THE OLYMPIAD CORNER

No. 190

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Since the "summer break" is coming up, we give three Olympiad contests from three different parts of the world. My thanks go to Bill Sands for collecting the contest materials for me when he was helping to coordinate marking of the IMO held in Toronto in 1995.

We first give the problems of the Grade XI and Grade XII versions of the Lithuanian Mathematical Olympiad.

44^{th} LITHUANIAN MATHEMATICAL OLYMPIAD (1995)

GRADE XI

1. You are given a set of 10 positive integers. Summing nine of them in ten possible ways we only get nine different sums: 86, 87, 88, 89, 90, 91, 93, 94, 95. Find those numbers.

2. What is the least possible number of positive integers such that the sum of their squares equals 1995?

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Replace the asterisks in the "equilateral triangle" by the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 so that, starting from the second line, each number is equal to the absolute value of the difference of the nearest two numbers in the line above.

Is it always possible to inscribe the numbers $1, 2, \ldots, n$, in the way required, into the equilateral triangle with the sides having n asterisks?

4. A function $f : \mathbb{N} \to \mathbb{N}$ is such that

$$f(f(m) + f(n)) = m + n$$

for all $m, n \in \mathbb{N}$ ($\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of all positive integers). Find all such functions.

5. In a trapezium $ABCD$, the bases are $AB = a$, $CD = b$, and the diagonals meet at the point O. Find the ratio of the areas of the triangle ABO and trapezium.

GRADE XII

1. Consider all pairs (x, y) of real numbers satisfying the inequalities

$$-1 \leq x + y \leq 1, \quad -1 \leq xy + x + y \leq 1.$$

Let M denote the largest possible value of x.

(a) Prove that $M \leq 3$.
(b) Prove that $M \leq 2$.
(c) Find M.

2. A positive integer n is called an *ambitious* number if it possesses the following property: writing it down (in decimal representation) on the right of any positive integer gives a number that is divisible by n. Find:

(a) the first 10 ambitious numbers;
(b) all the ambitious numbers.

3. The area of a trapezium equals 2; the sum of its diagonals equals 4. Prove that the diagonals are mutually orthogonal.

4. 100 numbers are written around a circle. Their sum equals 100. The sum of any 6 neighbouring numbers does not exceed 6. The first number is 6. Find the remaining numbers.

5. Show that, at any time, moving both the hour-hand and the minute-hand of the clock symmetrically with respect to the vertical $(6 - 12)$ axis results in a possible position of the clock-hands. How many straight lines containing the centre of the clock-face possess the same property?

Next we give the problems of the Korean Mathematical Olympiad.

8^{th} KOREAN MATHEMATICAL OLYMPIAD
First Round

Morning Session — 2.5 hours

1. Consider finitely many points in a plane such that, if we choose any three points A, B, C among them, the area of $\triangle ABC$ is always less than 1. Show that all of these finitely many points lie within the interior or on the boundary of a triangle with area less than 4.

2. For a given positive integer m, find all pairs (n, x, y) of positive integers such that m, n are relatively prime and $(x^2 + y^2)^m = (xy)^n$, where n, x, y can be represented by functions of m.

3. Let A, B, C be three points lying on a circle, and let P, Q, R be the midpoints of arcs BC, CA, AB, respectively. AP, BQ, CR intersect BC, CA, AB at L, M, N, respectively. Show that

$$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} \geq 9.$$

For which triangle ABC does equality hold?

4. A partition of a positive integer n is a sequence $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Each λ_i is called a summand. For example, $(4, 3, 1)$ is a partition of 8 whose summands are distinct. Show that, for a positive integer m with $n > \frac{1}{2}m(m+1)$, the number of all partitions of n into distinct m summands is equal to the number of all partitions of $n - \frac{1}{2}m(m+1)$ into r summands $(r \leq m)$.

Afternoon Session — 2.5 hours

5. If we select at random three points on a given circle, find the probability that these three points lie on a semicircle.

