THE OLYMPIAD CORNER

No. 188

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This number we begin with the problems of The Final Round of the Japan Mathematical Olympiad. My thanks go to Richard Nowakowski for collecting these when he was Canadian Team Leader at the 35th IMO in Hong Kong.

JAPAN MATHEMATICAL OLYMPIAD

Final Round

February 1994 — 4 hours

1. For positive integer $n$, let $a_n$ be the nearest positive integer to $\sqrt{n}$, and let $b_n = n + a_n$. Dropping all $b_n (n = 1, 2, \ldots)$ from the set of all positive integers $N$, we get a sequence of positive integers in ascending order $\{c_n\}$. Represent $c_n$ by $n$.

2. Five points are located on the plane. Any three of those points are not collinear. Let $l_1, \ldots, l_{10}$ be the length of the ten segments obtained by joining every two points of the five points. Assume that $l_1^2, \ldots, l_{10}^2$ are rational numbers. Prove that $l_{10}^2$ is also a rational number.

3. There is a triangle $A_0A_1A_2$ and seven points $P_0, \ldots, P_6$ on the plane. Assume that any $P_i$ and $P_{i+1}$ are symmetric with center $A_k$, where $k$ is the remainder of $i$ divided by 3.

   (a) Prove that $P_0 = P_3$.

   (b) Describe the possible position of $P_0$ under the additional assumption that every segment connecting $P_i$ and $P_{i+1}$ does not intersect with the interior of the triangle $A_0A_1A_2$.

4. We consider a triangle $ABC$ such that $\angle MAC = 15^\circ$, where $M$ is the midpoint of $BC$. Determine the possible maximum value of $\angle B$.

5. There are $N$ persons and $N$ pieces of lot cards. Each number 1 through $N$ is written on a card. When the $N$ persons draw these cards, their order is determined by the numbers on their cards. After repeating this draw two times, we give gifts by the following rule.

   Rule: A person $X$ gets the gift, if there is no person $Y$ such that $Y$ is prior to $X$ both times. Otherwise $X$ cannot get the gift.
For example, if \( X \) is at the top in the first lot, \( X \) always gets the gift whatever he draws in the second lot.

Then determine the expected number of persons who get gifts.

As a second Olympiad for your puzzling pleasure we give problems of the 30th Spanish Mathematical Olympiad, First Round, November 26–27, 1993. My thanks go to Richard Nowakowski, Canadian Team Leader to the 35th IMO in Hong Kong and to Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain, for copies of the contest.

30th SPANISH MATHEMATICAL OLYMPIAD
First Round — November 26–27, 1993
Proposed by the Royal Spanish Mathematical Society

Time allowed: 4 hours each day. Each problem carries 10 points.

First Day

1. Show that, for all \( n \in \mathbb{N} \), the fractions
\[
\frac{n - 1}{n}, \quad \frac{n}{2n + 1}, \quad \frac{2n + 1}{2n^2 + 2n}
\]
are irreducible.

2. A sphere of radius \( R \), and a right cone with base a meridian of the sphere and vertex external to the sphere, are given. Find the radius of the circle of intersection of the sphere and the cone, given that the volume of the cone is half of the volume of the sphere.

3. Solve the following system of equations:
\[
x \cdot |x| + y \cdot |y| = 1, \quad [x] + [y] = 1,
\]
in which \(|t|\) and \([t]\) represent the absolute value and the integer part of the real number \(t\).

4. Let \( AD \) the internal bisector of the triangle \( ABC \) (\( D \in BC \)), \( E \) the point symmetric to \( D \) with respect to the midpoint of \( BC \), and \( F \) the point of \( BC \) such that \( \angle BAF = \angle EAC \). Show that \( \frac{BF}{FC} = \frac{c^2}{a^2} \).

Second Day

5. Find all the natural numbers \( n \) such that the number
\[
n(n + 1)(n + 2)(n + 3)
\]
has exactly three prime divisors.
6. An ellipse is drawn taking as major axis the biggest of the sides of a given rectangle, such that the ellipse passes through the intersection point of the diagonals of the rectangle.

Show that, if a point of the ellipse, external to the rectangle, is joined to the extreme points of the opposite side, then three segments in geometric progression are determined on the major axis.

