PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (*) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard 8½"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than 1 September 1998. They may also be sent by email to crux-editors@cms.math.ca. (It would be appreciated if email proposals and solutions were written in T\LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.

2301. Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.
Suppose that \( \triangle ABC \) is a triangle with sides \( a, b, c \), that \( P \) is a point in the interior of \( \triangle ABC \), and that \( AP \) meets the circle \( BPC \) again at \( A' \). Define \( B' \) and \( C' \) similarly.
Prove that the perimeter \( P \) of the hexagon \( AB'C'A'BC' \) satisfies
\[
P \geq 2 \left( \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \right).
\]

2302. Proposed by Toshio Seimiya, Kawasaki, Japan.
Suppose that the bisector of angle \( A \) of triangle \( ABC \) intersects \( BC \) at \( D \). Suppose that \( AB + AD = CD \) and \( AC + AD = BC \).
Determine the angles \( B \) and \( C \).

2303. Proposed by Toshio Seimiya, Kawasaki, Japan.
Suppose that \( \triangle ABC \) is a triangle with angles \( B \) and \( C \) satisfying \( C = 90^\circ + \frac{1}{2}B \), that the exterior bisector of angle \( A \) intersects \( BC \) at \( D \), and that the side \( AB \) touches the incircle of \( \triangle ABC \) at \( E \).
Prove that \( CD = 2AE \).
2304. Proposed by Toshio Seimiya, Kawasaki, Japan.
An acute angled triangle \( ABC \) is given, and equilateral triangles \( ABD \) and \( ACE \) are drawn outwardly on the sides \( AB \) and \( AC \). Suppose that \( CD \) and \( BE \) meet \( AB \) and \( AC \) at \( F \) and \( G \) respectively, and that \( CD \) and \( BE \) intersect at \( P \).

Suppose that the area of the quadrilateral \( AFPG \) is equal to the area of the triangle \( PBC \). Determine angle \( BAC \).

2305. Proposed by Richard I. Hess, Rancho Palos Verdes, California, USA.
An integer sided triangle has angles \( p\theta \) and \( q\theta \), where \( p \) and \( q \) are relatively prime integers. Prove that \( \cos \theta \) is rational.

2306. Proposed by Vedula N. Murty, Visakhapatnam, India.
(a) Give an elementary proof of the inequality:
\[
\left( \frac{\sin \pi x}{2} \right)^2 > \frac{2x^2}{1 + x^2}; \quad (0 < x < 1).
\]

(b) Hence (or otherwise) show that
\[
\tan \pi x \begin{cases} < \frac{\pi x(1-x)}{1-2x}; & (0 < x < \frac{1}{2}), \\
> \frac{\pi x(1-x)}{1-2x}; & (\frac{1}{2} < x < 1).
\end{cases}
\]

(c) Find the maximum value of \( f(x) = \frac{\sin \pi x}{x(1 - x)} \) on the interval \((0, 1)\).

It is known that every regular \( 2n \)-gon can be dissected into \( \binom{n}{2} \) rhombuses with the same side length.

(a) How many different classes of rhombuses are there?
(b) How many rhombuses are there in each class?

2308. Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.
A sequence \( \{v_n\} \) has initial value \( v_0 = 1 \) and, for \( n \geq 0 \), satisfies the recurrence relation
\[
v_{n+1} = 2^{n+1} - \sum_{k=0}^{n} v_k v_{n-k};
\]
Find a formula for \( v_n \) in terms of \( n \).
2309. Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.

Suppose that $ABC$ is a triangle and that $P$ is a point of the circumcircle, distinct from $A$, $B$ and $C$. Denote by $S_A$ the circle with centre $A$ and radius $AP$. Define $S_B$ and $S_C$ similarly. Suppose that $S_A$ and $S_B$ intersect at $P$ and $P_C$. Define $P_B$ and $P_A$ similarly.

Prove that $P_A$, $P_B$ and $P_C$ are collinear.

2310. Proposed by K.R.S. Sastry, Dodballapur, India.

Let $n \in \mathbb{N}$. I call a positive integral divisor of $n$, say $d$, a unitary divisor if $\gcd(d, n/d) = 1$.

Let $\Upsilon(n)$ denote the sum of the unitary divisors of $n$.

Find a characterization of $n$ so that $\Upsilon(n) \equiv 2 \pmod{4}$.

2311. Proposed by K.R.S. Sastry, Dodballapur, India.

Let $\Upsilon_e(n)$ denote the sum of the even unitary divisors, and $\Upsilon_o(n)$, the sum of the odd unitary divisors, of $n$. Assume that $\Upsilon_e(n) - \Upsilon_o(n) = n$.

(a) If $n$ is composed of powers of exactly two distinct primes, show that $n$ must be the product of two consecutive integers, one of which is a Mersenne prime.

(b) Give an example of a natural number $n$ that is composed of powers of more than two distinct primes.

2312. Proposed by K.R.S. Sastry, Dodballapur, India.

The $r$th $n$-gonal number is given by $P(n, r) = \frac{(n - 2)r^2}{2} - (n - 4)\frac{r}{2}$, where $n \geq 3, r = 1, 2, \ldots$.

Prove that, in the interval $[P(n, r), P(n, r + 1)]$, there is an $(n - 1)$-gonal number.

2313. Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.

Let $N$ be a non-negative integer and let $a$ and $b$ be complex numbers with $a, b \not\in \{0, -1, -2, \ldots, -(n - 1)\}$. Find a closed form expression for

$$\sum_{k=0}^{n} \frac{(-1)^k}{(a)_k \cdot (b)_{n-k}},$$

where $(a)_k$ denotes the Pochhammer symbol, defined by $(a)_0 = 1$, $(a)_k = a(a + 1) \ldots (a + k - 1)$, $k \in \mathbb{N}$. 