6. Show that any positive integer $n > 1$ can be expressed by a finite sum of numbers satisfying the following conditions:

(i) they do not have factors except 2 or 3;
(ii) any two of them are neither a factor nor a multiple of each other.

That is,

$$n = \sum_{i=1}^{N} 2^{\alpha_i} 3^{\beta_i},$$

where α_i, β_i $(i = 1, 2, \dots, N)$ are nonnegative integers and, whenever $i \neq j$, the condition $(\alpha_i - \alpha_j)(\beta_i - \beta_j) < 0$ is satisfied.

7. Find all real valued functions f defined on real numbers except 0 such that

$$\frac{1}{x} f(-x) + f\left(\frac{1}{x}\right) = x, \quad x \neq 0.$$

8. Two circles O_1, O_2 of radii r_1, r_2 $(r_1 < r_2)$, respectively, intersect at two points A and B. P is any point on circle O_1. Lines PA, PB and circle O_2 intersect at Q and R, respectively.

(i) Express $y = QR$ in terms of r_1, r_2, and $\theta = \angle APB$.

(ii) Show that $y = 2r_2$ is a necessary and sufficient condition that circle O_1 be orthogonal to circle O_2.

Final Round

First Day — 4.5 hours

1. For any positive integer m, show that there exist integers a, b satisfying

$$|a| \leq m, \quad |b| \leq m, \quad 0 < a + b\sqrt{2} \leq \frac{1+\sqrt{2}}{m+2}\,.$$

2. Let A be the set of all non-negative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

(i) for any $m, n \in A$,

$$2f(m^2+n^2) = \{f(m)\}^2 + \{f(n)\}^2;$$

(ii) for any $m, n \in A$ with $m \geq n$,

$$f(m^2) \geq f(n^2).$$

3. Let $\triangle ABC$ be an equilateral triangle of side length 1, let D be a point on BC, and let r_1, r_2 be inradii of triangles ABD, ADC, respectively. Express r_1r_2 in terms of $p = BD$, and find the maximum of r_1r_2.

Second Day — 4.5 hours

4. Let O and R be the circumcentre and the circumradius of $\triangle ABC$, respectively, and let P be any point on the plane of ABC. Let perpendiculars PA_1, PB_1, PC_1, be dropped to the three sides BC, CA, AB. Express

$$\frac{[A_1B_1C_1]}{[ABC]}$$

in terms of R and $d = OP$, where $[ABC]$ denotes the area of $\triangle ABC$.

5. Let p be a prime number such that

(i) p is the greatest common divisor of a and b;

(ii) p^2 is a divisor of a. Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be decomposed into the product of two polynomials with integral coefficients, whose degrees are greater than one.

6. Let m, n be positive integers with $1 \leq n \leq m - 1$. A box is locked with several padlocks, all of which must be opened to open the box, and all of which have different keys. A total of m people each have keys to some of the locks. No n people of them can open the box but any $n + 1$ people can open the box. Find the smallest number l of locks and in that case find the number of keys that each person has.

Now, problems selected from the 1995 Israel Mathematical Olympiads.

Selected Problems From
ISRAEL MATHEMATICAL OLYMPIADS 1995

1. n positive integers $d_1, d_2, \ldots, d_n$ divide 1995. Prove that there exist d_i and d_j among them, such that the denominator of the reduced fraction $\frac{d_i}{d_j}$ is at least n.

2. Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with O. Then, Player II chooses another square and marks it with X. They play until one of the players marks a whole row or a whole column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the result of the game is a tie. Show that Player II can prevent Player I from winning.

3. Two thieves stole an open chain with $2k$ white beads and $2m$ black beads. They want to share the loot equally, by cutting the chain to pieces in such a way that each one gets k white beads and m black beads. What is the minimal number of cuts that is always sufficient?

4. α and β are two given circles that intersect each other at two points. Find the geometric locus of the centres of all circles that are orthogonal to both α and β.

