SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2179★. [1996: 318] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For real numbers $a \geq -1$, we consider the sequence

$$F(a) := \left\{ \left(1 + \frac{1}{n}\right)^{\sqrt{n(n+a)}}, \quad n \geq 1 \right\} .$$

Determine the sets D, respectively I, of all a, such that $F(a)$ strictly decreases, respectively increases.

Solution by Michael Lambrou, University of Crete, Crete, Greece.

We show that the sequence $F(a)$ is strictly increasing if and only if

$$-1 \leq a \leq 1$$

and strictly decreasing if and only if

$$a \;>\; \frac{4(\log 1.5)^2 - (\log 2)^2}{(\log 2)^2 - 2(\log 1.5)^2} \;\approx\; 1.168188898,$$

where all logarithms here and below are natural logarithms.

We approach this as follows. For a fixed a, the sequence $F(a)$ is strictly decreasing if and only if for all integers $n \geq 1$ we have

$$\left(1 + \frac{1}{n+1}\right)^{\sqrt{(n+1)(n+1+a)}} \;<\; \left(1 + \frac{1}{n}\right)^{\sqrt{n(n+a)}} .$$

Taking logarithms and squaring, this is equivalent (since all quantities involved are positive) to

$$(n+1)(n+1+a)\left(\log \frac{n+2}{n+1}\right)^2 \;<\; n(n+a)\left(\log \frac{n+1}{n}\right)^2 ,$$

and so

$$a\left[(n+1)\left(\log \frac{n+2}{n+1}\right)^2 - n\left(\log \frac{n+1}{n}\right)^2\right]$$
$$< \; n^2\left(\log \frac{n+1}{n}\right)^2 - (n+1)^2\left(\log \frac{n+2}{n+1}\right)^2 .$$

The quantity in square brackets on the left is negative, because the function

$$f_1(x) \;=\; x\left(\log \frac{x+1}{x}\right)^2 \tag{1}$$

has derivative

$$\begin{aligned} f_1'(x) &= \left(\log\frac{x+1}{x}\right)\left(\log\frac{x+1}{x}-\frac{2}{x+1}\right) \\ &< \left(\log\frac{x+1}{x}\right)\left(\frac{1}{x}-\frac{2}{x+1}\right) < 0 \quad \text{for } x\in[1,\infty) \end{aligned}$$

[the first inequality follows because $\log(1+z) < z$ for all $z > 0$] so f_1 decreases for $x \in [1,\infty)$. Thus the sequence $F(a)$ is strictly decreasing if and only if

$$a > \frac{n^2\left(\log\frac{n+1}{n}\right)^2-(n+1)^2\left(\log\frac{n+2}{n+1}\right)^2}{(n+1)\left(\log\frac{n+2}{n+1}\right)^2-n\left(\log\frac{n+1}{n}\right)^2} \tag{2}$$

for all integers $n \geq 1$.

Next we show that the right-hand side of (2) is a decreasing function of n. It is more convenient to study the right-hand side as a function of a continuous variable x on $[1,\infty)$. Thus we are to show

$$\frac{(x+1)^2\left(\log\frac{x+2}{x+1}\right)^2-(x+2)^2\left(\log\frac{x+3}{x+2}\right)^2}{(x+2)\left(\log\frac{x+3}{x+2}\right)^2-(x+1)\left(\log\frac{x+2}{x+1}\right)^2} < \frac{x^2\left(\log\frac{x+1}{x}\right)^2-(x+1)^2\left(\log\frac{x+2}{x+1}\right)^2}{(x+1)\left(\log\frac{x+2}{x+1}\right)^2-x\left(\log\frac{x+1}{x}\right)^2}.$$

We work on a more general case of an inequality, for appropriate functions $f\colon [1,\infty)\to\mathbb{R}$, of the form

$$\frac{(x+1)f(x+1)-(x+2)f(x+2)}{f(x+2)-f(x+1)} < \frac{xf(x)-(x+1)f(x+1)}{f(x+1)-f(x)}. \tag{3}$$

Here putting f equal to the function f_1 in (1) gives the desired inequality.

We shall need the following easy lemma.

Lemma. If $f : [b,\infty) \to \mathbb{R}$ is a strictly monotonic, twice differentiable function with $f(x) > 0$ for all x in its domain, and further satisfies $f(x)f''(x) < 2\left(f'(x)\right)^2$ there, then inequality (3) holds.

Proof. Indeed, observe that the function $g(x) = 1/f(x)$ satisfies

$$g''(x) = \frac{2\left(f'(x)\right)^2 - f(x)f''(x)}{\left(f(x)\right)^3} > 0,$$

so g is strictly convex. Applying convexity to x, $x+2$ and $\frac{1}{2}(x+(x+2)) = x+1$ we see that

$$\frac{1}{f(x+1)} \;<\; \frac{1}{2}\left(\frac{1}{f(x)} + \frac{1}{f(x+2)}\right);$$

that is,

$$2f(x)f(x+2) \;<\; \left[f(x)+f(x+2)\right] f(x+1). \tag{4}$$

If f is strictly monotonic, then the quantity

$$\big(f(x+1)-f(x)\big)\big(f(x+2)-f(x+1)\big)$$

is strictly positive. Multiplying (3) by this quantity and simplifying, we get (4), which is therefore equivalent to (3). □

Returning to the proof, we will show that the function f_1 given by (1), which we have already shown is strictly monotonic, satisfies the rest of the conditions of the lemma. Clearly $f_1(x) > 0$ on $[1,\infty)$, and we need to show $f_1 f_1'' < 2(f_1')^2$. Here

$$f_1'(x) \;=\; \left(\log\frac{x+1}{x}\right)^2 - \frac{2}{x+1}\log\frac{x+1}{x}$$

and

$$f_1''(x) \;=\; \frac{2}{x(x+1)^2} - \frac{2}{x(x+1)^2}\log\frac{x+1}{x}\,.$$

Thus upon cancellation we need to show

$$1-\log\frac{x+1}{x} \;<\; \left[(x+1)\log\frac{x+1}{x} - 2\right]^2 \tag{5}$$

on $[1,\infty)$.

This unfortunately is a little tedious, as the sides of (5) are almost equal for large x, so we will delay its proof until the end. Assuming (5) and returning to inequality (2), if we call its right-hand side a_n, then we have shown that (a_n) is strictly decreasing. Hence the condition $a > a_n$ for all integers $n \geq 1$ is equivalent to

$$a \;>\; a_1 \;=\; \frac{4(\log 1.5)^2 - (\log 2)^2}{(\log 2)^2 - 2(\log 1.5)^2} \;\approx\; 1.168188898.$$

Conclusion: $F(a)$ is strictly decreasing if and only if $a > a_1 = 1.16818889\ldots$.

Similarly, $F(a)$ is strictly increasing if and only if $a < a_n$ for all integers $n \geq 1$. But as (a_n) decreases and is bounded below (by zero), this is equivalent to $a \leq \lim a_n$. We show that $\lim a_n = 1$. This can be done by l'Hôpital's Rule taking $x \to \infty$ on the continuous analogue of a_n; or, arguing asymptotically, we have

$$\begin{aligned}\left(\log \frac{n+1}{n}\right)^2 &= \left(\log\left(1+\frac{1}{n}\right)\right)^2 = \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + O(1/n^4)\right)^2 \\ &= \frac{1}{n^2}\left(1 - \frac{1}{n} + \frac{11}{12n^2} + O(1/n^3)\right)\end{aligned}$$

and hence

$$\begin{aligned}a_n &= \frac{\left(1 - \frac{1}{n} + \frac{11}{12n^2} + O(1/n^3)\right) - \left(1 - \frac{1}{n+1} + \frac{11}{12(n+1)^2} + O(1/n^3)\right)}{\left(\frac{1}{n+1} - \frac{1}{(n+1)^2} + O(1/n^3)\right) - \left(\frac{1}{n} - \frac{1}{n^2} + O(1/n^3)\right)} \\ &= \frac{\frac{-1}{(n+1)^2} + O(1/n^3)}{\frac{-1}{(n+1)^2} + O(1/n^3)} \longrightarrow 1,\end{aligned}$$

as claimed.

