PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"$\times$11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 November 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2306. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

CORRECTION to (a) Give an elementary proof of the inequality:

$$\left(\sin\left(\frac{\pi x}{2}\right)\right)^2 > \frac{2x^2}{1+x^2}; \qquad (0 < x < 1). \tag{1}$$

2326*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that if A, B and C are the angles of a triangle, then

$$\frac{2}{\pi} < \sum_{\text{cyclic}} \frac{\left(1-\sin\frac{A}{2}\right)\left(1+2\sin\frac{A}{2}\right)}{\pi - A} \leq \frac{9}{\pi}.$$

2327. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and

$$a_{n+1} = a_n - a_{n-1} + \frac{a_n^2}{a_{n-2}}, \qquad n \geq 3.$$

Prove that each $a_n \in \mathbb{N}$, and that no a_n is divisible by 4.

2328*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

It is known from Wilson's Theorem that the sequence $\{y_n : n \geq 0\}$, with $y_n = \dfrac{n!+1}{n+1}$, contains infinitely many integers; namely, $y_n \in \mathbb{N}$ if and only if $n+1$ is prime.

(a) Determine all integer members of the sequences $\{y_n(a) : n \geq 0\}$, with $y_n = \dfrac{n!+a}{n+a}$, in the cases $a = 2, 3, 4$.

(b) Determine all integer members of the sequences $\{y_n(a) : n \geq 0\}$, with $y_n = \dfrac{n!+a}{n+a}$, in the cases $a \geq 5$.

2329*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that p and $t > 0$ are real numbers. Define

$$\lambda_p(t) := t^p + t^{-p} + p^p \qquad \text{and} \qquad \kappa_p(t) := \left(t+t^{-1}\right)^p + 2\,.$$

(a) Show that $\lambda_p(t) \leq \kappa_p(t)$ for $p \geq 2$.

(b) Determine the sets of p: A, B and C, such that

1. $\lambda_p(t) \leq \kappa_p(t)$,
2. $\lambda_p(t) = \kappa_p(t)$,
3. $\lambda_p(t) \geq \kappa_p(t)$.

2330. *Proposed by Florian Herzig, student, Perchtoldsdorf, Austria.*

Prove that

$$e = 3 - \frac{1!}{1\cdot 3} + \frac{2!}{3\cdot 11} - \frac{3!}{11\cdot 53} + \frac{4!}{53\cdot 309} - \frac{5!}{309\cdot 2119} + \dots\,,$$

where

$$\begin{array}{rcrcl} 11 &=& 3\cdot 3 &+& 2\cdot 1,\\ 53 &=& 4\cdot 11 &+& 3\cdot 3,\\ 309 &=& 5\cdot 53 &+& 4\cdot 11,\\ 2119 &=& 6\cdot 309 &+& 5\cdot 53,\\ &&\vdots&& \end{array}$$

[Ed: There is enough information here to deduce the general term.]

2331. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let p be an odd prime. Show that there is at most one non-degenerate integer triangle with perimeter $4p$ and integer area. Characterize those primes for which such triangles exist.

2332. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose x and y are integers. Solve the equation

$$x^2y^2 - 7x^2y + 12x^2 - 21xy - 4y^2 + 63x + 70y - 174 = 0.$$

2333. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

You are given that D and E are points on the sides AC and AB respectively of $\triangle ABC$. Also, DE is not parallel to CB. Suppose F and G are points of BC and ED respectively such that

$$\overline{BF} : \overline{FC} = \overline{EG} : \overline{GD} = \overline{BE} : \overline{CD}.$$

Show that GF is parallel to the angle bisector of $\angle BAC$.

2334. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that ABC is a triangle with incentre I, and that BI, CI meet AC, AB at D, E respectively. Suppose that P is the intersection of AI with DE. Suppose that $PD = PI$.

Find angle ACB.

2335. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Triangle ABC has circumcircle Γ. A circle Γ' is internally tangent to Γ at P, and touches sides AB, AC at D, E respectively. Let X, Y be the feet of the perpendiculars from P to BC, DE respectively.

Prove that $PX = PY \sin \frac{A}{2}$.

2336. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The bisector of angle A of a triangle ABC meets BC at D. Let Γ and Γ' be the circumcircles of triangles ABD and ACD respectively, and let P, Q be the intersections of AD with the common tangents to Γ, Γ' respectively.

Prove that $PQ^2 = AB \cdot AC$.

2337. *Proposed by Iliya Bluskov, Simon Fraser University, Burnaby, BC.*

Let $F(1) = \left\lceil \dfrac{n^2+2n+2}{n^2+n+1} \right\rceil$, and, for each $i > 1$, let
$F(i) = \left\lceil \dfrac{n^2+2n+i+1}{n^2+n+i} F(i-1) \right\rceil$.

Find $F(n)$.

Some readers have pointed out that problem 2287 [1997: 501] is the same as problem 2234 [1997: 168], and that problem 2288 [1997: 501] is the same as problem 2251 [1997: 299]. Also part (a) of problem 2306 [1998: 46; 175] is the same as the first part of 2296 [1997; 503]. The editors missed these duplications. Proposers are asked not to submit the same problem more than once.