5. Four points are given in space, in a general position (that is, they are not contained in a single plane). A plane π is called "an equalizing plane" if all four points have the same distance from π. Find the number of equalizing planes.

6. n is a given positive integer. A_n is the set of all points in the plane, whose x and y coordinate are positive integers between 0 and n. A point (i, j) is called "internal" if $0 < i, j < n$. A real function f, defined on A_n, is called a "good function" if it has the following property: for every internal point x, the value of $f(x)$ is the mean of its values on the four neighbouring points (the neighbouring points of x are the four points whose distance from x equals 1).

If f and g are two given good functions and $f(a) = g(a)$ for every point a in A_n which is not internal (that is, a boundary point), prove that $f \equiv g$.

7. Solve the system

$$\begin{aligned} x + \log(x + \sqrt{x^2+1}) &= y \\ y + \log(y + \sqrt{y^2+1}) &= z \\ z + \log(z + \sqrt{z^2+1}) &= x \end{aligned}$$

8. Prove the inequality

$$\frac{1}{kn}+\frac{1}{kn+1}+\frac{1}{kn+2}+\cdots+\frac{1}{(k+1)n-1}\geq n\left(\sqrt[n]{\frac{k+1}{k}}-1\right)$$

for any positive integers k, n.

9. PQ is a diameter of a half circle H. The circle O is tangent to H from the inside and touches diameter PQ at the point C. A is a point on H and B is a point on PQ such that AB is orthogonal to PQ and is also tangent to the circle O. Prove that AC bisects the angle PAB.

10. α is a given real number. Find all functions $f:(0,\infty)\mapsto(0,\infty)$ such that the equality

$$\alpha x^2 f\left(\frac{1}{x}\right)+f(x)=\frac{x}{x+1}$$

holds for all real $x>0$.

Next we turn to solutions to problems posed in the February 1997 number of the Corner. Some new solutions arrived after we went to press last issue, and at least one other batch of solutions was incorrectly filed and just turned up! Pavlos Maragoudakis, Pireas, Greece sent in solutions to problems 1, 3 and 4 of the Final Grade, Third Round, and also to Problem 1 of the 1^{st} Selection Round. The misplaced solutions were from D.J. Smeenk, who gave solutions to Problem 3 of the Final Grade, 3^{rd} Round and to Problem 3 of each of the second and third Selection Rounds. Because they are interesting and different, we give his two solutions to Problem 3 of the Final Grade. (Look at all the 3's in the above!)

3. [1997: 78, 1998: 13–14] *Latvian* 44 *Mathematical Olympiad*.

It is given that $a>0$, $b>0$, $c>0$, $a+b+c=abc$. Prove that at least one of the numbers a, b, c exceeds $17/10$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

In any triangle ABC the following identity holds

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$$

Hence, in this problem a, b, c can be considered to be the tangents of angles of an acute angled triangle. At least one of the angles is at least $\frac{\pi}{3}$, and $\tan\left(\frac{\pi}{3}\right)=\sqrt{3}>\frac{17}{10}$.

Second solution.

We may suppose $a \geq b \geq c$,

$$abc \;=\; a + b + c \;\geq\; 3c.$$

So $ab \geq 3$, $a \geq b$ and $a \geq \sqrt{3} > \frac{17}{10}$.

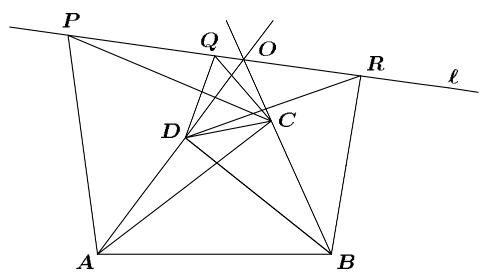
Now we turn to readers' solutions of problems from the March 1997 Corner and the 3rd Mathematical Olympiad of the Republic of China (Taiwan) [1997: 66].