To complete the proof we must show (5). Set $w = \frac{1}{x+1}$ for $x \geq 1$ so that $w \leq 1/2$ and $\frac{x}{x+1} = 1 - w$. Now (5) becomes

$$1 + \log(1-w) < \left[\frac{1}{w}\log(1-w) + 2\right]^2,$$

and hence we have to show on $0 < w \leq 1/2$ that

$$[\log(1-w)]^2 + (4w - w^2)\log(1-w) + 3w^2 > 0. \tag{6}$$

Now we need the following estimates of $\log(1-w)$: for $0 < w \leq 1/2$ we have

$$\begin{aligned}&-w - \frac{w^2}{2} - \frac{w^3}{3} - \frac{w^4}{4} - \frac{w^5}{5} - \frac{13w^6}{42} \\ &\qquad < \log(1-w) < -w - \frac{w^2}{2} - \frac{w^3}{3} - \frac{w^4}{4} - \frac{w^5}{5} - \frac{w^6}{6}.\end{aligned} \tag{7}$$

The second inequality is simply a truncation of the Taylor series of $\log(1-w)$. The first simply says

$$\frac{13w^6}{42} > \frac{w^6}{6} + \frac{w^7}{7} + \frac{w^8}{8} + \cdots;$$

this is true enough, for $0 < w \leq 1/2$ implies $0 < \frac{w}{1-w} \leq 1$, and we have

$$\begin{aligned}\frac{w^6}{6} + \frac{w^7}{7} + \frac{w^8}{8} + \cdots &< \frac{w^6}{6} + \frac{w^7}{7}(1 + w + w^2 + \cdots) \\ &= \frac{w^6}{6} + \frac{w^7}{7}\left(\frac{1}{1-w}\right) \leq \frac{w^6}{6} + \frac{w^6}{7} = \frac{13w^6}{42}.\end{aligned}$$

Substituting inequalities (7) into the left-hand side of (6), we get that it is enough to show for $0 < w \le 1/2$ that

$$\left(w+\frac{w^2}{2}+\frac{w^3}{3}+\frac{w^4}{4}+\frac{w^5}{5}+\frac{w^6}{6}\right)^2 + (4w-w^2)\left(-w-\frac{w^2}{2}-\frac{w^3}{3}-\frac{w^4}{4}-\frac{w^5}{5}-\frac{13w^6}{42}\right)+3w^2 \;>\; 0.$$

The lengthy but routine calculation gives

$$\frac{1}{12}w^4+\frac{1}{6}w^5+\frac{19}{90}w^6-\frac{71}{210}w^7+\text{(positive terms)} \;>\; 0.$$

But this last inequality is certainly true since, for $0 < w \le 1/2$, we have

$$\frac{1}{12}w^4-\frac{71}{210}w^7 \;\ge\; \frac{1}{12}w^4-\frac{71}{210}w^4\left(\frac{1}{2}\right)^3 \;=\; \frac{69w^4}{1680} > 0.$$

[*Editor's note*: some minor adjustments seemed to be needed to Lambrou's asymptotic proof that $\lim a_n = 1$, and also to one of his coefficients toward the end of the solution. These have been made in the above writeup.]

Also solved by CON AMORE PROBLEM GROUP, The Royal Danish School of Educational Studies, Copenhagen, Denmark; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA. Two other readers sent in incorrect or incomplete solutions.

A couple of readers mentioned that $F(a)$ is eventually *decreasing (for large enough n) whenever $a > 1$, as can be seen from the above solution.*

One reader pointed out the similar problem 442 *in the* College Mathematics Journal, *solution in Vol.* **23** (1992) *pp.* 71–72, *in which the exponent is $n + a$ instead of $\sqrt{n(n+a)}$.*

2180. [1996: 318] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Prove that if $a > 0, x > y > z > 0, n \ge 0$ (natural), then

1. $a^x(yz)^n(y-z)+a^y(xz)^n(z-x)+a^z(xy)^n(x-y) \ge 0$,
2. $a^x\cosh x(y-z)+a^y\cosh y(z-x)+a^z\cosh z(x-y) \ge 0$.

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA.

A characterization for a function f to be convex is that, for $x > y > z > 0$,

$$(y-z)f(x)+(z-x)f(y)+(x-y)f(z) \;\ge\; 0.$$

[See, for example, D.S. Mitrinović, *Analytic Inequalities*, Springer, Berlin, 1970, p. 16.]

Since result 1 is equivalent to

$$(y-z)\frac{a^x}{x^n} + (z-x)\frac{a^y}{y^n} + (x-y)\frac{a^z}{z^n} \ \geq\ 0,$$

it is sufficient to show that $f(t) = \dfrac{a^t}{t^n}$ is a convex function of t for $n \geq 0$.

The case $n = 0$ is easy, so suppose that $n > 0$, and without loss of generality, suppose that $a = e$. Then

$$f''(t) \ =\ \frac{e^t\left(t^2 - 2t + n(n+1)\right)}{t^{n+2}}.$$

The discriminant of the quadratic equation $t^2 - 2t + n(n+1)$ is $-4(n^2 + n - 1) < 0$. Hence $f''(t) > 0$, and so f is convex.

For result 2, we must show that $f(t) = a^t \cosh t$ is convex. Again, assume without loss of generality, that $a = e$. Thus

$$f''(t) \ =\ 2e^t\left(\cosh t + \sinh t\right) \ =\ 2e^{2t} \ >\ 0.$$

Thus f is convex, and the inequality follows.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta (first part only); MICHAEL LAMBROU, University of Crete, Crete; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Klamkin read the second part as $\cosh((x(y-z))$, *etc., which, on reflection, is a reasonable interpretation, and gives a trivial result. We wonder how many other readers did this?*

2181. [1996: 318] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove that the product of eight consecutive positive integers cannot be the fourth power of any positive integer.

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA. [Slightly modified by the editor.]

Without loss of generality, let

$$P = \ (n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)(n+4).$$

Then

$$P = n^8 + 4n^7 - 14n^6 - 56n^5 + 49n^4 + 196n^3 - 36n^2 - 144n.$$

If P is a 4^{th} power, then it would be of the form

$$F = (n^2 + an + b)^4,$$

where a, b are constants. But then $b = 0$ since P has a zero constant term. Thus $F = n^4(n+a)^4$, and clearly $a \neq 0$.

But this implies that the coefficients of n^3, n^2 and n in F are all zero. However, P has these coefficients non-zero. Hence P cannot be of the form F.

It has been pointed out by many readers that this problem has appeared before. Most readers referred to the American Mathematical Monthly, 1936, p. 310 *for the solution to #3703 (posed by Victor Thébault in 1934, p. 522). Another reference was made to Honsberger's monograph* Mathematical Morsels, *where it appears on p. 156 as "A Perfect 4th Power". Several readers also made reference to the general problem of proving that the product of (two or more) consecutive integers is never a square, which was established in 1975 by Erdős and Selfridge* [1]. *Because the solution by Cautis was quite different from any of these published solutions, we have decided to publish it here. The interested reader is directed to these other sources for a different solution.*

Comments and/or solutions were submitted also by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, Scarborough, Ontario; GORAN CONAR, student, Varaždin, Croatia; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Greece; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

Seiffert remarks that A. Guibert proved the result in 1862. This is stated by L.E. Dickson in his History of the Theory of Numbers, Vol. II, 1952, pp. 679-680.

Janous notes the following deep theorem by Erdős and Selfridge:

The product of two or more consecutive positive integers is never a power of a positive integer; that is, the Diophantine equation

$$(n+1)(n+2)\ldots(n+k) = x^k$$

has no integer solutions with $k, l \geq 2$ and $n \geq 0$.

Reference

1. *P. Erdős and J.L. Selfridge,* The product of consecutive integers is never a power, *Illinois J. Math.* **19** *(1975), 292–301.*

2182. [1996: 318] *Proposed by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.*

Many ***CRUX*** readers are familiar with the card game "Crazy Eights", of which there are many variations. We define the game of "Solo Crazy Eights" in the following manner:

We are given a standard deck of 52 cards, and are dealt k of these at random, $1 \leq k \leq 52$. We then attempt to arrange these k cards according to three rules:

1. Any card can be chosen as the first card of a sequence;
2. A card can be succeeded by any card of the same suit, or the same number, or by any eight;
3. Anytime in the sequence that an eight appears, any suit can be "called", and the succeeding card must be either of the called suit, or another eight. (This means that, in effect, any card can follow an eight).

The game is won if all dealt cards can be ordered into a sequence according to rules 1–3. If no such sequence is possible, the game is lost.

What is the largest value of k for which it is possible to lose the game?

Solution by Michael Lambrou, University of Crete, Greece.

We show that if 34 or fewer cards are dealt, then it is possible to lose the game, but for any 35 cards or more, the game always ends successfully. Thus $k = 34$.

For example, the 34 or fewer cards may be chosen as follows; (a) ace of hearts, (b) no eights, (c) any subset of the 33 cards consisting of any suit which is not a heart and any rank which is neither an ace nor an eight. In this situation the ace cannot be linked to any other card (as there are no other aces nor hearts nor eights); thus we have a losing hand.

Let us now show that for 35 cards or more, the game can end successfully. Note that if eights were present, we could exchange them with any undealt cards, arrange the cards, and re-exchange the eights (if there are insufficient undealt cards, we could simply place the eights at the end). Thus it is sufficient to show that we can obtain success on 35 or more cards which do not include eights.

