THE ACADEMY CORNER

No. 19

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Solutions by Solomon W. Golomb, USC, Los Angeles, CA, USA, who writes: "It's reassuring to know that I can still do freshman/high school mathematics, after all these years".

1. Determine whether or not the following system has any real solutions. If so, state how many real solutions exist.

$$x+\frac{1}{x}=y, \qquad y+\frac{1}{y}=z, \qquad z+\frac{1}{z}=x. \qquad (1)$$

Solution. If (1) is to have real solutions, then $xyz \neq 0$. Hence

$$x^2+1=xy, \qquad y^2+1=yz, \qquad z^2+1=zx.$$

Multiplying gives $\left(x^2+1\right)\left(y^2+1\right)\left(z^2+1\right) \;=\; x^2y^2z^2$.

But, for real x, y, z, we have $x^2+1>x^2$, $y^2+1>y^2$, $z^2+1>z^2$, so there are no real solutions.

2. The surface area of a closed cylinder is twice the volume. Determine the radius and height of the cylinder given that the radius and height are both integers.

Solution. "The surface area of a closed cylinder is twice the volume" is absurd. It is dimensionally incorrect. One has units of area, the other has units of volume.

What was no doubt intended was $2\pi rh+2\pi r^2=2\pi r^2h$, $h+r=rh$, $\frac{1}{r}+\frac{1}{h}=1$, with no integer solution except $r=h=2$. But what are the *units* in which $h+r=rh$? (No physical object corresponds to this solution, independent of the arbitrary choice of units.)

3. Prove that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} < 2.$$

Solution (a).

$$\begin{aligned} 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} &< \sum_{k=1}^{\infty}\frac{1}{k^2} \\ &= \frac{\pi^2}{6} < \frac{10}{6} \\ &= \frac{5}{3} < 2. \end{aligned}$$

Solution (b). (For those who do not know the value of $\zeta(2)$.)

$$\begin{aligned} 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} &< 1+\int_1^{n+1}\frac{dt}{t^2} \\ &< 1+\int_1^{\infty}\frac{dt}{t^2} = 1-\frac{1}{t}\bigg|_1^{\infty} \\ &= 1+1 = 2. \end{aligned}$$

4. Describe the set of points (x, y) in the plane for which

$$\sin(x+y) = \sin x + \sin y.$$

Solution. Since $\sin(x+y) = \sin x \cos y + \sin y \cos x$, it follows that $\sin(x+y) = \sin x + \sin y$ has no solutions in the first quadrant, in which $\cos x < 1$, $\cos y < 1$, so that $\sin(x+y) < \sin x + \sin y$.

There are no solutions in the third quadrant, in which

$$\sin(x+y) = \sin x \cos y + \sin y \cos x > 0 > \sin x + \sin y.$$

The line $y = -x$, which bisects the second and fourth quadrants, is clearly a solution line: $\sin(x+(-x)) = 0$ and $\sin x + \sin(-x) = 0$.

To show that there are no other solutions, observe:

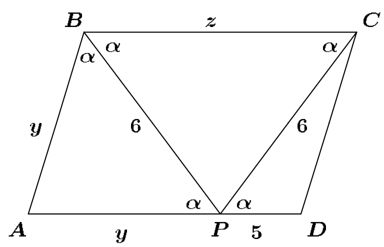
any solution (x, y) in the second or fourth quadrant corresponds to a first-quadrant solution of $\sin(x-y) = \sin x - \sin y$.

If $x \neq y$, suppose (without loss of generality) that $x > y$, so that $\sin x > \sin y$ while $\cos y > \cos x$. But then

$$\sin(x-y) = \sin x \cos y - \sin y \cos x > \sin x - \sin y.$$

5. In a parallelogram $ABCD$, the bisector of angle ABC intersects AD at P. If $PD = 5$, $BP = 6$ and $CP = 6$, find AB.

Solution.



We label the parallelogram as shown.

All angles α are equal, because BCP is an isosceles triangle, BP bisects $\angle ABC$, and alternate interior angles are equal.

Thus BPA is also an isosceles triangle, similar to triangle BCP, and $z = 5 + y$ because $BC = AD$.

From similar triangles, we see that $\frac{6}{z} = \frac{y}{6}$, $36 = yz = y(5+y)$, and $y^2 + 5y - 36 = 0$. By the quadratic formula, the positive root is $y = 4$, which is the length of side AB. (Triangle CDP is a 4–5–6 triangle!)

6. Show that, where $k + n \le m$,

$$\sum_{i=0}^{n} \binom{n}{i}\binom{m}{k+i} = \binom{m+n}{n+k}.$$

Solution. With $k+n \le m$, we represent $\binom{m+n}{n+k}$ by taking j elements from n, and the remaining $(n+k) - j$ elements from m, for every j, $0 \le j \le n$. Thus

$$\begin{aligned}\binom{m+n}{n+k} &= \sum_{j=0}^{n} \binom{n}{j}\binom{m}{n+k-j} \\ &= \sum_{i=n}^{0} \binom{n}{n-i}\binom{m}{k+i} \\ &= \sum_{i=0}^{n} \binom{n}{i}\binom{m}{k+i},\end{aligned}$$

where we substituted $i = n - j$, and used $\binom{n}{a} = \binom{n}{n-a}$.

Solutions were also received from D. KIPP JOHNSON, Beaverton, Oregon, USA and D.J. SMEENK, Zaltbommel, the Netherlands.