3rd MATHEMATICAL OLYMPIAD OF THE REPUBLIC OF CHINA (Taiwan)

First Day — April 14, 1994

1. Let $ABCD$ be a quadrilateral with $\overline{AD} = \overline{BC}$ and let $\angle A + \angle B = 120^\circ$. Three equilateral triangles $\triangle ACP$, $\triangle DCQ$ and $\triangle DBR$ are drawn on $\overline{AC}$, $\overline{DC}$ and $\overline{DB}$ away from $\overline{AB}$. Prove that the three new vertices P, Q and R are collinear.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let O be the intersection of AD and BC. Since $\angle A + \angle B = 120^\circ$, we get $\angle AOB = 60^\circ$. Let l be the exterior bisector of angle AOB. Since $\angle APC = 60^\circ = \angle AOC$, we have that O, P, A, C are concyclic. Hence $\angle POA = \angle PCA = 60^\circ$. The exterior angle of $\angle AOB$ is 120°, showing that PO bisects the exterior angle of $\angle AOB$. Thus P lies on l. Similarly Q and R lie on l. Hence, P, Q and R are collinear.

Comment. As is shown in the proof, the condition $AD = BC$ is not necessary.

2. Let a, b, c be positive real numbers, α be a real number. Suppose that

$$f(\alpha) = abc(a^\alpha + b^\alpha + c^\alpha)$$

$$g(\alpha) = a^{\alpha+2}(b + c - a) + b^{\alpha+2}(a - b + c) + c^{\alpha+2}(a + b - c)$$

Determine the magnitude between $f(\alpha)$ and $g(\alpha)$.

Solution by Panos E. Tsaoussoglou, Athens, Greece.

$$\begin{aligned}
&bca^{\alpha+1} + acb^{\alpha+1} + abc^{\alpha+1} - a^{\alpha+2}(b+c-a) \\
&\qquad\quad -b^{\alpha+2}(a-b+c) - c^{\alpha+2}(a+b-c) \\
&= a^{\alpha+1}(bc - a(b+c) + a^2) + b^{\alpha+1}(ac - b(a+c) + b^2) \\
&\qquad\quad +c^{\alpha+1}(ab - c(a+b) + c^2) \\
&= a^{\alpha+1}(a-b)(a-c) + b^{\alpha+1}(b-a)(b-c) + c^{\alpha+1}(c-a)(c-b) \\
&\geq 0
\end{aligned}$$

which is an inequality of Schur.

Next we move to solutions of Selected Problems from the Israel Mathematical Olympiads, 1994 [1997: 131].

SELECTED PROBLEMS FROM THE ISRAEL MATHEMATICAL OLYMPIADS, 1994

1. p and q are positive integers. f is a function defined for positive numbers and attains only positive values, such that $f(xf(y)) = x^p y^q$. Prove that $q = p^2$.

Solutions by Pavlos Maragoudakis, Pireas, Greece; and Michael Selby, University of Windsor, Windsor, Ontario. We give the solution of Maragoudakis.

For $x = \dfrac{1}{f(y)}$, we get $f(y) = \dfrac{y^{q/p}}{(f(1))^{1/p}}$.

For $y = 1$, we get $f(1) = 1$, so $f(y) = y^{q/p}$. Hence $f(x \cdot y^{q/p}) = x^p \cdot y^q$. For $y = z^{p/q}$ we get $f(x \cdot z) = x^p z^p$ or $f(x) = x^p$.

Thus $\frac{q}{p} = p$, whence $q = p^2$.

2. The sides of a polygon with 1994 sides are $a_i = \sqrt{4 + i^2}$, $i = 1, 2, \ldots, 1994$. Prove that its vertices are not all on integer mesh points.

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

One assumes integer mesh points are lattice points.

Assume that P_i, the i^{th} vertex, has coordinates (x_i, y_i) where x_i, y_i are integers. Let $\overrightarrow{d_i} = (x_{i+1} - x_i, y_{i+1} - y_i) = (\alpha_i, \beta_i)$ be a vector representation of the i^{th} side. (The indices are read cyclically.) Then $|\overrightarrow{d_i}|^2 = a_i^2 = 4 + i^2$. Also $\overrightarrow{d_{1994}} = (x_1 - x_{1994}, y_1 - y_{1994})$.