The algorithm below is based on two simple observations:

1. Any set of ranks each of which appears 3 times among the dealt cards (that is, is represented by 3 suits) can be arranged in a sequence following the rules of the game. This is so because any two such ranks have at least two suits in common so, running through all suits of a fixed rank, we can link it to the next rank utilising a common suit which we can go through similarly leaving last a suit that links it with a further rank in the set. (Clearly a common suit between the second and the third rank, which is different from the suit that linked the first to the second rank, always exists). This can be repeated until we exhaust the set.

2. Any rank which appears twice among the dealt cards can be linked to a rank which appears three times, since at least one suit is common to both.

Let A_i, $(i = 0, 1, 2, 3, 4)$, denote the set of ranks that appear i times (that is, represented with i suits) among the dealt cards, and let $n_i = |A_i|$, the number of elements in A_i. Since we are excluding eights, we have

$$n_0 + n_1 + n_2 + n_3 + n_4 \ = \ 12$$

and we also have

$$n_1 + 2n_2 + 3n_3 + 4n_4 \ \geq \ 35.$$

We will show that $n_1 + n_2 \leq n_4 + 1$ and that if $n_2 = 0$ this can be improved to $n_1 \leq n_4$. We argue by contradiction. Suppose that $n_4 + 1 < n_1 + n_2$. Then

$$\begin{aligned}
35 &\leq n_1 + 2n_2 + 3n_3 + 4n_4 \\
&= n_1 + 2n_2 + 3(12 - n_0 - n_1 - n_2 - n_4) + 4n_4 \\
&= 36 - 3n_0 - 2n_1 - n_2 + n_4 \\
&\leq 36 - 2n_1 - n_2 + n_4 \\
&< 36 - 2n_1 - n_2 + (n_1 + n_2 - 1) \\
&= 35 - n_1,
\end{aligned}$$

which is impossible since $n_1 \geq 0$. If further we have $n_2 = 0$, and we assume that $n_4 < n_1$ we would get

$$\begin{aligned}
35 &\leq n_1 + 2n_2 + 3n_3 + 4n_4 \\
&= n_1 + 0 + 3(12 - n_0 - n_1 - 0 - n_4) + 4n_4 \\
&= 36 - 3n_0 - 2n_1 + n_4 \\
&\leq 36 - 2n_1 + n_4 \\
&< 36 - 2n_1 + n_1 \\
&= 36 - n_1,
\end{aligned}$$

which forces $n_1 < 1$, but then $0 \leq n_4 < n_1$ is impossible.

We are now in position to describe an algorithm arranging all 35 cards according to rules 1–3. Suppose first that $n_2 = 0$. Then $n_1 \leq n_4$ and we may write $A_1 = \{b_1, b_2, \ldots, b_{n_1}\}$ and $A_4 = \{c_1, c_2, \ldots, c_{n_4}\}$. We start with a card, say of rank b_1, in A_1. We link progressively this card with each card of A_1 having the same suit. After this suit has been exhausted within A_1, we link with a card, say of rank c_1, of A_4 of the same suit (as the elements of A_4 have all four suits dealt, this is always possible). We then run through all four cards of rank c_1, in some order ending with a suit for which there still exist cards in A_1. (This is always possible unless A_1 is exhausted.) Then exhaust the cards of A_1 with the same suit and jump back to A_4 (this is always possible as $n_1 \leq n_4$). Repeat this process until A_1 is exhausted, then go through the rest of A_4 ending with a suit that is the same as some suit for some rank in A_3. Finally jump to A_3 and complete the process, which is possible from observation 1 above.

On the other hand, if $n_2 \neq 0$, and thus $n_1 + n_2 \leq n_4 + 1$, we modify the algorithm from above as follows. After we exhaust A_1 and return to A_4, we then go to A_2, select a rank, exhaust the (two) suits of this rank, and jump back to A_4. Continue back and forth between A_4 and A_2, running each rank through all its suits, being sure to order the suits for each rank from A_4 so that it matches a suit for a rank still remaining in A_2. If A_2 is exhausted first we complete the rest of A_4 and jump to A_3 as described for the first case. If A_4 is exhausted first we go from A_2 to A_3 which can be done by observation 2 above, as long as we have taken care to end up for the last rank of A_2 with a suit which matches the suit for at least one rank from A_3. (This may take a little bit of planning for the ordering of the last rank of A_4 as well!)

Also solved by the proposer. There were two incomplete solutions.

Geretschläger also asks about a generalization to replace $52 = 4 \cdot 12 + 4$ *by* $n \cdot k + j$. *He observes that for* $n = 4$ *(that is,* $4k + j$*), an analogous argument to the one presented would yield a maximum number of* $3k - 2$. *He then asks for general conditions such that the resulting maximum number is* $(n-1)(k-1)+1$, *or to find out what other numbers (if any) could turn up.*

2183. [1996: 319] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that A, B, C are the angles of a triangle and that k, l, $m \geq 1$. Show that:

$$\begin{aligned} 0 &< \sin^k A \, \sin^l B \, \sin^m B \\ &\leq k^k l^l m^m S^{\frac{S}{2}} \left((Sk^2 + P)^{-\frac{k}{2}}\right) \left((Sl^2 + P)^{-\frac{l}{2}}\right) \left((Sm^2 + P)^{-\frac{m}{2}}\right), \end{aligned}$$

where $S = k + l + m$ and $P = klm$.

Editor's and Proposer's comments.

This problem has already been posed; see **908** [1984: 19] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine the maximum value of

$$P \equiv \sin^\alpha A \cdot \sin^\beta B \cdot \sin^\gamma C,$$

where A, B, C are the angles of a triangle and α, β, γ are given positive numbers.

A solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, is given in [1985: 93]. He sent in a solution this time pointing out that he had done so before!

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

2184. [1996: 319] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer and let a_n denote the sum

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}.$$

Prove that the sequence $\{a_n : n \geq 0\}$ is periodic.

Composite solution from Michael Lambrou, University of Crete, Crete, Greece and Kee-Wai Lau, Hong Kong.

Since

$$\binom{n-k}{k} = \binom{n-1-k}{k} + \binom{n-1-k}{k-1},$$

we have

$$\begin{aligned} a_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-1-k}{k} + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-1-k}{k-1} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-1-k}{k} - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-2-k}{k} \\ &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} - \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \binom{n-2-k}{k} \\ &= a_{n-1} - a_{n-2}. \end{aligned}$$

It follows that $a_{n+3} = a_{n+2} - a_{n+1} = (a_{n+1} - a_n) - a_{n+1} = -a_n$, and so $a_{n+6} = -a_{n+3} = a_n$, showing that the sequence $\{a_n : n \geq 0\}$ is periodic with period 6.

In fact, the sequence takes consecutively the values $1, 1, 0, -1, -1, 0$, indefinitely.

Also solved by PAUL BRACKEN, CRM, Université de Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Bradley and Hess commented that the corresponding sum without the factor $(-1)^k$ *would yield the sequence of Fibonacci numbers. [Ed.: This can be found in many elementary books on Combinatorics, for example, Exercise 21 on page 87 of* Basic Techniques of Combinatorial Theory *by Daniel I.A. Cohen.] Flanigan pointed out that this problem appears, with technical differences in the definition of* a_n*, in the book* Concrete Mathematics *by Graham, Knuth and Patashuik, (1989), 177–179. Lau pointed out that with the recurrence relation* $a_n = a_{n-1} - a_{n-2}$ *and the initial values* $a_0 = a_1 = 1$*, one can prove easily, by induction, that*

$$a_n = \cos\left(\frac{n\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right), \qquad n = 0, 1, 2, \ldots .$$

This fact was also derived by the proposer.

2185. [1996: 319] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Notice that

$$2^2 + 4^2 + 6^2 + 8^2 + 10^2 = 4\cdot5 + 5\cdot6 + 6\cdot7 + 7\cdot8 + 8\cdot9;$$

that is, the sum of the first n (in this case 5) even positive squares is equal to the sum of some n consecutive products of consecutive pairs of positive integers.

Find another value of n for which this happens.

(NOTE: this problem was suggested by a final exam that I marked recently.)

I. *Solution by Tim Cross, King Edward's School, Birmingham, England.*

We require to find positive integers n, k for which

$$2^2 + 4^2 + \cdots + (2n)^2 = k(k+1) + (k+1)(k+2) + \cdots + (k+n-1)(k+n),$$

which is equivalent to

$$4\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n}(k+r-1)(k+r) = k(k-1)\sum_{r=1}^{n} 1 + (2k-1)\sum_{r=1}^{n} r + \sum_{r=1}^{n} r^2,$$

$$3 \cdot \frac{n}{6}(n+1)(2n+1) = k(k-1)n + (2k-1)\frac{n}{2}(n+1),$$

$$\frac{1}{2}(n+1)(2n+1) + \frac{1}{2}(n+1) = k(k-1) + k(n+1),$$

and finally

$$k^2 + nk - (n+1)^2 = 0. \tag{1}$$

We thus look for positive integer solutions

$$k = \frac{-n + \sqrt{n^2 + 4(n+1)^2}}{2}$$

and we require the discriminant $\Delta = 5n^2 + 8n + 4$ to be a perfect square, say $\Delta = \alpha^2$ for some positive integer α. This condition leads to a Pell equation $(5n+4)^2 - 5\alpha^2 = -4$.