7. Let \( a \in \mathbb{R} \) given. Find the real numbers \( x_1, \ldots, x_n \) which satisfy the system of equations

\[
\begin{align*}
x_1^2 + ax_1 + \left(\frac{a-1}{2}\right)^2 &= x_2 \\
x_2^2 + ax_2 + \left(\frac{a-1}{2}\right)^2 &= x_3 \\
&\vdots \\
x_{n-1}^2 + ax_{n-1} + \left(\frac{a-1}{2}\right)^2 &= x_n \\
x_n^2 + ax_n + \left(\frac{a-1}{2}\right)^2 &= x_1.
\end{align*}
\]

8. (The Sisyphus's myth) There are 1001 steps going up, with rocks on some of them (no more than 1 rock on each step). Sisyphus may pick any rock and raise it one or more steps up to the nearest empty step. Then his opponent Hades rolls a rock (with an empty step directly below it) down one step. There are 500 rocks, originally located on the first 500 steps. Sisyphus and Hades move rocks in turn, Sisyphus making the first move. His goal is to place a rock on the top step. Can Hades stop him?

We now turn to readers' solutions received before January 1st, to problems for consideration by the International Jury at the 36th IMO in Canada [1996: 299–301].

4. Let \( A, B \) and \( C \) be non-collinear points. Prove that there is a unique point \( X \) in the plane of \( ABC \) such that \( XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 +XA^2 + CA^2 \).

**Solutions by D.J. Smeenk, Zaltbommel, the Netherlands; and by Vasiliou Meletis, Elefsis, Greece. We give Meletis' solutions.**

**First Solution.**
From the hypothesis we have

\[ AX^2 + AB^2 = CX^2 + CB^2. \]  

(1)

If \( B_1 \) is the midpoint of \( BX \), applying the first theorem of the median in the triangles \( \triangle ABX, \triangle CBX \) we get

\[ 2AB_1^2 + 2BB_1^2 = 2CB_1^2 + 2BB_1^2 \quad \text{or} \quad AB_1 = CB_1. \]  

(2)

This indicates that the perpendicular bisector of the side \( AC \) passes through the point \( B_1 \). Let \( A_1, C_1 \) be the midpoints of \( AX \) and \( CX \), respectively.

Similarly, we obtain that the perpendicular bisectors of \( BC \) and \( AB \) pass through the midpoints \( A_1 \) and \( C_1 \), respectively.

Furthermore we obtain \( AB \parallel A_1B_1, AC \parallel A_1C_1 \) and \( BC \parallel B_1C_1 \).

From (3) and (4) we get that the circumcentre \( O \) of \( ABC \) is the orthocentre \( H_1 \) of \( A_1B_1C_1 \).

Also from (4) the triangles \( ABC \) and \( A_1B_1C_1 \) are similar with \( X \) the centre of similarity and ratio \( \frac{1}{2} \).

So, their orthocentres \( H \) and \( H_1 \) lie in the same straight line with the point \( X \) and \( HH_1 = HX \).

Combining (5) and (7) we get \( HO = OX \); that is the point \( X \) is known (constant), because \( X \) is symmetric to \( H \) with respect to the orthocentre \( O \) of \( ABC \).

**Second Solution.**

The conditions of the problem are equivalent to the system of equations

\[ XB^2 - XC^2 = AC^2 - AB^2 \]  

(1)

\[ XC^2 - XA^2 = BA^2 - BC^2. \]  

(2)

Now, taking equation (1) gives a locus of points \( X \) satisfying the condition.

The relation reminds us of the second theorem of the median in a triangle.

Let \( AA_1, XA_2 \) be the altitudes of the triangles \( ABC \) and \( XBC \) respectively on side \( BC \) (extended). Let \( M \) be the midpoint of the side \( BC \).

If we suppose

\[ AB \leq AC \leq BC, \]  

(3)

for illustration, we get

\[ XC \leq XB \leqXA, \]
and furthermore the point \( M \) lies between the points \( A_1 \) and \( A_2 \) \hspace{1cm} (4)

But

\[
 XB^2 - XC^2 = 2BC \cdot MA_2, \quad \text{and} \\
 AC^2 - AB^2 = 2BC \cdot MA_1.
\]

Hence \( MA_1 = MA_2 \) and \( A_2 \) is a constant point on \( BC \) because it is symmetric to \( A_1 \) with respect to the midpoint \( M \).

Consequently, if (1) holds, the point \( X \) lies on the line \( E_1 \) perpendicular to \( BC \) at \( A_2 \). Similarly, if (2) holds, the point \( X \) lies on the line \( E_2 \) perpendicular to \( AC \) at \( B_2 \) (where \( BB_1 \perp AC \) and \( AB_1 = CB_2 \)).

Hence, the required point \( X \) lies at the intersection of \( E_1 \) and \( E_2 \).

5. The incircle of \( ABC \) touches \( BC, CA \) and \( AB \) at \( D, E \) and \( F \) respectively. \( X \) is a point inside \( ABC \) such that the incircle of \( XBC \) touches \( BC \) at \( D \) also, and touches \( CX \) and \( XB \) at \( Y \) and \( Z \), respectively. Prove that \( EFZY \) is a cyclic quadrilateral.