We know that

$$\sum_{i=1}^{1994} (\overrightarrow{d_i})^2 = \sum_{i=1}^{1994} (4 + i^2), \qquad \text{or}$$

$$\sum_{i=1}^{1994}(\alpha_i^2+\beta_i^2) = 4\cdot 1994 + \frac{(1994)(1995)(3989)}{6}$$
$$= 4\cdot 1994 + (997)(665)(3989),$$

which is an odd integer.

However, we know $\sum_{i=1}^{1994}\alpha_i = \sum_{i=1}^{1994}\beta_i = 0$, since $\sum_{i=1}^{1994}\overrightarrow{d_i} = \overrightarrow{0}$.

Therefore $\left(\sum_{i=1}^{1994}(\alpha_i+\beta_i)\right)^2 = 0$.

Thus

$$\sum_{i=1}^{1994}(\alpha_i^2+\beta_i^2) + 2\left(\sum_{ij}\alpha_i\beta_j + \sum_{i<j}\alpha_i\alpha_j + \sum_{i<j}\beta_i\beta_j\right) = 0$$

and

$$2\left(\sum_{i,j}\alpha_i\beta_j + \sum_{i<j}\alpha_i\alpha_j + \sum_{i<j}\beta_i\beta_j\right) = -\left(\sum_i(\alpha_i^2+\beta_i^2)\right),$$

giving an odd integer on the right, and an even one on the left, a contradiction.

Therefore, not all the vertices can be lattice points.

5. Find all real coefficients polynomials $p(x)$ satisfying

$$(x-1)^2 p(x) = (x-3)^2 p(x+2)$$

for all x.

Solutions by F.J. Flanigan, San Jose State University, San Jose, California, USA; and Michael Selby, University of Windsor, Windsor, Ontario. We give Flanigan's solution.

We consider polynomials $p(x)$ with coefficients in a field $\mathbb{F}$ of *arbitrary* characteristic and find as follows:

(i) If char($\mathbb{F}$) $= 0$, (in particular, if $\mathbb{F} = \mathbb{R}$), then $p(x) = a(x-3)^2$, where a is any scalar (possibly 0) in $\mathbb{F}$;

(ii) If char($\mathbb{F}$) $= 2$, then *every* $p(x)$ satisfies the equation (clear);

(iii) If char($\mathbb{F}$) $=$ an odd prime, l, then there are infinitely many solutions, including all $p(x) = a(x-3)^2(x^{l^\nu} - x + c)$ with $a, c \in \mathbb{F}$, and $\nu = 0, 1, 2, \ldots$. (Note that $p(x)$ has the form $a(x-3)^2$ if $\nu = 0$.)

To prove this, observe that if char($\mathbb{F}$) $\neq 2$, then $x-1$ and $x-3$ are coprime, whence $p(x) = (x-3)^2 q(x)$ in $\mathbb{F}[x]$.

Thus our equation becomes

$$(x-1)^2(x-3)^2 q(x) = (x-3)^2(x-1)^2 q(x+2) \qquad (*)$$

whence $q(x) = q(x+2)$, as polynomials; that is, elements of $\mathbb{F}[x]$.

Now if char($\mathbb{F}$) $= 0$, then $(*)$ has only constant solutions.

(The most elementary proof of this: without loss of generality, $q(x) = x^n + ax^{n-1} + \cdots$. Then $q(x+2) - q(x) = 2nx^{n-1} + \cdots$, and this is non-zero if $n \geq 1$. Another proof: $(*)$ implies that $q(x)$ is periodic, which forces equations $q(x) = c$ to have infinitely many roots x, a contradiction).

This establishes the assertion (i).

