearly, the negative square root is to be rejected.

14. A game is played with a deck of ten cards numbered from 1 to 10. Shuffle the deck thoroughly.

(i) Take the top card. If it is numbered 1, you win. If it is numbered k, where $k > 1$, go to (ii).

(ii) If this is the third time you have taken a card, you lose. Otherwise, put the card back into the deck at the k^{th} position from the top and go to (i).

What is the probability of winning?

Solution. **(c)** If card number 1 is initially in the first or second position from the top, you will win. You can also win if it is in the third, and card number 2 is not in the first. Hence the winning probability is $\frac{1}{10} + \frac{1}{10} + \frac{1}{10}\left(\frac{8}{9}\right)$.

15. Five of the angles of a convex polygon are each equal to $108°$. In which of the following five intervals does the maximum angle of all such polygons lie?

Solution. **(a)** The sum of the exterior angles of the five given angles is $5(180° - 108°) = 360°$. Hence these five angles are the only angles of the convex polygon.

16. Which one of the following numbers cannot be expressed as the difference of the squares of two integers?

Solution. **(b)** Suppose $k = m^2 - n^2 = (m - n)(m + n)$. If k is odd, we can set $m - n = 1$ and $m + n = k$. If $k = 4l$, we can set $m - n = 2$ and $m + n = 2l$. However, if $k = 4l + 2$, then $m - n$ and $m + n$ cannot have the same parity.

As a contest this number we give the Fifteenth W.J. Blundon Contest, which was taken by students in Newfoundland and Labrador. The contest is supported in part by the Canadian Mathematical Society. My thanks go to Bruce Shawyer for forwarding me a copy.

15^{th} W.J. BLUNDON CONTEST
February 18, 1998

1. (a) Find the exact value of

$$\frac{1}{\log_2 36} + \frac{1}{\log_3 36}\,.$$

(b) If $\log_{15} 5 = a$, find $\log_{15} 9$ in terms of a.

2. (a) If the radius of a right circular cylinder is increased by 50% and the height is decreased by 20%, what is the change in the volume?

(b) How many digits are there in the number $2^{1998} \cdot 5^{1988}$?

3. Solve: $3^{2+x} + 3^{2-x} = 82$.

4. Find all ordered pairs of integers such that $x^6 = y^2 + 53$.

5. When one-fifth of the adults left a neighbourhood picnic, the ratio of adults to children was $2 : 3$. Later, when 44 children left, the ratio of children to adults was $2 : 5$. How many people remained at the picnic?

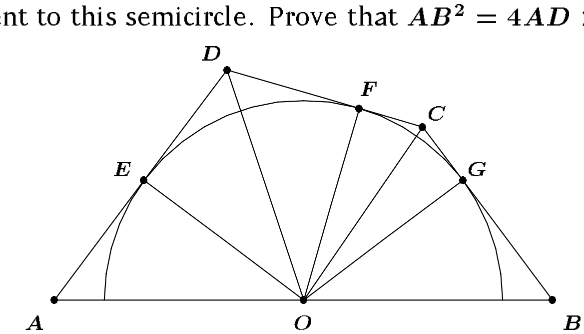
6. Find the area of a rhombus for which one side has length 10 and the diagonals differ by 4.

7. In how many ways can 10 dollars be changed into dimes and quarters, with at least one of each coin being used?

8. Solve: $\sqrt{2 + 10} + \sqrt[4]{x + 10} = 12$.

9. Find the remainder when the polynomial $x^{135} + x^{125} - x^{115} + x^5 + 1$ is divided by the polynomial $x^3 - x$.

10. Quadrilateral $ABCD$ below has the following properties: (1) The midpoint O of side AB is the centre of a semicircle; (2) sides AD, DC and CB are tangent to this semicircle. Prove that $AB^2 = 4AD \times BC$.



Now a correction to a solution given in the October number of the Skoliad Corner.

4. [1997:347] *1995 P.E.I. Mathematics Competition.*
An *autobiographical number* is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells you how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's, and so on. For example, $6,210,001,000$ is an autobiographical number. Find the smallest autobiographical number and prove that it is the smallest.

Correction by Vedula N. Murty, Visakhapatnam, India. The answer given is 1201, but the smallest example is 1210.

Editor's Note: Dyslexia strikes again!

That is all for this issue of the Skoliad Corner. I need your suitable materials and your suggestions for directions for this feature.

Advance Announcement

The 1999 Summer Meeting of the Canadian Mathematical Society will take place at Memorial University in St. John's, Newfoundland, from Saturday, 29 May 1999 to Tuesday, 1 June 1999.

The Special Session on Mathematics Education will feature the topic
What Mathematics Competitions do for Mathematics.
The invited speakers are

Ed Barbeau (University of Toronto),
Ron Dunkley (University of Waterloo),
Tony Gardiner (University of Birmingham, UK), and
Rita Janes (Newfoundland and Labrador Senior Mathematics League).

Requests for further information, or to speak in this session, as well as suggestions for further speakers, should be sent to the session organizers:

Bruce Shawyer and Ed Williams
CMS Summer 1999 Meeting, Education Session
Department of Mathematics and Statistics, Memorial University
St. John's, Newfoundland, Canada A1C 5S7