Solutions by Toshio Seimiya, Kawasaki, Japan; and by Vasilis Meletis, Elefsis, Greece. We give Seimiya's argument.

Let \( P \) be the intersection of \( EF \) with \( BC \). Then by Menelaus' Theorem we have

\[
 \frac{BP}{PC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1. \hspace{1cm} (1)
\]

Since \( CE = CD, EA = AF \), and \( FB = BD \), we get

\[
 \frac{BP}{PC} \cdot \frac{CD}{BD} = 1
\]

so that

\[
 \frac{BP}{PC} = \frac{BD}{CD}. \hspace{1cm} (2)
\]
Since $XZ = XY$, $BZ = BD$ and $CY = CD$, we have from (2)

\[
\frac{BP}{PC} \cdot \frac{CY}{YX} \cdot \frac{XZ}{ZB} = \frac{BD}{CD} \cdot \frac{CD}{XY} \cdot \frac{XY}{BD} = 1.
\]

Hence by Menelaus' Theorem $P$, $Z$ and $Y$ are collinear. Since $PF \cdot PE = PD^2$ and $PZ \cdot PY = PD^2$ we have $PF \cdot PE = PZ \cdot PY$.

Hence $EFZY$ is a cyclic quadrilateral.

Comment. If $AB = AC$ then $BD = DC$ and then it can easily be proved that $AD$ is the perpendicular bisector of $EF$ and $YZ$ so that $EFZY$ is an isosceles trapezoid, and is a cyclic trapezoid.

6. An acute triangle $ABC$ is given. Points $A_1$ and $A_2$ are taken on the side $BC$ (with $A_2$ between $A_1$ and $C$), $B_1$ and $B_2$ on the side $AC$ (with $B_2$ between $B_1$ and $A$) and $C_1$ and $C_2$ on the side $AB$ (with $C_2$ between $C_1$ and $B$) so that

\[
\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.
\]

The lines $AA_1$, $BB_1$, and $CC_1$ bound a triangle, and the lines $AA_2$, $BB_2$ and $CC_2$ bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

Solutions by Toshio Seimiya, Kawasaki, Japan; by D.J. Smeenk, Zaltbommel, the Netherlands; and by Vasiliou Meletis, Elefsis, Greece. We give Meletis' solution.

![Diagram](image)

Let $AA_1$, $BB_1$ meet at the point $E$; $AA_1$, $CC_2$ meet at the point $F$; and $BB_1$, $CC_1$ meet at the point $I$. Also

\[
\angle A_1AA_2 = \angle B_1BB_2 = \angle C_1CC_2 = 2x. \quad (1)
\]

The bisectors of the angles at $A_1$, $B_1$ and $C$ in triangles $\triangle A_1AA_2$, $\triangle B_1BB_2$ and $\triangle C_1CC_2$ respectively are perpendicular to their respective
bases. Hence they are the altitudes of $\triangle ABC$. Let $H$ be the orthocentre of $\triangle ABC$.

Since $\angle A_1AH = \angle B_1BH = x$ and $\angle A_1AH = \angle C_1CH = x$ each one of the quadrilaterals $AHEB$, $AHDC$ is inscribable in a circle.

These two circles have a common chord, the segment $AH$ and since $\angle ABH = \angle ACH = 90^\circ - \angle BAC$, then the circles have equal radii.

Thus, since the inscribed angles $\angle EAH$, $\angle DAH$ are equal, the corresponding chords $HE$ and $HD$ are equal.

Therefore $HE = HD$. Similarly, we prove that $HD = HI$, and so on for all six vertices of these two triangles of the problem.

Thus, all six vertices lie at the same distance from the point $H$, and the points are concyclic.

Next we give a rather novel solution by Meletis to the second problem of the 37th IMO itself [1996: 303].

2. Let $P$ be a point inside triangle $ABC$ such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$ Let $D, E$ be the incentres of triangles $APB, APC$ respectively. Show that $AP, BD$ and $CE$ meet at a point.

Solution by Meletis Vasiliou, Elefsis, Greece.

\begin{center}
\includegraphics[width=0.5\textwidth]{figure1.png}
\end{center}

Figure 1.

We will first prove the converse of the proposition and then apply it to prove the stated problem.

Stage I. The equality $\angle APB - \angle APC = \angle ACB - \angle ABC$ is equivalent to

$$\angle A_1 - \angle A_2 = \angle C_3 - \angle B_4$$

(see figure 1).

