ONE PROBLEM – SIX SOLUTIONS

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The three ingredients necessary to solve a problem are insight, persistence and technique. In the following, all three of these play a role as six different solutions to the following problem are presented.

The Problem:

On the three sides of triangle $ABC$, construct squares facing outwards. Let $A'$, $B'$, $C'$ be the centres of the squares constructed on sides $BC$, $CA$ and $AB$, respectively. Prove that $\text{dist}(A, A') = \text{dist}(B', C')$ and $AA' \perp B'C'$.

Solution 1: (Analytic Geometry)

It is important to make the coordinates work for you. Place the origin at $B$, and the $x$–axis along $BC$. Let $C = (2a, 0)$ and $A = (2b, 2c)$. Then label the vertices of the three squares as shown.

It is easy to show that $B'' = (2a+2c, 2a-2b)$, $A'' = (0, -2a)$ and $C''' = (-2c, 2b)$. Now we quickly obtain $A' = (a, -a)$, $B' = (a + b + c, a - b + c)$ and $C' = (b - c, b + c)$.
Now, we have that \( \text{dist}(A, A') = \sqrt{(2b - a)^2 + (2c + a)^2} = \text{dist}(B', C') \), slope \( AA' = \frac{2c + a}{2b - a} \) and slope \( B'C' = \frac{-2b - a}{2c + a} \), showing that the lines are perpendicular.

**Solution 2: (Geometry)**

Let \( M \) be the midpoint of \( BC \). Consider triangles \( \triangle ABA' \) and \( \triangle C'BM \). We have \( \overline{AB}A' = B + 45^\circ = C'B'M, \overline{AB} = c, \overline{C'B} = \frac{c}{\sqrt{2}}, \overline{BA'} = \frac{a}{\sqrt{2}} \) and \( BM = \frac{p}{2} \).

So these triangles are similar, and \( \triangle C'BM \) is obtained from \( \triangle ABA' \) by rotating through \( 45^\circ \) about \( B \), and scaling by a factor of \( \frac{1}{\sqrt{2}} \).

This means that \( \overline{CM} = \frac{\overline{AA'}}{\sqrt{2}} = \frac{p}{\sqrt{2}} \), and that \( \angle PMC' = 45^\circ \). Likewise, triangles \( \triangle ACA' \) and \( \triangle B'CM \) are similar, and \( \triangle B'CM \) is obtained from \( \triangle ACA' \) by rotating through an angle of \( -45^\circ \) about \( C \) and dilating by \( \frac{1}{\sqrt{2}} \).

Hence \( \overline{B'M} = \frac{p}{\sqrt{2}} \) and \( \angle MQP = 45^\circ \).

But this now tells us that \( B'MC' = 90^\circ \), and so

\[
\overline{C'B'}^2 = \overline{CM}^2 + \overline{B'M}^2 = \frac{p^2}{2} + \frac{p^2}{2} = p^2;
\]

hence \( \overline{C'B'} = p = \overline{AA'} \).

Also, since \( \triangle C'MB' \) is a right-angled isosceles triangle, we note that \( MC'B' = 45^\circ \).
In $\triangle TC'P$, we have $\hat{P} = \hat{C}' = 45^\circ$ and so $\hat{T} = 90^\circ$, telling us that $C'B' \perp AA'$.

**Solution 3:** (Trigonometry)

Consider the hexagon $AB'C'A'B'C'$ and all six of its diagonals.

Now we use the cosine law:

\[
BF'^2 = AC'^2 + AB'^2 - 2AC' \cdot AB' \cos(\hat{A} + 90^\circ)
\]

\[
= c^2 + \frac{b^2}{2} + bc \sin(\hat{A}).
\]

Also

\[
AA'^2 = AC^2 + A'A'^2 - 2AC \cdot A'A' \cos(\hat{C} + 45^\circ)
\]

\[
= b^2 + \frac{a^2}{2} - ab \left( \cos(\hat{C}) - \sin(\hat{C}) \right)
\]

\[
= \frac{b^2}{2} + \frac{c^2}{2} + ab \sin(\hat{C}).
\]

But recall that

\[
bc \sin(\hat{A}) = ac \sin(\hat{C}) = 2 \times (\text{area } \triangle ABC).
\]

Hence $BF'^2 = AA'$.

Likewise, $C'A' = BB'$ and $A'B' = CC'$.

Now, consider triangles $B'C'B$ and $A'AC'$. These triangles are congruent since $BF'C' = AA'$, $C'B = AC'$ and $BB' = C'A'$. Hence $x = x'$. Likewise,
\( y = y' \) and \( z = z' \). But then \( 2x + 2y + 2z = 180^\circ \) and hence \( x + y + z = 90^\circ \).

It follows, looking at triangle \( A'B'T \), that \( A'A \perp B'C' \).

**Solution 4: (Vectors)**

![Diagram of triangle with vectors](image)

Here, we let the vertices of the triangle be represented by vectors \( \overrightarrow{a} \), \( \overrightarrow{b} \) and \( \overrightarrow{c} \).

Let \( \overrightarrow{n} \) be the unit vector which is perpendicular to the plane containing the diagram.

