A Note on Special Numerals in Arbitrary Bases

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1 Introduction

In [1], at the end of a section devoted to the Pigeonhole Principle, the author posed the following problem:

> "4.2.28 Show that, for any integer n, there exists a multiple of n that contains only the digits 7 and 0."

Plainly the author is referring to the representation of the multiple as a numeral in base 10. Thus the reader of the problem is encouraged to believe that the validity of the statement depends in some way on the numbers 7 and 10 and their mutual relationship. Our analysis of the problem shows that, in fact, we may replace 10 by any base $b \geq 2$, and 7 by any digit t in base b. (Of course, we assume $t \neq 0$ to avoid absurd triviality.) Moreover, we prefer to restrict ourselves to numerals consisting of a sequence of k t's followed by l 0's; and, if we assume, as we may without real loss of generality, that $n > 0$, then we have $k \geq 1$, $l \geq 0$. We then give two arguments for the conclusion. The first does employ the Pigeonhole Principle but does not give us a means of calculating all pairs (k, l) from the data. Our second approach yields the minimal pair (k, l) and shows, as expected, that an arbitrary pair (K, L) in the set of solutions is obtained from the minimal pair by taking K to be an arbitrary multiple of k, and L an arbitrary integer satisfying $L \geq l$.

We prefer to replace the modulus n of the problem as stated by m, thus freeing n to stand for the multiple we are seeking. Also we point out that, of course, the analysis we carry out does not depend on the condition $t < b$ imposed by the requirement that t be a digit in base b. Thus the number n we seek may be represented, for arbitrary t and base b, by the expression

$$n \quad = \quad t\,\frac{(b^k - 1)b^l}{b - 1}\,. \tag{1.1}$$

2 The two main arguments

Let b be an arbitrary base and let t, $0 < t \leq b - 1$, be an arbitrary non-zero digit in base b.

Theorem 1 *For any integer $m > 0$, there exists a positive integer n, written as a numeral in base b consisting of a sequence of t's followed by a (possibly empty) sequence of 0's, such that $n \equiv 0 \bmod m$.*

We give two proofs, the first uses the Pigeonhole Principle, the second being an exercise in modular arithmetic.

Proof A.

Consider the sequence of numerals 0, t, tt, ttt, $\ldots$. The remainders $\bmod\ m$ of these integers lie in a set containing m elements. Thus after at most $(m+1)$ such numerals, some remainder must have been repeated. If $ttt\ldots$ ($k+l$ occurrences), and $ttt\ldots$, (l occurrences), $k > 0$ yield the same remainders, then the numeral $ttt\ldots 000\ldots$ (k occurrences of t, l zeros) fulfils the conclusion of the theorem.

This elegant proof has the single defect that it gives no clue as to how one finds the minimal values of k and l as functions of b, t and m. Our second proof is not so neat, but gives more information.

Proof B.

Let $z = b - 1$. We show first how to find a smallest number $n = n_z$, whose numeral in base b consists of a sequence of z's followed by a sequence of 0's, such that $n \equiv 0 \bmod m$.

Write $m = vw$, where v is prime to b and, for every prime p, $p|w$ implies that $p|b$. Note that this factorization of m is unique. Let k be the order of $b \bmod v$ and let l be the smallest non-negative integer such that $w|b^l$. Then the number n, represented in base b by a sequence of k z's followed by a sequence of l 0's, is obviously a positive integer n_z of the form

$$n_z = zzz\ldots 000\ldots \tag{2.1}$$

divisible by m. We will show below that n_z, given by (2.1), is minimal for this property.

It is now a trivial matter to complete the proof of the theorem. For if we replace m by mz in the argument above, we find a number $n_z = zzz\ldots 000\ldots$ such that $n_z \equiv 0 \bmod mz$, so that n_1, given by $zn_1 = n_z$ gives us

$$n_1 = 111\ldots 000\ldots \equiv 0 \bmod m. \tag{2.2}$$

But then

$$n = tn_1 = ttt\ldots 000\ldots \equiv 0 \bmod m. \tag{2.3}$$

However, if we want to make sure we have the *smallest* number n of the required form, we should proceed more cautiously.

We first prove that if $n = n_z$ is chosen as in (2.1), then it is the smallest integer of the given form to satisfy $n \equiv 0 \bmod m$.

Now if n is of the form $zzz\ldots000\ldots$, with K z's and L 0's, then [see (1.1)] $n = (b^K - 1)b^L$. Moreover, since v, w are coprime, we have

$$n \equiv 0 \bmod m \iff n \equiv 0 \bmod v \text{ and } n \equiv 0 \bmod w.$$

Since v is prime to b,

$$n \equiv 0 \bmod v \iff b^K - 1 \equiv 0 \bmod v \iff k|K.$$

Further, since, for all primes p, $p|w$ implies that $p|b$, we have that w is prime to $b^K - 1$, so that

$$n \equiv 0 \bmod w \iff b^L \equiv 0 \bmod w \iff L \geq l.$$

Now suppose that n is chosen to be the smallest integer of the form $zzz\ldots000\ldots$ such that $n \equiv 0 \bmod mz$, and let $n = zn_1$. Then since

$$zn_1 \equiv 0 \bmod mz \iff n_1 \equiv 0 \bmod m, \tag{2.4}$$

it follows that n_1, given by (2.2), is the smallest integer of the form $111\ldots000\ldots$ to satisfy $n_1 \equiv 0 \bmod m$.