Stage II. Assume the conclusion of the problem; that is, assume that the bisectors of the angles $\angle PCA$ and $\angle PBA$ concur at a point $I$ on $AP$. Then we have

$$\frac{AC}{PC} = \frac{AI}{PI} = \frac{AB}{PB}$$

or

$$\frac{AC}{AB} = \frac{PC}{PB}.$$

(2)
This ratio indicates that if $AD$ is the bisector of the angle $\angle BAC$, then $PD$ is the bisector of the angle $\angle BPC$, or equivalently

"the points $A, P$ belong to the “circle of Apollonius” whose diameter $DD'$ lies on the line $CB$, with $D, D'$ harmonic conjugates to the points $C, B$".

**Stage III.** If (3) holds then (1) holds. Since $AD$ bisects $\angle CAB$, then

$$\angle PAD = x = \frac{\angle A_1 - \angle A_2}{2}.$$  

(4)

Since $PD$ bisects $\angle CPB$ then

$$\angle PDB - \angle PDC = \angle C_3 - \angle B_4$$

and if we draw the bisector $DE$ of the straight angle $\angle BDC$ we get

$$\angle PDE = y = \frac{\angle C_3 - \angle B_4}{2}.$$  

(5)

Because of (3), and since $DE \perp CD$ we obtain that $DE$ is tangent at $D$ to the circumcircle of the triangle $\triangle APD$.

Hence $x = y$.

Combining (4), (5) and (6) we get that

$$\angle A_1 - \angle A_2 = \angle C_3 - \angle B_4$$

so the converse of the proposition is true.

**Stage IV.** If (1) holds then (3) holds.

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**Figure 2.**

Consider the circle of Apollonius with respect to the angle $A$ of the triangle $ABC$ with $DD'$ as diameter. Let $P_1$ be the point at which the circle intersects line $CP$. We want to show that $P \equiv P_1$ to complete the proof.

Suppose $P \not\equiv P_1$. We distinguish two cases:

**Case a.** $P_1$ is external to the segment $CP$ (see figure 2). Denote

$$\angle A_1 = \angle CAP, \quad \angle A_2 = \angle BAP,$$

$$\angle A_6 = \angle CAP_1, \quad \angle A_7 = \angle BAP_1,$$
\[ \angle B_4 = \angle CBP \text{ and } \angle B_5 = \angle CBP. \]

Then we have

\[
\begin{align*}
\angle A_1 &< \angle A_6 \\
\angle A_2 &> \angle A_7 \\
\angle B_4 &< \angle B_5.
\end{align*}
\]

(7) (8)

So from the hypothesis and the conclusion of Stage III

\[
\angle C_3 - \angle B_4 = \angle A_1 - \angle A_2 \\
< \angle A_6 - \angle A_7 \\
= \angle C_3 - \angle B_5 \\
< \angle C_3 - \angle B_4.
\]

or

\[
\angle C_3 - \angle B_4 < \angle C_3 - \angle B_4,
\]

a contradiction.

**Case b.** \( P_1 \) is internal to the segment \( CD \). The proof is similar. This completes the proof.

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**To complete this number of the corner and our file of solutions submitted by the readers to problems from 1996 numbers of the Corner, we give one solution to a problem proposed to the jury but not used at the 36th IMO.**

5. [1996: 348] 36th IMO Problems proposed to the jury and not used.

Let \( ABC \) be a triangle. A circle passing through \( B \) and \( C \) intersects the sides \( AB \) and \( AC \) again at \( C' \) and \( B' \), respectively. Prove that \( BB' \), \( CC' \) and \( HH' \) are concurrent, where \( H \) and \( H' \) are the orthocentres of triangles \( ABC \) and \( AB'C' \) respectively.

**Solution by Toshio Seimiy, Kawasaki, Japan.**

(See diagram on the next page.) Let \( BH, CH \) meet \( AC, AB \) at \( D, E \) respectively. Since \( \angle BDC = \angle BEC = 90^\circ \), \( B, C, D, E \) are concyclic and thus \( \angle ADE = \angle ABC \).

Now, since \( B, C, B', C' \) are concyclic we have

\[ \angle AB'C' = \angle ABC. \]

Thus \( \angle ADE = \angle AB'C' \), so that \( DE \parallel B'C' \). Let \( Q \) be the intersection of \( DC' \) with \( EB' \). As \( HE \perp AB \), \( B'H' \perp AB \), we get \( HE \parallel H'B' \). Similarly we have \( HD \parallel H'C' \).
Since $DE\parallel C'B'$, $HD\parallel H'C'$ and $HE\parallel H'B'$, $DC'$, $EB'$ and $HH'$ are concurrent, so that $H$, $Q$, $H'$ are collinear.

By Pappus' Theorem $H$, $P$, $Q$ are collinear.

Hence $H$, $P$, $Q$, $H'$ are collinear. So $BB'$, $CC'$, and $HH'$ are concurrent.

That is the Corner for this issue! Send me your National and Regional Olympiad Contest materials for use in the Corner.