Recall that for any two vectors \( \overrightarrow{u} \), \( \overrightarrow{v} \), the vector \( \overrightarrow{u} \times \overrightarrow{v} \) is perpendicular to both \( \overrightarrow{u} \) and \( \overrightarrow{v} \), and points in the direction given by the right-hand rule.

It follows that

\[
\overrightarrow{AC} = \overrightarrow{n} \times \overrightarrow{BA} = \overrightarrow{n} \times (\overrightarrow{a} - \overrightarrow{b}), \\
\overrightarrow{BA} = \overrightarrow{n} \times \overrightarrow{CB} = \overrightarrow{n} \times (\overrightarrow{b} - \overrightarrow{c}), \\
\overrightarrow{CB} = \overrightarrow{n} \times \overrightarrow{AC} = \overrightarrow{n} \times (\overrightarrow{c} - \overrightarrow{a}).
\]

We next compute:

\[
\overrightarrow{AC} = \frac{1}{2} \left[ \overrightarrow{b} - \overrightarrow{a} + \overrightarrow{n} \times (\overrightarrow{a} - \overrightarrow{b}) \right], \\
\overrightarrow{AB} = \frac{1}{2} \left[ \overrightarrow{c} - \overrightarrow{a} + \overrightarrow{n} \times (\overrightarrow{c} - \overrightarrow{a}) \right], \\
\overrightarrow{BA} = \frac{1}{2} \left[ \overrightarrow{c} - \overrightarrow{b} + \overrightarrow{n} \times (\overrightarrow{b} - \overrightarrow{c}) \right].
\]
Let \( \vec{u} \) be the vector from \( \vec{C} \) to \( \vec{B} \); let \( \vec{v} \) be the vector \( \vec{AA}' \).

Now:
\[
\vec{u} = \vec{AB}' - \vec{AC}' = \frac{1}{2} \left[ \vec{c} - \vec{b} + \vec{n} \times \left( \vec{c} + \vec{b} - 2 \vec{a} \right) \right],
\]
and
\[
\vec{v} = \frac{1}{2} \left[ \vec{c} - \vec{b} + \vec{n} \times \left( \vec{b} - \vec{c} \right) \right] - \left( \vec{a} - \vec{b} \right) = \frac{1}{2} \left[ \vec{c} + \vec{b} - 2 \vec{a} + \vec{n} \times \left( \vec{b} - \vec{c} \right) \right].
\]

But now:
\[
\vec{n} \times \vec{v} = \frac{1}{2} \left[ \vec{n} \times \left( \vec{c} + \vec{b} - 2 \vec{a} \right) + \left( \vec{c} - \vec{b} \right) \right] = \vec{u}.
\]
And since \( |\vec{n}| = 1 \), we conclude that \( |\vec{v}| = |\vec{u}| \) and that \( \vec{v} \perp \vec{u} \).

**Solution 5:** (Complex Numbers)

Place the diagram in the complex plane, with the origin at \( B \). Let \( A \) and \( C \) represent the complex numbers at vertices \( A \) and \( C \).

Recall that for any complex number \( z \), the product \( z' = ze^{i\theta} \) is the complex number which is obtained by rotating \( z \) about the origin through the angle \( \theta \).

If instead, we wish to rotate \( w \) about \( z \) through an angle \( \theta \), then
\[
w' = z + (w - z)e^{i\theta}.
\]
Using this, we obtain the following:

\[
B'' = C + (A - C)(-i) = C(1 + i) - Ai,
\]

\[
C'' = Ai,
\]

\[
A'' = -C i.
\]

Also, \( A' = \frac{1}{2}C(1 - i), \ B' = \frac{1}{2}A(1 - i) + C(1 + i) \) and \( C' = \frac{1}{2}A(1 + i) \).

Let \( \vec{u} \) be the vector from \( C \) to \( B' \); let \( \vec{v} \) be the vector from \( A' \) to \( A \).

Now \( \vec{u} = \frac{1}{2}C(1 + i) - Ai \) and \( \vec{v} = A - \frac{1}{2}C(1 - i) \).

But now note that \( \vec{u} \ i = A - \frac{1}{2}C(1 - i) = \vec{v} \).

So \( A'A \) is obtained by rotating \( C'B' \) through \( 90^\circ \), which proves both parts of the result that we seek.

**Solution 6:** (Transformations)

Rotate triangle \( ACA' \) about \( C \) through \( -45^\circ \) and dilate by \( \sqrt{2} \) to obtain triangle \( B''B' \).

This tells us that \( B''B' \) is inclined at \( 45^\circ \) with respect to \( AA' \).
Now rotate triangle $AB''B$ about $A$ through $-45^\circ$ and dilate by $\frac{1}{\sqrt{2}}$ to obtain triangle $AB'C'$.

This tells us that $B'C' = \frac{1}{\sqrt{2}} B''B$, and that $B'C'$ is inclined at $45^\circ$ with respect to $B''B$.

But this proves our result.

So, there it is — one problem, six different approaches. Which solution is the best? The late Professor Paul Erdős used to talk about "God's Little Black Book", in which could be found the perfect solution to every problem. Of the solutions presented here, perhaps the last one is the closest candidate for inclusion in this "Little Black Book": it is economical, to the point and carries with it a wonderful element of surprise. Those are the properties of beautiful mathematics.

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