Examining the more general form $x^2 - 5y^2 = -4$, we find solutions

$$(x, y) = (1, 1), (4, 2), (11, 5), (29, 13), \ldots\ .$$

The solution $x = 5n + 4 = 29$ gives $n = 5$, the example given.

Pell equations have solution-pairs which satisfy similar second-order recurrence relations. In this case, x_k and y_k both satisfy

$$u_k = 3u_{k-1} - u_{k-2}, \quad k \geq 3, \tag{2}$$

with $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$. (Notice that the sequence $\{y_k\} = 1, 2, 5, 13, \ldots$ is that of alternate Fibonacci numbers.)

If we take the sequence $\{x_k\} = 1, 4, 11, 29, \ldots$ we see that terms are alternately $\equiv 1 \bmod 5$ and $\equiv 4 \bmod 5$. Since we need $x \equiv 4 \bmod 5$, we put $v_k = x_{2k}$ and can then derive from (2) the sequence

$$v_k = 7v_{k-1} - v_{k-2}, \quad k \geq 3, \qquad \text{with } v_1 = 4 \quad \text{and } v_2 = 29.$$

[*Editor's note*. For example, from (2) we get

$$\begin{aligned} x_{2k} &= 3x_{2k-1} - x_{2k-2} = 3(3x_{2k-2} - x_{2k-3}) - x_{2k-2} \\ &= 7x_{2k-2} - (3x_{2k-3} - x_{2k-2}) = 7x_{2k-2} - x_{2k-4} \end{aligned}$$

and the recurrence for the v's follows.] Then, since $n_k = (v_k - 4)/5$, we can deduce the sequence $\{n_k\}$ defined by

$$n_k = 7n_{k-1} - n_{k-2} + 4, \quad k \geq 3, \qquad \text{with } n_1 = 0 \quad \text{and } n_2 = 5.$$

This gives the sequence

$$\{n_k\} = 0 \text{ (trivially)},\ 5,\ 39,\ 272,\ 1869,\ 12815,\ \ldots$$

of suitable values of n.

II. *Solution by John Oman and Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA.*

[*Editor's note*: Oman and Prielipp first derived equation (1), which however they wrote in the form

$$n^2 - (k-2)n - (k^2-1) = 0. \tag{3}$$

Here and below, their notation has been changed to agree with Solution I.]

Considering equation (3) as a quadratic in n, a necessary condition for it to have integer solutions is for the discriminant $k(5k-4)$ to be a perfect square. Thus $5k-4 = x^2$ and $k = y^2$ for some positive integers x and y. [*Editor's note*. Since $\gcd(k, 5k-4) = 1, 2$ or 4, the only alternative is $5k-4 = 2x^2$ and $k = 2y^2$, which implies $5y^2 - 2 = x^2$ and thus $x^2 \equiv 3 \bmod 5$, impossible. Note that we thus get the same equation $x^2 = 5y^2 - 4$ as in Solution I.]

The following sequences for k_m and n_m solve (3) and provide additional solutions to the problem:

$$k_m = y_m^2 \qquad \text{and} \qquad n_m = \frac{k_m - 2 + \sqrt{k_m(5k_m-4)}}{2} = \frac{y_m^2 - 2 + x_m y_m}{2},$$

where

$$x_m = (2+\sqrt{5})\left(\frac{3+\sqrt{5}}{2}\right)^m + (2-\sqrt{5})\left(\frac{3-\sqrt{5}}{2}\right)^m$$

and

$$y_m = \left(\frac{5+2\sqrt{5}}{5}\right)\left(\frac{3+\sqrt{5}}{2}\right)^m + \left(\frac{5-2\sqrt{5}}{5}\right)\left(\frac{3-\sqrt{5}}{2}\right)^m.$$

The first few values generated by these formulas are

m	x	y	k	n
0	4	2	4	5
1	11	5	25	39
2	29	13	169	272
3	76	34	1156	1869

[*Editor's note*. Oman and Prielipp then noted that

$$x_m = L_{2m+3} \qquad \text{and} \qquad y_m = F_{2m+3},$$

the $(2m+3)$rd Lucas and Fibonacci numbers, respectively (where, as usual, $F_1 = F_2 = 1$ and $L_1 = 1$, $L_2 = 3$, both sequences then generated by the familiar Fibonacci recurrence). This follows because

$$\frac{3 \pm \sqrt{5}}{2} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^2 \qquad \text{and} \qquad 2 \pm \sqrt{5} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^3,$$

and so

$$x_m = \left(\frac{1+\sqrt{5}}{2}\right)^{2m+3} + \left(\frac{1-\sqrt{5}}{2}\right)^{2m+3} = L_{2m+3}$$

and

$$y_m = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2m+3} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2m+3} = F_{2m+3}.$$

Thus $k_m = F_{2m+3}^2$ and (after some manipulations) $n_m = F_{2m+3}F_{2m+4} - 1$.]

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, Scarborough, Ontario; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. One incorrect solution was sent in.

Besides Oman and Prielipp, and (to a smaller extent) Cross, no other readers seem to have noticed the presence of the Fibonacci numbers in the solution to this problem. One more occurrence, which Oman and Prielipp don't mention, is that the largest number $k+n$ on the right-hand side of the given equation is F_{2m+4}^2, which fits nicely with the fact that the smallest number on the right-hand side is $k = F_{2m+3}^2$. In fact, only Brown commented on the fact that these numbers are squares! So for example, the equation arising from the next-smallest solution $n = 39$ is

$$2^2 + 4^2 + \cdots + 78^2 = 25 \cdot 26 + 26 \cdot 27 + \cdots + 63 \cdot 64,$$

where $25 = F_5^2$ *and* $64 = F_6^2$ *(and* $78 = 2F_5F_6 - 2$*, as follows from Solution* II). *In general, the required equation reads*

$$2^2 + 4^2 + \cdots + (2F_{2t-1}F_{2t} - 2)^2$$

$$= F_{2t-1}^2(F_{2t-1}^2 + 1) + (F_{2t-1}^2 + 1)(F_{2t-1}^2 + 2) + \cdots + (F_{2t}^2 - 1)F_{2t}^2$$

for any integer $t \geq 2$.

2186. [1996: 319] *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Let a, b, c respectively denote the lengths of the sides BC, CA, AB of triangle ABC. Let G denote the centroid, let I denote the incentre, let R denote the circumradius, r denote the inradius, and let s denote the semi-perimeter.

Prove that

$$GI^2 = \frac{1}{9(a+b+c)}\Big((a-b)(a-c)(b+c-a)$$

$$+ (b-c)(b-a)(c+a-b) + (c-a)(c-b)(a+b-c)\Big).$$

Deduce the (known) result

$$GI^2 = \frac{1}{9}\left(s^2 + 5r^2 - 16Rr\right).$$

Solution by Kee-Wai Lau, Hong Kong.

Let A be the origin, $\mathbf{AB} = \mathbf{u}$ and $\mathbf{AC} = \mathbf{v}$. It is well known that

$$\mathbf{AG} = \frac{1}{3}(\mathbf{u} + \mathbf{v}) \text{ and } \mathbf{AI} = \frac{b}{a+b+c}\mathbf{u} + \frac{c}{a+b+c}\mathbf{v}.$$

Hence

$$\mathbf{GI} = \left(\frac{b}{a+b+c} - \frac{1}{3}\right)\mathbf{u} + \left(\frac{c}{a+b+c} - \frac{1}{3}\right)\mathbf{v}.$$

Since $\mathbf{u}\cdot\mathbf{u} = c^2$, $\mathbf{v}\cdot\mathbf{v} = b^2$ and $\mathbf{u}\cdot\mathbf{v} = bc\cos A = \frac{1}{2}(b^2+c^2-a^2)$, so

$$\begin{aligned} GI^2 &= \left(\frac{b}{a+b+c}-\frac{1}{3}\right)^2 c^2 + \left(\frac{c}{a+b+c}-\frac{1}{3}\right)^2 b^2 \\ &\quad + \left(\frac{b}{a+b+c}-\frac{1}{3}\right)\left(\frac{c}{a+b+c}-\frac{1}{3}\right)(b^2+c^2-a^2) \\ &= \frac{1}{9(a+b+c)^2}\Big((a+c-2b)^2c^2+(a+b-2c)^2b^2 \\ &\quad + (a+c-2b)(a+b-2c)(b^2+c^2-a^2)\Big) \\ &= \frac{1}{9(a+b+c)}\Big(-a^3-b^3-c^3+2a^2b+2ab^2 \\ &\quad + 2b^2c+2bc^2+2c^2a+2ca^2-9abc\Big) \\ &= \frac{1}{9(a+b+c)}\Big((a-b)(a-c)(b+c-a) \\ &\quad + (b-c)(b-a)(c+a-b)+(c-a)(c-b)(a+b+c)\Big) \end{aligned}$$

as required. Since

$$s = \frac{a+b+c}{2}, \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad \text{and} \quad R = \frac{abc}{4rs},$$

we have

$$\begin{aligned} &\frac{1}{9}\left(s^2+5r^2-16Rr\right) \\ &= \frac{1}{9}\left(\frac{(a+b+c)^2}{4}+\frac{5(b+c-a)(a+c-b)(a+b-c)}{4(a+b+c)}-\frac{8abc}{a+b+c}\right) \\ &= \frac{1}{9(a+b+c)}\left(\frac{(a+b+c)^3+5(b+c-a)(a+c-b)(a+b-c)-32abc}{4}\right) \\ &= \frac{1}{9(a+b+c)}\Big(-a^3-b^3-c^3+2a^2b+2ab^2 \\ &\quad + 2b^2c+2bc^2+2c^2a+2ca^2-9abc\Big) \\ &= GI^2, \end{aligned}$$

as required.