We want now to find the smallest integer n_t represented in base b by a sequence of t's followed by a sequence of 0's [compare (2.3)] to satisfy $n_t \equiv 0 \bmod m$.

Our recipe for constructing n_t is as follows. Let $d = \gcd(m, t)$, $m = m'd$, $t = t'd$. Then if n_1 has the form $111\ldots000\ldots$,

$$tn_1 \equiv 0 \bmod m \iff t'n_1 \equiv 0 \bmod m' \iff n_1 \equiv 0 \bmod m'. \tag{2.5}$$

Thus we construct n as in (2.1) to be minimal satisfying $n \equiv 0 \bmod m'z$. Then $n = zn_1$, and, by (2.4), we have that n_1 is minimal of the required form to satisfy $n_1 \equiv 0 \bmod m'$, so that finally, by (2.5), we have that $n_t = tn_1$ is minimal of the form $ttt\ldots000\ldots$ to satisfy $n_t \equiv 0 \bmod m$.

Example 1.

Let us work in base $b = 10$ and look for the smallest number n_6 of the form

$$n_6 = 666\ldots000\ldots$$

divisible by 99. We have $b = 10$, $z = 9$, $m = 99$, $t = 6$, so $d = 3$, $m' = 33$, $m'z = 297$. To find n, minimal of the form $999\ldots000\ldots$, to satisfy $n \equiv 0 \bmod 297$, we factorize 297, as in the construction above, as $297 = 297 \times 1$, since 297 is prime to 10. We find $k = 6$ (that is, the order of 10 mod 297 is 6) and, of course, $l = 0$. Thus $n = 999999$, so $n_1 = 111111$, $n_6 = 666666$.

Example 1 brings out the important practical point that, to find n_t, we simply find the minimal n of the form $zzz\ldots 000\ldots$ to satisfy $n \equiv 0 \bmod m'z$, and then replace z by t in the numeral for n. Indeed, all we have to do is to find the minimal values of k and l which yield n (see Section 3).

It is also plain that, having obtained our minimal n_t, involving k t's followed by l 0's, we obtain *all* solutions of the congruence $N \equiv 0 \bmod m$ of the required form by taking K t's followed by L 0's, where $k|K$ and $L \geq l$. Notice that this implies that every solution of the congruence $N \equiv 0 \bmod m$, of the required form, is a multiple of the minimum solution.

3 The algorithm

We extract from the analysis in Section 2 the algorithm for finding the minimal pair (k, l) as functions of b, t and m.

Given: b (base), t (digit) and m (modulus).

- **Set** $\gcd(m, t) = d$, and $m = m'd$.
- **Write** $m'(b-1) = vw$ where v is prime to b and, for all primes p,
$$p|w \quad \Longrightarrow \quad p|b.$$
- **Finally**, let k be the order of $b \bmod v$ (that is, $b^k \equiv 1 \bmod v$, with k positive minimal) and let l be minimal such that $w|b^l$. Then
$$n_t = \overbrace{ttt\ldots}^{k \text{ times}}\overbrace{000\ldots}^{l \text{ times}}$$
is the minimal numeral n, in base b, consisting of a sequence of k t's followed by a sequence of l 0's, such that $m|n$.

Example 2.

Given: $b = 7$, $t = 5$ and $m = 2499$.

- $\gcd(2499, 5) = 1$, so that $m = m'$.
- **Write** $2499 \times 6 = 306 \times 49$ (thus, $v = 306$, $w = 49$).
- **Finally**, let k be the order of $7 \bmod 306$ (that is, $7^k \equiv 1 \bmod 306$, with k positive minimal), and let l be minimal such that $49|7^l$. Using modular arithmetic[1] we see that $k = 48$, and it is clear that $l = 2$. Thus
$$n_5 = \overbrace{555\ldots}^{48 \text{ times}}00$$

[1] $306 = 2 \times 9 \times 17$. The order of $7 \bmod 2$ is 1; the order of $7 \bmod 9$ is 3; the order of $7 \bmod 17$ is 16. To see the last, observe that, by Fermat's Theorem, $7^{16} \equiv 1 \bmod 17$; but $7^2 \equiv -2 \bmod 17$, so $7^8 \equiv -1 \bmod 17$. Thus the order of $7 \bmod 306$ is 48.

is the minimal numeral, in base 7, consisting of a sequence of 5's followed by a sequence of 0's such that $2499|n_5$.

Example 2 shows that it may sometimes be very tedious to apply the Pigeonhole Principle to obtain k and l.

Notice that, in the special case $t = b-1$, we may simplify the algorithm by cutting out the first step and replacing $m'(b-1)$ by m in the second step.

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Reference [1] Zeitz, Paul, *The Art and Craft of Problem Solving*, John Wiley & Sons, 1998.

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