Re: assertion (iii). Let char($\mathbb{F}$) $= l$ and $q(x) = x^{l^\nu} - x + c$.

Then for $x = 0, 1, \ldots, l-1$, (that is for each element of the prime field), we have $q(x) = c$ and so $q(x) = q(x+1) = q(x+2) = \ldots$, yielding polynomials of degree greater than or equal to l which satisfy $(*)$. This establishes the assertion (iii).

We next turn to solutions to Problems From the Bi-National Israel–Hungary Competition, 1994 [1997: 132].

PROBLEMS FROM THE BI-NATIONAL ISRAEL-HUNGARY COMPETITION, 1994

1. $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n$ are positive numbers $(k < n)$. Suppose that the values of $a_{k+1}, \ldots, a_n$ are fixed. How should one choose the values of $a_1, \ldots, a_n$ in order to minimize $\sum_{i,j,i\neq j} \frac{a_i}{a_j}$?

Solutions by F.J. Flanigan, San Jose State University, San Jose, California, USA; and Michael Selby, University of Windsor, Windsor, Ontario. We give Flanigan's solution.

To minimize the given rational function, choose

$$a_i = \left(\frac{a_{k+1} + \cdots + a_n}{\frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n}}\right)^{1/2} = (\mathbb{A} \cdot \mathbb{H})^{1/2}, \quad i = 1, 2, \ldots, k$$

where $\mathbb{A}$ is the arithmetic and $\mathbb{H}$ the harmonic mean of $a_{k+1}, \ldots, a_n$.

To prove this, we will be forgiven if we change notation: let $x_i = a_i$, $i = 1, 2, \ldots, k$ and $b_r = a_{k+r}$, $r = 1, \ldots, m$ with $k + m = n$, and denote the given rational function $F(x_1, \ldots, x_k)$. Then we have $F(x_1, \ldots, x_k) = X + Y + B$, where

$$\begin{aligned} X &= \sum_{1\leq i<j\leq k} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i}\right), \\ Y &= \sum_{1\leq i\leq k} \sum_{1\leq r\leq m} \left(\frac{x_i}{b_r} + \frac{b_r}{x_i}\right), \\ B &= \sum_{1\leq r<s\leq m} \left(\frac{b_r}{b_s} + \frac{b_s}{b_r}\right). \end{aligned}$$

Note that B is fixed and Y can be improved to

$$\begin{aligned} Y &= \sum_{1\le i\le k}\left(\left(\sum_{1\le r\le m}\frac{1}{b_r}\right)x_i + \left(\sum_{1\le r\le m} b_i\right)\frac{1}{x_i}\right) \\ &= \sum_i \left(\frac{m}{\mathbb{H}}x_i + \frac{m\mathbb{A}}{x_i}\right) \end{aligned}$$

where $\mathbb{A}$ is the arithmetic mean and $\mathbb{H}$ is the harmonic mean of the b_r.

Now we recall that the simple function $\alpha x + \frac{\beta}{x}$ (with α, β, x all positive) assumes its minimum when $\alpha x = \frac{\beta}{x}$; that is $x = \sqrt{\beta/\alpha}$. Thus each of the terms in Y (and so Y itself) assumes its minimum when we choose, for $i = 1, 2, \ldots, k$,

$$x_i = \sqrt{\frac{m\mathbb{A}}{(m/\mathbb{H})}} = \sqrt{\mathbb{A}\mathbb{H}},$$

as asserted.

But there is more. It is also known that each term in X, (and so X itself) assumes its minimum when $x_i = x_j$, with $1 \le i < j \le k$. Thus choosing all $x_i = \sqrt{\mathbb{A}\mathbb{H}}$ minimizes *both* X *and* Y and, since B is fixed, minimizes $F(x_1, \ldots, x_k)$ as claimed.

Comments:

(1) It is unusual to minimize the sum of two terms in the same variables by minimizing each term simultaneously.

(2) When $m = 2$, then $\sqrt{\mathbb{A}\mathbb{H}} = \mathbb{G}$, the geometric mean of b_1, b_2.