Bellot Rosado notes that a variation of this problem was proposed by Cezar Coşniţă, solved by T.C. Esty, with the second form of GI given by D.L. MacKay [Problem E415, American Mathematical Monthly (1940), solution p. 712].

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, *Rancho Palos Verdes, California, USA;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* MICHAEL LAMBROU, *University of Crete, Crete, Greece;* BOB PRIELIPP, *University of Wisconsin–Oshkosh, Wisconsin, USA;* PANOS E. TSAOUSSOGLOU, *Athens, Greece;* and the proposer.

2187. [1996: 320] *Proposed by Syd Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is easy to show that the maximum number of bishops that can be placed on an 8×8 chessboard, so that no two of them attack each other, is 14.

(a) Prove or disprove that in any configuration of 14 non-attacking bishops, all the bishops must be on the boundary of the board.

(b) Describe all of the configurations with 14 non-attacking bishops.

Solution by Kee-Wai Lau, Hong Kong.

(a) We prove the result.

Clearly we need consider only the black bishops. The diagram below shows the black squares of a chessboard rearranged so that the bishops will now move vertically or horizontally. [The diagram is labelled the same as the standard labelling of the squares of a chessboard, with the rows (ranks) numbered 1 to 8 and the columns (files) labelled a to h. — *Ed.*]

			$h8$			
		$f8$	$g7$	$h6$		
	$d8$	$e7$	$f6$	$g5$	$h4$	
$b8$	$c7$	$d6$	$e5$	$f4$	$g3$	$h2$
$a7$	$b6$	$c5$	$d4$	$e3$	$f2$	$g1$
	$a5$	$b4$	$c3$	$d2$	$e1$	
		$a3$	$b2$	$c1$		
			$a1$			

In any column and any row there is at most one bishop. [Thus there are at most 7 black bishops and similarly at most 7 white bishops, for the given maximum of 14. Moreover, to attain this maximum, there must be a black bishop in each column in the above diagram. — *Ed.*]

Denote the bishop in the kth column by B_k. If B_1 is on $a7$ then B_7 must be on $h2$, and if B_1 is on $b8$ then B_7 must be on $g1$. Next, B_2 must be on $a5$ or $d8$ and correspondingly B_6 must be on $h4$ or $e1$. Next B_3 must be on $a3$ or $f8$ and correspondingly B_5 must be on $h6$ or $c1$. Finally B_4 must be on $a1$ or $h8$. This shows that the non-attacking bishops must be on the boundary of the board.

(b) From part (a) we see that there are $2^4 = 16$ ways to locate the black bishops. Similarly there are 16 ways to locate the white bishops. Thus there are 256 configurations with 14 non-attacking bishops.

Also solved (usually the same way) by SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposers. Part (a) *was also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria.*

2188. [1996: 320] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Suppose that a, b, c are the sides of a triangle with semi-perimeter s and area $\triangle$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{s}{\triangle}.$$

I. *Solution Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let C be the angle opposite to side c of the triangle. Since $\triangle=\frac{1}{2}ab\sin C\le\frac{1}{2}ab$, we have $\frac{1}{a}\le\frac{b}{2\triangle}$. Here, equality holds if and only if the angle opposite to c is a right angle.

Similarly, $\frac{1}{b}\le\frac{c}{2\triangle}$, and $\frac{1}{c}\le\frac{a}{2\triangle}$. Since equality cannot hold simultaneously, we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{b+c+a}{2\triangle}=\frac{s}{\triangle}.$$

II. *Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

I will prove the stronger inequality:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{\sqrt{3}}{2}\cdot\frac{s}{\triangle}.$$

Let $a=x+y,\ b=y+z,\ c=z+x$, where $x,y,z>0$. Then

$$\begin{aligned}
\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &= \left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)^2\\
&\le \left(\frac{1}{2\sqrt{xy}}+\frac{1}{2\sqrt{yz}}+\frac{1}{2\sqrt{zx}}\right)^2\\
&= \frac{1}{4}\left(\frac{1}{\sqrt{xy}}+\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}\right)^2\\
&\le \frac{3}{4}\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right) \text{ (by Cauchy-Schwarz)}
\end{aligned}$$

Now $\frac{s}{\triangle}=\frac{x+y+z}{\sqrt{(x+y+z)xyz}}=\sqrt{\frac{x+y+z}{xyz}}=\sqrt{\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}}$.

Thus $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{\sqrt{3}}{2}\cdot\frac{s}{\triangle}$. The equality holds only when $a=b=c$.

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADAM BROWN, Scarborough, Ontario; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; STEFAN and ALEXANDER LAMBROU, students, Crete, Greece; MICHAEL LAMBROU, University of Crete, Crete, Greece; CAN AN+H MINH, University of California, Berkeley, California; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (2 solutions); BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; ADRIÁN UBIS MATIÍNEZ, student, I.B. Sagasta, Logroño, Spain; MELETIS VASILIOU, Elefsis, Greece; and the proposer.

2189. [1996: 361] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The incircle of a triangle ABC touches BC at D. Let P and Q be variable points on sides AB and AC respectively such that PQ is tangent to the incircle. Prove that the area of triangle DPQ is a constant multiple of $BP \cdot CQ$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let E, F, T be the points of contact of the incircle with CA, AB and PQ; define $x := AP$, $y := AQ$ as well as $z := PQ$. Then

$$\begin{aligned} x+y+z &= x+y+PT+QT \\ &= (x+PF)+(y+QE) \\ &= AF+AE \;=\; 2(s-a) \end{aligned} \qquad (1)$$

where s is the semi-perimeter of $\triangle ABC$. Applying the cosine rule to $\triangle APQ$ yields

$$z^2 \;=\; x^2+y^2-2xy\cos A \;=\; x^2+y^2-2xy\cdot\frac{b^2+c^2-a^2}{2bc}.$$

From equation (1) we get

$$4(s-a)^2-4(s-a)(x+y)+x^2+2xy+y^2 \;=\; x^2+y^2-xy\cdot\frac{b^2+c^2-a^2}{bc}$$

which is equivalent to

$$\begin{aligned} xy\cdot\left(\frac{b^2+2bc+c^2-a^2}{bc}\right)+4(s-a)^2 &= 4(s-a)(x+y) \\ xy\left((b+c)^2-a^2\right)+4bc(s-a)^2 &= 4bc(s-a)(x+y) \\ 4xys(s-a)+4bc(s-a)^2 &= 4bc(s-a)(x+y) \\ xys+bc(s-a) &= bc(x+y). \qquad (2) \end{aligned}$$

Now let R be the circumradius of $\triangle ABC$ and use the sine law [in the second equality] and (2) [in the third] to get

$$\begin{aligned} [DPQ] &= [ABC]-\frac{AP\cdot AQ\cdot\sin A}{2}-\frac{BP\cdot BD\cdot\sin B}{2} \\ &\quad -\frac{CD\cdot CQ\cdot\sin C}{2} \\ &= \frac{1}{4R}\Big(abc-xya-(c-x)(s-b)b-(b-y)(s-c)c\Big) \\ &= \frac{1}{4R}\Big[abc-axy-bc(s-b)-bc(s-c)+bx(s-b) \\ &\quad +\; cy(s-c)+\Big(xys+bc(s-a)-bc(x+y)\Big)\Big] \\ &= \frac{1}{4R}\Big(abc-axy-abc+bx(s-b-c)+cy(s-b-c) \\ &\quad +xys+bc(s-a)\Big) \\ &= \frac{1}{4R}\Big(xy(s-a)-bx(s-a)-cy(s-a)+bc(s-a)\Big) \\ &= \frac{(s-a)}{4R}(c-x)(b-y) \;=\; \frac{(s-a)}{4R}\cdot BP\cdot CQ. \end{aligned}$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. Several solvers mentioned that this problem generalizes the proposer's earlier 1862 [1993: 203], [1994: 172-173].

2190. [1996: 361] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the range of

$$\frac{\sin^2 A}{A}+\frac{\sin^2 B}{B}+\frac{\sin^2 C}{C}$$

where A, B, C are the angles of a triangle.

Solution by Kee-Wai Lau, Hong Kong.

Denote the function of the problem by $f(A,B,C)$. We show that

$$0 \;<\; f(A,B,C) \;\le\; \frac{27}{4\pi}.$$

The first inequality follows from the definitions. By considering the degenerate triangle $A=\pi$, $B=0$, $C=0$, we see that it is also sharp. We now prove the second inequality.

For $0<x\le\frac{\pi}{2}$, let $g(x)=\frac{\sin^2 x}{x}$. We have

$$\frac{dg}{dx} \;=\; \frac{\sin x(2x\cos x-\sin x)}{x^2} \quad\text{and}$$

$$\frac{d^2g}{dx^2} \;=\; \frac{h(x)}{x^3},\quad\text{where}\quad h(x)=1-\cos(2x)+2x^2\cos(2x)-2x\sin(2x).$$

Since $\frac{dh}{dx}=-4x^2\sin(2x)\le 0$ and $h(0)=0$, we have $h(x)\le 0$. It follows that $\frac{d^2g}{dx^2}\le 0$, so that g is concave. Hence, if $\triangle ABC$ is acute angled, then

$$f(A,B,C)\;\le\;3g\left(\frac{A+B+C}{3}\right)\;=\;3g\left(\frac{\pi}{3}\right)\;=\;\frac{27}{4\pi}.$$

Equality holds if and only if $A=B=C=\frac{\pi}{3}$.

Now, suppose that one of the angles of $\triangle ABC$ is obtuse. We may assume that $\angle A>\frac{\pi}{2}$, $\angle B+\angle C<\frac{\pi}{2}$.

Hence $f(A,B,C)\;<\;\frac{2}{\pi}+\frac{\sin^2 B}{B}+\frac{\sin^2 C}{C}\;\le\;\frac{2}{\pi}+2\frac{\sin^2\left(\frac{B+C}{2}\right)}{\frac{B+C}{2}}$

[by the concavity of $g(x)$ together with $\frac{\sin^2 A}{A}<\frac{1}{A}<\frac{2}{\pi}$ for $A>\frac{2}{\pi}$].

It is easy to show that $\tan x\le 2x$ for $0\le x\le\frac{\pi}{4}$. Hence $\frac{dg(x)}{dx}\ge 0$ for $0\le x\le\frac{\pi}{4}$. It follows that

$$f(A,B,C)\;<\;\frac{2}{\pi}+2\frac{\sin^2\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}}\;=\;\frac{6}{\pi}\;<\;\frac{27}{4\pi}.$$

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and the proposer.

Three unsatisfactory solutions were received. Two of these depended on an inequality that was passed off as "known". Such claims must be accompanied by a reference. Otherwise, how are the editors supposed to know that the claim is true? More importantly, the purpose of the solution should be to explain **why** *the result is true; consequently any reference should be accessible to* ***CRUX with MAYHEM*** *readers.*

A key step in the third rejected submission was claimed to be "obvious". The only thing obvious to this editor was that such claims do not belong in mathematical arguments.

2191. [1996: 361] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all positive integers n, that satisfy the inequality

$$\frac{1}{3} < \sin\left(\frac{\pi}{n}\right) < \frac{1}{2\sqrt{2}}.$$

Editor's composite solution based on the ones submitted by the solvers whose names appear below.

Since $\sin\pi = 0$, $n = 1$ is clearly not a solution.

Since $\sin\left(\frac{\pi}{n}\right)$ is decreasing for $n \geq 2$ and since

$$\sin\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}} > \frac{1}{2\sqrt{2}}$$

we have $\sin\left(\frac{\pi}{n}\right) > \frac{1}{2\sqrt{2}}$ for $2 \leq n \leq 8$.

On the other hand, since $\sin x < x$ for $x > 0$, we have, for all $n \geq 10$, $\sin\left(\frac{\pi}{n}\right) < \frac{\pi}{n} \leq \frac{\pi}{10} < \frac{1}{3}$.

We now show that

$$\frac{1}{3} < \sin\left(\frac{\pi}{9}\right) < \frac{1}{2\sqrt{2}} \qquad (1)$$

and so $n = 9$ is the only solution. [Ed: Though (1) can be verified numerically by using a calculator, as many solvers did, the proposer's original intention was an "analytic" proof like the one presented below.]

Let $\theta = \frac{\pi}{9}$ and let $r = \sin\theta$. Then from

$$\frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right) = \sin(3\theta) = 3\sin\theta - 4\sin^3\theta = 3r - 4r^3,$$

we see that r is a positive root of the polynomial $f(x) = 4x^3 - 3x + \frac{\sqrt{3}}{2}$.

Note that

$$f(-1) = -1 + \frac{\sqrt{3}}{2} < 0, \qquad f(0) = \frac{\sqrt{3}}{2} > 0,$$

$$f\left(\frac{1}{3}\right) = -\frac{23}{27} + \frac{\sqrt{3}}{2} > 0, \qquad f\left(\frac{1}{2\sqrt{2}}\right) = \frac{-5}{4\sqrt{2}} + \frac{\sqrt{3}}{2} < 0,$$

and

$$f(1) = 1 + \frac{\sqrt{3}}{2} > 0.$$

Since f is a continuous function, we conclude that it has three real roots, one in each of the three intervals: $(-1, 0)$, $\left(\frac{1}{3}, \frac{1}{2\sqrt{2}}\right)$ and $\left(\frac{1}{2\sqrt{2}}, 1\right)$. But $\sin\left(\frac{\pi}{9}\right) < \frac{\pi}{9} \approx 0.349 < \frac{1}{2\sqrt{2}}$, and so (1) follows.

Solved by GERALD ALLEN, CHARLES DIMINNIE, TREY SMITH and ROGER ZARNOWSKI (jointly) Angelo State University, San Angelo, TX, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DIEGO SOTÉS and JAVIER GUTIÉRREZ, students, University of Rioja, Logroño, Spain; and DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA.

There were also nine incomplete or (partially) incorrect solutions. Though these solutions all give the correct answer $n = 9$, they contain various errors. Most of these errors pertain to the analysis of the three roots of $f(x)$ which is actually more subtle than it appears to be. Some solvers erroneously claimed that one of the roots is greater than one. Using **MAPLE**, *we find easily that the three roots are: -0.9848077, 0.3420201, and 0.6427876. Only one (joint) solver stated that the three roots are, in fact, $\cos\left(\frac{19\pi}{18}\right)$, $\sin\left(\frac{\pi}{9}\right)$, and $\sin\left(\frac{7\pi}{9}\right)$. However, they made the mysterious and wrong statement that $\sin\left(\frac{7\pi}{9}\right) < \frac{1}{3}$.*

Some other solvers, after checking that $f\left(\frac{1}{3}\right)$ and $f\left(\frac{1}{2\sqrt{2}}\right)$ have opposite signs, jump to the conclusion that $\sin\left(\frac{\pi}{9}\right)$ must be the root in the interval $\left(\frac{1}{3}, \frac{1}{2\sqrt{2}}\right)$. Clearly, to make this argument valid, one has to show that the other positive root is not in this interval.

2192. [1996: 362] *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Let $\{a_n\}$ be a sequence defined as follows:

$$a_{n+1} + a_{n-1} = \left(\frac{a_2}{a_1}\right) a_n, \qquad n \geq 1.$$

Show that if $\left|\frac{a_2}{a_1}\right| \geq 2$, then $\left|\frac{a_n}{a_1}\right| \geq n$.

Solution by Can Anh Minh, University of California, Berkeley.

By the triangle inequality we have

$$|a_{n+1}| + |a_{n-1}| \geq |a_{n+1} + a_{n-1}| = \left|\frac{a_2}{a_1} a_n\right| = \left|\frac{a_2}{a_1}\right| |a_n| \geq 2|a_n|$$

Thus we have $|a_{n+1}| - |a_n| \geq |a_n| - |a_{n-1}|$, and therefore

$$|a_n| - |a_{n-1}| \geq |a_{n-1}| - |a_{n-2}| \geq \cdots \geq |a_2| - |a_1| \geq |a_1|$$

Thus we have

$$|a_n| - |a_1| = \sum_{k=2}^{n} (|a_k| - |a_{k-1}|) \geq (n-1)|a_1|$$

It follows that $|a_n| \geq n|a_1|$, or equivalently

$$\left|\frac{a_n}{a_1}\right| \geq n.$$

Also solved by GERALD ALLEN, CHARLES DIMINNIE, TREY SMITH, and ROGER ZARNOWSKI (jointly), Angelo State University, San Angelo, Texas; CHETAN T. BALWE, Pune, India; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer.