(3) $F_{\min} = k(k-1) + 2k(n-k)^2\sqrt{\mathbb{A}/\mathbb{H}} + B$.

3. m, n are 2 different natural numbers. Show that there exists a real number x, such that $\frac{1}{3} \le \{xn\} \le \frac{2}{3}$ and $\frac{1}{3} \le \{xm\} \le \frac{2}{3}$, where $\{a\}$ is the fractional part of a.

Solution by F.J. Flanigan, San Jose State University, San Jose, California, USA.

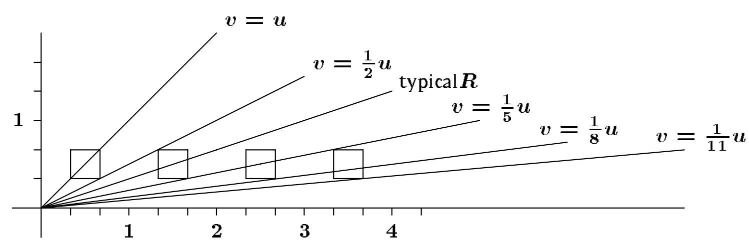
We work in the first quadrant of the standard uv–plane, studying the ray

$$R = \{(u, v) = (xm, xn) : x > 0\}.$$

The key is to observe that the problem is equivalent to showing that the ray R contacts at least one of the "small" $\frac{1}{3}$ by $\frac{1}{3}$ squares in the centres of the large standard 1 by 1 lattice squares (considered as a 3×3 checkerboard). (For if (xm, xn) lies in one of these small squares then $(\{xm\}, \{xn\})$ lies in the small square $\{(u, v) : \frac{1}{3} \le u, v \le \frac{2}{3}\}$ closest to the origin, as desired.)

To establish this contact, we assume, without loss of generality, that $0 < n < m$, so that R is given by $v = \frac{n}{m}u$, $u > 0$.

Consider the sequence of rays $v = \frac{1}{2}u$, $v = \frac{1}{5}u$, $v = \frac{1}{8}u$, $v = \frac{1}{11}u, \ldots$, with $u > 0$. These rays are determined by the lower right corners of the 1^{st}, 2^{nd}, 3^{rd}, $4^{\text{th}}, \ldots$, small central square.



It is now apparent that our ray R lies between, (or on) the ray $v = u$ and $v = \frac{1}{2}u$, **or** $v = \frac{1}{2}u$ and $v = \frac{1}{5}u$, **or** $\ldots$, and hence R will contact the first **or** the second **or** $\ldots$, small square, as required.

Comment: We can now estimate the least x for given m, n in the sequence $1, \frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ of slopes and use this to determine which small square is the first to be contacted by the ray R. From this one can estimate the coordinates of (xm, xn) in various ways.

4. An "n-m society" is a group of n girls and m boys. Show that there exist numbers n_0 and m_0 such that every n_0-m_0 society contains a subgroup of 5 boys and 5 girls in which all of the boys know all of the girls or none of the boys knows none of the girls.

Solution by Michael Lebedinsky, student, Henry Wise Wood School, Calgary, Alberta.

We will show that we can take $n_0 = 9$. For $n_0 \geq 9$, observe that for each girl there must be at least 5 boys whom she knows, or 5 boys whom she does not know. We associate to each girl an ordered pair, the first element of which is a subset of 5 of the boys all of whom she knows or all of whom she does not know, and the second element of which is 0 or 1 according as she knows the boys or not. There are $\binom{9}{5} \times 2 = 252$ such pairs. Invoking the Pigeonhole Principle, if $m_0 \geq 4 \times 252 + 1 = 1009$, at least 5 girls must be assigned the same ordered pair, producing 5 girls and 5 boys for which each girl knows each boy, or no girl knows any of the boys.

That completes this Corner for this issue. Enjoy solving problems over the next weeks — and send me your nice solutions as well as your Olympiad materials.