2193. [1996: 362] *Proposed by Luis V. Dieulefait, IMPA, Rio de Janeiro, Brazil.*

(a) Prove that every positive integer is the difference of two relatively prime composite positive integers.

(b) Prove that there exists a positive integer n_0 such that every positive integer greater than n_0 is the sum of two relatively prime composite positive integers.

Solution to (a) *by Michael Lambrou, University of Crete, Crete, Greece, modified slightly by the editor.*

Note that for all natural numbers k, we have:

$$2k + 1 = (k + 1)^2 - k^2 \tag{1}$$

and

$$2k = (2k + 1)(8k + 1) - (4k + 1)^2 \tag{2}$$

Since $\gcd(k, k + 1) = 1$, (1) gives a required representation for all odd integers greater than 3. But $1 = 9 - 8$ and $3 = 25 - 22$, and so a required representation exists for all odd natural numbers. On the other hand, straightforward computations show that

$$-(8k + 4)(2k + 1)(8k + 1) + (8k + 5)(4k + 1)^2 = 1$$

and so

$$\gcd\left((2k + 1)(8k + 1), (4k + 1)^2\right) = 1.$$

Thus (2) gives a required representation for all even natural numbers.

Solution to (b) *by the proposer, modified by the editor.*

Let $\phi(n)$ denote Euler's totient function and let $\pi(n)$ denote the prime-counting function. [Ed: that is, $\pi(n)$ is the number of primes p such that $p \leq n$.] It is known (see Hardy and Wright, *An Introduction to the Theory of Numbers*) that

$$\liminf_{n\to\infty} \frac{\phi(n)}{\dfrac{n}{\log\log n}} = e^{-\gamma} > \frac{1}{2},$$

where γ is Euler's constant. Hence, for sufficiently large n, we have

$$\phi(n) > \frac{2}{2\log\log n}. \tag{1}$$

On the other hand, by Tschebycheff's theorem, we have

$$\pi(n) < \frac{6}{5}\frac{n}{\log n} \tag{2}$$

if n is large enough. From (1) and (2), we obtain

$$\lim_{n\to\infty} \frac{\phi(n)}{\pi(n)} > \lim_{n\to\infty} \frac{5\log n}{12\log\log n} = \infty,$$

which implies, for large enough n, that

$$\phi(n) \geq 2\left(\pi(n)+1\right). \tag{3}$$

Note that, if $n = a + b$, then $(a, n) = 1$ is equivalent to $(b, n) = 1$, and to $(a, b) = 1$.

Consider the $\phi(n)$ ordered decompositions of $n : n = k + (n - k)$, where $1 \leq k \leq n$, such that $\gcd(k, n) = 1$. If we strike out those pairs in which the first or second summand is a prime or 1, then we are deleting at most $2(\pi(n) + 1)$ pairs, and so from (3), we conclude that there is at least one decomposition $n = a + b$ where a and b are relatively prime composite natural numbers, and our proof is complete.

Solved (both parts) by RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer. Part (a) *only was solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.*

Regarding (a), *Janous actually obtained a stronger result by showing that there are, in fact, infinitely many required representations. Regarding* (b), *Hess remarked that a computer search seems to indicate that* $n_0 \leq 210$. *Using arguments similar to those given in the proof above, Lambrou proved a stronger result, namely:*

For any given integer $k \geq 1$, there exists a positive integer m_k such that every integer greater than m_k can be written as the sum of two relatively prime composite natural numbers in at least m_k different ways.

It is not difficult to see that this result also follows easily from the proposer's proof presented above.

2194. [1996: 362] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Prove or disprove that it is possible to find a triangle ABC and a transversal NML with N lying between A and B, M lying between A and C, and L lying on BC produced, such that BC, CA, AB, NB, MC, NM, ML, and CL are all of integer length, and $NMCB$ is a cyclic inscriptable quadrilateral.

Editor's comment. There is a terminology problem here. It appears (from his solution) that the proposer had intended the word *inscriptable* to mean that a circle can be inscribed in the quadrangle $NMCB$. Although several of the solvers assumed this to be the case, all reference books that I consulted disagree. The *Oxford English Dictionary* and Nathan Altshiller Court (in his 1925 *College Geometry*) both say that an *inscriptible quadrangle* (note the spelling) is one that can be inscribed in a circle; today one more commonly calls such quadrangles (that can be inscribed in circles) *cyclic, concyclic*, or *inscribable* — take your choice. On the other hand, one must keep in mind that English (the language in which, for example, *inflammable* means *flammable* and *unravel* means *ravel*) is so unpredictable that the proposer might well have references to back up his terminology. For a quadrangle that contains an inscribed circle, one would say *circumscribing* or avoid the issue and just say "a quadrangle with an inscribed circle". The matter really becomes muddy here: Court calls the latter quadrangle *circumscriptible*, while the *OED* agrees in one place (under *inscriptible*), but contradicts itself in another, where a quadrangle is said to be *circumscribable* if it "may be circumscribed by a circle". The moral of this story: be circumspect (or maybe, circumspectable). As it turns out, our featured solutions all produce cyclic quadrangles having inscribed circles!

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece* (somewhat edited).

We show that *any* triangle ABC with rational sides and $\angle A \neq \angle C$ has a transversal LMN such that $NMCB$ satisfies both circle conditions, and has all relevant lengths rational. Hence, multiplying by an appropriate number, we can obtain a similar triangle with the required properties.

Let a triangle ABC with rational sides be given. By renaming, we may assume that B, which may be acute, right or obtuse, is larger than C. Note that B cannot equal C, as the transversal — in order to produce a cyclic quadrangle — would then be parallel to BC.

Consider as transversal LMN such that

$$\angle AMN = B \text{ (making } NMCB \text{ cyclic), and} \qquad (4)$$

$$LM \text{ is tangent to the incircle of triangle } ABC \qquad (5)$$

(forcing $NMCB$ to have an inscribed circle). We must show that the lengths NB, MC, NM, ML and CL are all rational.

We have that the triangles ANM and ACB are similar. Let $\lambda = AM/AB$ be the similarity ratio. We use the fact that the sides of triangle ABC are rational to show that $\lambda \in \mathbb{Q}$.

$$\begin{aligned} &\lambda(BC + AB + AC) \\ &\quad = MN + AM + AN \\ &\quad = MN + (AC - MC) + (AB - NB) \\ &\quad = [(MN + BC) - (NB + MC)] + AB + AC - BC \\ &\quad = 0 + AB + AC - BC \ \in \mathbb{Q}. \end{aligned}$$

But $BC + AB + AC \in \mathbb{Q}$, so that $\lambda \in \mathbb{Q}$ as well.

Next, by Menelaus's Theorem applied to triangle ABC with transversal LMN, it follows that

$$\frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = 1 \in \mathbb{Q},$$

so that $BL/CL \in \mathbb{Q}$. Since $BC/CL = (BL - CL)/CL$, we have $LC \in \mathbb{Q}$ and $BL \in \mathbb{Q}$.

Similarly, by Menelaus's Theorem applied to triangle MLC with transversal AB, we see that $LN \in \mathbb{Q}$, and the desired result follows.

II. *Families of solutions by Michael Lambrou, University of Crete, Crete, Greece and (independently) by Richard I. Hess, Rancho Palos Verdes, California, USA.*

Let AMN and LBN be congruent right triangles with right angles at M and B. Denote the legs by p ($= AM = LB$), q ($= MN = BN$), and the hypotenuse by r ($= NL = NA = \sqrt{p^2+q^2}$). The solution is achieved by choosing p, q, r so that p divides $r+q$. Hess provided the examples in column 2, and Lambrou, column 3.

	Hess	Lambrou
$p = AM = LB$	$2k+1$	λ^2,
$q = MN = BN$	$2k(k+1)$	$\lambda\mu$,
$r = NL = NA$	$(k+1)^2 + k^2$	$\lambda\nu$, where $\lambda^2 + \mu^2 = \nu^2$,
$r + q = AB = LM$	$(2k+1)^2$	$(\mu+\nu)\lambda$,
$q(q+r)/p = MC = BC$	$2k(k+1)(2k+1)$	$(\mu+\nu)\mu$,
$r(q+r)/p = AC = LC$	$(2k+1)\times((k+1)^2+k^2)$	$(\mu+\nu)\nu = \lambda^2+\mu^2+\mu\nu$.

In each case, $NMCB$ has a circumscribed circle because of the right angles at B and M, while it has an inscribed circle because of its symmetry (so that $NM + CB = NB + CM$).

Lambrou also sent in a family of asymmetric solutions for positive integers $m > n$:

$$\begin{aligned} AB &= nm^2(m-n)(n^2+1), \\ AC &= n^2m(m-n)(m^2+1), \\ BC &= mn(m-n)(m+n)(mn-1), \quad \text{and} \\ AN &= \frac{1}{mn}AC, \quad AM = \frac{1}{mn}AB. \end{aligned}$$

We leave it to the reader to verify the details. (Remember to check the various triangle inequalities.)

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2195. [1996: 362] *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

A barrel contains $2n$ balls, numbered 1 to $2n$. Choose three balls at random, one after the other, and with the balls replaced after each draw.

What is the probability that the three-element sequence obtained has the properties that the smallest element is odd and that only the smallest element, if any, is repeated?

For example, the sequences 453 and 383 are acceptable, while the sequences 327 and 388 are not.

(NOTE: this problem was suggested by a final exam that I marked recently.)

I. *Solution by David Hankin, Hunter College Campus Schools, New York, NY, USA.*

Let u_k be the number of acceptable sequences chosen from balls numbered 1 to $2k$, and let a_k be the number of these acceptable sequences that contain 1. We first show that, for $k \geq 1$,

$$u_k - a_k = u_{k-1},$$

where $u_0 = 0$. Note that $u_k - a_k$ is the number of acceptable sequences that do not contain 1. The number of these sequences is the same as the number of acceptable sequences that can be chosen from balls numbered 3 to $2k$. Clearly, this is equal to u_{k-1}.

To find a_k, note that sequences that contain 1 must have one, two or three 1's. There is one sequence with three 1's. For sequences with two 1's, there are $2k-1$ ways to choose the third element and 3 arrangements of the three elements; so there are $3(2k-1)$ such sequences. Similarly, there are $6\binom{2k-1}{2} = 6(2k-1)(k-1)$ sequences with one 1. Thus

$$a_k = 1 + 3(2k-1) + 6(2k-1)(k-1) = 12k^2 - 12k + 4 = 4[k^3 - (k-1)^3].$$

Now (since $u_0 = 0$)

$$u_n = \sum_{k=1}^{n} (u_k - u_{k-1}) = \sum_{k=1}^{n} a_k = 4\sum_{k=1}^{n} [k^3 - (k-1)^3] = 4n^3.$$

Thus the requested probability is

$$\frac{4n^3}{(2n)^3} = \frac{1}{2}\,.$$

II. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

We show that there is a one-to-one onto pairing that pairs each favourable triplet abc (which for convenience we will denote (a,b,c)) with an unfavourable one. Once this is done, the independence of events shows that the sought probability is $1/2$, as the favourable events exactly match the unfavourable ones.

We denote the smallest element among (a, b, c) by $2s - 1$, where $1 \leq s \leq n$. The pairing will be described for the case when $2s - 1$ occurs in the first position (and is perhaps repeated in one or both of the other positions). The other two possibilities are dealt with in a similar fashion, by cyclic change of order. Note that some care must be taken not to double count the triplets that belong to more than one situation.

Here is how we do our pairing.

- For fixed s and p with $2s - 1 < p \leq 2n - 1$, map, in any one-to-one onto fashion, the $2n - 2s$ triplets of the form

$$(2s-1, p, q) \qquad \text{where} \quad 2s - 1 < q \leq 2n \quad \text{and} \quad q \neq p$$

  to the $2n - 2s$ triplets of the form

$$(2s, p+1, v) \qquad \text{where} \quad 2s < v \leq 2n;$$

- for $2s - 1 < q \leq 2n - 1$, map $(2s-1, 2s-1, q)$ to $(2s, 2s, q+1)$;
- for $2s - 1 < q \leq 2n - 1$, map $(2s-1, 2n, q)$ to $(2s-1, q, q)$;
- map $(2s-1, 2s-1, 2n)$ to $(2n, 2n, 2s-1)$; and finally
- map $(2s-1, 2s-1, 2s-1)$ to $(2s, 2s, 2s)$.

It is easy to see that this map is not ambiguous. It is also easy to check that all favourable and all unfavourable triplets have been taken into account once and once only. This completes the argument.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; W. MOSER, McGill University, Montréal, Québec; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; D. N. SHETH, Sir Parashuram College, Pune, India; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and the proposer. Two incorrect solutions were sent in.

Lambrou was the only solver to find a "combinatorial" proof that the probability is exactly $1/2$. He also sent in two other solutions, one of which is similar to solution I. *Bradley's solution is also similar to solution* I.

2196. [1996: 362] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Find all solutions of the diophantine equation

$$2(x+y)+xy=x^2+y^2,$$

with $x>0$, $y>0$.

Solution by Sam Baethge, Nordheim, Texas, USA.

Let $y=rx$ with r rational. The given equation becomes a quadratic in r:

$$r^2x^2-r(x^2+2x)+(x^2-2x)=0$$

Then $r=\left(x+2\pm\sqrt{-3x^2+12x+4}\right)/2x$. The discriminant can be written as $16-3(x-2)^2$ and must be the square of an integer. Since x is a positive integer the only solutions are $(x,r)=(4,1)$, $(4,\frac{1}{2})$, or $(2,2)$. This produces $(x,y)=(4,4)$, $(4,2)$, or $(2,4)$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; GORAN CONAR, student, Varaždin, Croatia; PAUL-OLIVIER DEHAYE, Brussels, Belgium; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; C. FESTRAETS-HAMOIR, Brussels, Belgium; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; IGNOTUS, Villeta, Colombia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; CAN ANH MINH, University of California, Berkeley; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; ADRIÁN UBIS MATIÍNEZ, Logroño, Spain; DAVID C. VELLA, Skidmore College, Saratoga Springs, New York; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There were seven incorrect solutions and one incomplete solution.

Vella (in one of his two submitted solutions) used a change of variables to turn the given equation into the equation of an ellipse with centre on the

x-axis and major axis aligned with the x-axis, and then used a geometric approach to solve the problem. The proposer also suggests a generalization:

Solve the diophantine equation:

$$z(x+y)+xy = x^2+y^2$$

2197. [1996: 363] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer. Evaluate the sum:

$$\sum_{k=n}^{\infty} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}}.$$

Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.

The value of the sum is $\dfrac{1}{4^n}\dbinom{2n}{n}$. This follows directly from

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}} = 1 \tag{1}$$

and

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}} = 1 - \frac{1}{4^n}\binom{2n}{n}. \tag{2}$$

To prove (1), note that it is easy to show that

$$\frac{\binom{2k}{k}}{(k+1)2^{2k+1}} = -\binom{\frac{1}{2}}{k+1}(-1)^{k+1}$$

$\left(\text{they both simplify to } \dfrac{1\cdot 3\cdot 5\cdots(2k-1)}{2^{k+1}(k+1)!}\right)$. Therefore

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}} &= -\sum_{k=0}^{\infty}(-1)^{k+1}\binom{\frac{1}{2}}{k+1} = -\sum_{k=1}^{\infty}(-1)^{k}\binom{\frac{1}{2}}{k} \\ &= -\left(-1+\sum_{k=0}^{\infty}(-1)^k\binom{\frac{1}{2}}{k}\right) = 1-\sqrt{1+(-1)} = 1.\end{aligned}$$

To prove (2), we use induction. After checking at $n = 1$, we use the inductive assumption to get

$$\begin{aligned}\sum_{k=0}^{n} \frac{\binom{2k}{k}}{(k+1)2^{2k+1}} &= 1 - \frac{1}{4^n}\binom{2n}{n} + \frac{\binom{2n}{n}}{(n+1)2^{2n+1}} \\ &= 1 - \frac{1}{4^n}\binom{2n}{n}\left(1 - \frac{1}{2(n+1)}\right) = 1 - \frac{1}{4^n}\binom{2n}{n}\frac{2n+1}{2(n+1)} \\ &= 1 - \frac{1}{4^{n+1}}\binom{2n}{n}\frac{(2n+1)(2n+2)}{(n+1)^2} = 1 - \frac{1}{4^{n+1}}\binom{2n+2}{n+1},\end{aligned}$$

which proves the identity.

Also solved by PAUL BRACKEN, Université de Montréal, Montréal, Québec; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; STAN WAGON, Macalester College, St. Paul, Minnesota, USA; and the proposer.

Janous finds the sum by considering the generating function of the sequence $\{C_k\} = \left\{\frac{1}{k+1}\binom{2k}{k}\right\}$ *of Catalan numbers and in doing so, obtains, as a by-product, the recurrence relation*

$$C_n = \frac{1}{n+1}\left(4^n - 2\sum_{k=0}^{n-1} C_k \cdot 4^{n-1-k}\right).$$

He believes that this is a new identity, and wonders whether there is a purely combinatorial proof and/or interpretation of it.

Wagon obtained the sum by using MATHEMATICA, *and remarked that more sophisticated symbolic algebra software (work of Petkovsek, Wilf and Zeilberger) can provide certificates that the closed-form formula obtained is actually valid. He also commented that:* "Thus sums such as this are really best left to computers, ... ". *This editor is interested in knowing how many of our readers would agree with him.*